The spectral analysis and exponential stability of a bi-directional coupled wave-ODE system

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In this paper, an unstable linear time invariant (LTI) ODE system \( \dot{X}(t) = AX(t) \) is stabilized exponentially by the PDE compensator—a wave equation with Kelvin-Voigt (K-V) damping. Direct feedback connections between the ODE system and wave equation are established: The velocity of the wave equation enters the ODE through the variable \( v_t(1, t) \); meanwhile, the output of the ODE is fluxed into the wave equation. It is found that the spectrum of the system operator is composed of two parts: point spectrum and continuous spectrum. The continuous spectrum consists of an isolated point \(-\frac{1}{d}\), and there are two branches of asymptotic eigenvalues: the first branch approaches to \(-\frac{1}{d}\), and the other branch tends to \(-\infty\). It is shown that there is a sequence of generalized eigenfunctions, which forms a Riesz basis for the Hilbert state space. As a consequence, the spectrum-determined growth condition and exponential stability of the system are concluded.

KEYWORDS
exponential stability, Riesz basis, the spectral analysis, wave equation

1 INTRODUCTION

It is known that an unstable ODE/PDE system can be stabilized by a time-delay controller, see other works\textsuperscript{4}-\textsuperscript{6} and references therein. Due to the time-delay can be depicted by a first-order transport PDE, so the overall system can be seen as a PDE-ODE coupled system, where the PDE served as a compensator to stabilize the unstable ODE/PDE system. In Wang et al\textsuperscript{4} and Zhao and Wang,\textsuperscript{4,5} the exponential stability of a PDE-ODE coupled system, which is composed of a pendulum system and a time-delay controller, is established by the semigroup theory and spectral analysis. In Krstic,\textsuperscript{6} the control of an unstable reaction-diffusion PDE with input delays is studied, and a Smith predictor-like design is first proposed to stabilize the time-delay PDE coupled system by a backstepping transformation.

In recent years, the PDE compensators, such as heat equation and wave equation, are widely used to stabilize the ODE/PDE system. In previous studies,\textsuperscript{7-12} an LTI finite-dimensional ODE system is compensated by a diffusion heat and a wave equation, respectively. The explicit feedback laws and observers are presented, and the stability and robustness of the overall coupled system are proved by backstepping transformation, decoupling technique, and Lyapunov function. By the semigroup theory and the spectral analysis methods, the stabilization of an unstable reaction-diffusion PDE or a Schrödinger equation cascaded with a heat equation are established, in Wang et al.\textsuperscript{13,14}
In this paper, we attempt to show a simple and direct boundary feedback connection between the $n$–th ODE and a wave equation with K-V damping (see Figure 1) and analyze the spectral distribution and exponential stability for the coupled system:

\[
\begin{align*}
\dot{X}(t) &= AX(t) + Bv_t(1, t), \\ v_t(x, t) &= v_{xx}(x, t) + dv_{xx}(x, t), \\ v(0, t) &= 0, \\ v_x(1, t) + dv_{tx}(1, t) &= -BCX(t),
\end{align*}
\]

where $X \in \mathbb{C}^n$ and $v(x, t) \in \mathbb{C}$ are the states of ODE and the wave equation respectively, $t \geq 0$ is time, and $x \in [0, 1]$ is the spatial variable. The velocity of the wave equation enters the ODE through the variable $v_t(1, t)$, while the output $X(t)$ is fed into the boundary of the wave equation.

Throughout this paper, we made the following assumptions for the matrices $A, B, C$:

(I) Let the matrix $A \in \mathbb{R}^{n \times n}$ is invertible and $-\frac{1}{d}$ is not an eigenvalue of $A$. Without loss of generality, let $A$ to be a block diagonal matrix, that is

\[
A = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix},
\]

since any matrix can be transformed into the form through an invertible linear transformation, where $A_1 \in \mathbb{R}^{m \times m}$, $A_2 \in \mathbb{R}^{k \times k}$, $n = m + k$.

(II) Let $B^T = \begin{pmatrix} 0 & B_1 \end{pmatrix} \in \mathbb{R}^{1 \times n}$, where the first $m$ components of the matrix $B$ are 0, $B_1 \in \mathbb{R}^{1 \times k}$, and $C \in \mathbb{R}^{n \times n}$ is positive definite.

(III) To matrices $A, B,$ and $C$, $(A, B)$ is controllable, $A^HC + CA = 0$, and $B^TC = \begin{pmatrix} 0 & B_2 \end{pmatrix}$, where the first $m$ components of the matrix $B^TC$ are 0, $B_2 \in \mathbb{R}^{1 \times k}$.

The paper is organized as follows. In Section 2, we formulate the system (1) into an abstract evolution equation and prove the well-posedness of the system. Section 3 is devoted to the spectrum analysis. In Section 4, the asymptotic expressions of eigenvalues and eigenfunctions are presented, and the exponential stability of (1) is established.

# 2 WELL-POSEDNESS OF SYSTEM (1)

We consider system (1) in the energy space

\[
\mathcal{H} = \mathbb{C}^n \times H^1_L(0, 1) \times L^2(0, 1)
\]

equipped with the usual inner product:

\[
\langle Z_1, Z_2 \rangle = X^H_1CX_2 + \int_0^1 f_1'(x)f_2'(x)dx + \int_0^1 g_1(x)g_2(x)dx,
\]

where $H^1_L(0, 1) = \{ f \in H^1(0, 1) | f(0) = 0 \}$, $Z_1 = (X_1, f_1, g_1)$, $Z_2 = (X_2, f_2, g_2) \in \mathcal{H}, X^H_1$ denotes the adjoint of $X_1$, which is the conjugate of the transpose of $X_1$. 
Define a linear operator \( A : D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H} \) by:

\[
A(X, f, g) = \left( AX + Bg(1), g, (f + dg)'' \right),
\]

with

\[
D(A) = \{ (X, f, g) \in \mathcal{H} | A(X, f, g) \in \mathcal{H}, f'(1) + dg'(1) + B^T C X = 0 \}.
\]

Then (1) can be written as an evolution equation in \( \mathcal{H} \):

\[
\begin{align*}
Z(t) &= AZ(t), t > 0, \\
Z(0) &= Z_0,
\end{align*}
\]

where \( Z(t) = (X(t), \nu(t), v_l(t)) \).

Now, we have the following results on the properties of \( A \).

**Lemma 1.** Let \( A \) be given by (5) and (6). Then its adjoint operator \( A^* \) has the following form:

\[
A^*(\tilde{X}, \tilde{f}, \tilde{g}) = (-A\tilde{X} - B\overline{g}(1), -\overline{g}, -(\tilde{f} - d\overline{g})'').
\]

where

\[
D(A^*) = \{ (\tilde{X}, \tilde{f}, \tilde{g}) \in \mathcal{H} | A^*(\tilde{X}, \tilde{f}, \tilde{g}) \in \mathcal{H}, \tilde{f}'(1) - d\overline{g}'(1) + B^T C \tilde{X} = 0 \}.
\]

**Proof.** For each \( F = (X, f, g) \in D(A) \), \( G = (\tilde{X}, \tilde{f}, \tilde{g}) \in D(A^*) \), a direct computation yields

\[
\begin{align*}
\langle AF, G \rangle &= \langle (AX + Bg(1), g, (f + dg)''), \tilde{X}, \tilde{f}, \tilde{g} \rangle \\
&= \langle AX + Bg(1) \rangle^T C X + \int_0^1 g'(x) \tilde{f}'(x) dx + \int_0^1 (f + dg)''(x) \overline{g}(x) dx \\
&= X^T A^T C X + B^T C X g(1) + \int_0^1 \tilde{f}'(1) g(1) - \tilde{f}'(0) g(0) - \int_0^1 g(x) \tilde{f}''(x) dx \\
&+ f'(1) \overline{g}(1) + dg'(1) \overline{g}(1) - f'(0) \overline{g}(0) - dg'(0) \overline{g}(0) - \int_0^1 (f'(x) + dg'(x)) \overline{g}(x) dx \\
&= X^T A^T C X + B^T C X g(1) + \int_0^1 \tilde{f}'(1) g(1) - \tilde{f}'(0) g(0) - \int_0^1 g(x) \tilde{f}''(x) dx \\
&+ f'(1) \overline{g}(1) + dg'(1) \overline{g}(1) - f'(0) \overline{g}(0) - dg'(0) \overline{g}(0) - \int_0^1 f'(x) \overline{g}(x) dx \\
&- d(g(1) \overline{g}(1) - g(0) \overline{g}(0) - \int_0^1 g(x) \overline{g}(x) dx) \\
&= X^T C(-A) \tilde{X} + (B^T C X + \tilde{f}'(1) - d\overline{g}'(1)) g(1) - \int_0^1 f'(x) \overline{g}(x) dx \\
&- \int_0^1 g(\tilde{f} - d\overline{g})'' dx + (f'(1) + dg'(1)) \overline{g}(1) \\
&= X^T C(-A) \tilde{X} - \int_0^1 f'(x) \overline{g}(x) dx - \int_0^1 g(\tilde{f} - d\overline{g})'' dx + (X^T C B \overline{g}(1) \\
&= X^T C(-A) \tilde{X} - B\overline{g}(1)) - \int_0^1 f'(x) \overline{g}(x) dx - \int_0^1 g(\tilde{f} - d\overline{g})'' dx \\
&= \langle F, A^* G \rangle,
\end{align*}
\]

and hence, we obtain \( A^* \) given by (8) and (9).

**Lemma 2.** Let \( A \) be given by (5) and (6), \( A^* \) be given by (8) and (9). Then \( A \) and \( A^* \) are dissipative in \( \mathcal{H} \) and \( A \) generates a \( C_0 \)-semigroup \( e^{At} \) of contractions on \( \mathcal{H} \).
Proof. For each $Z = (X, v, v_t) \in D(A)$, we have
\[
\langle AZ, Z \rangle + \langle Z, AZ \rangle
\]
\[
= \langle (AX + Bv_t(1,t), v_t, (v + dv_t)''), (X, v, v_t) \rangle + \langle (X, v, v_t), (AX + Bv_t(1,t), v_t, (v + dv_t)''') \rangle
\]
\[
= (AX + Bv_t(1,t))'\overline{X} + \int_0^1 v_xv_{\alpha} dx + \int_0^1 (v + dv_t)'''v_t dx
\]
\[
+ X'' C(AX + Bv_t(1,t)) + \int_0^1 v_xv_{\alpha} dx + \int_0^1 v_t(v + dv_t)'' dx
\]
\[
= X'' A'\overline{X} + B'' C\overline{X}v_t(1,t) + \int_0^1 v_xv_{\alpha} dx + v_x(1,t)\overline{v_t(1,t)} - \int_0^1 v_xv_{\alpha} dx
\]
\[
+ (\overline{v_t(1,t)} - d) \int_0^1 v_xv_{\alpha} dx + X'' CAX + X'' C\overline{Bv_t(1,t)}
\]
\[
= X'' (A'' C + CA)X + B'' C\overline{X}v_t(1,t) + X'' C\overline{Bv_t(1,t)} + v_x(1,t)\overline{v_t(1,t)} + v_t(1,t)v_x(1,t)
\]
\[
+ (\overline{v_t(1,t)} - d) \int_0^1 v_xv_{\alpha} dx
\]
\[
= (B'' C + v_x(1,t) + d)\overline{v_t(1,t)} + (X'' CB + v_x(1,t) + dv_x(1,t))\overline{v_t(1,t)}
\]
\[
- 2d \int_0^1 |v_x| dx
\]
\[
= -2d \int_0^1 |v_x| dx,
\]
so
\[
\text{Re}(AZ, Z) = -d \int_0^1 |v_x| dx \leq 0. \quad (11)
\]

Similarly, for any $\tilde{Z} = (\tilde{X}, \tilde{f}, \tilde{g}) \in D(A^*)$,
\[
\text{Re}(A^*\tilde{Z}, \tilde{Z}) = -d \int_0^1 |\tilde{v}_\alpha| dx \leq 0. \quad (12)
\]

Therefore, both $A$ and $A^*$ are strongly dissipative in $H$. By the Lumer-Philips Theorem,$^{15}$ $A$ generates a $C_0$-semigroup $e^{At}$ of contractions on $H$.

3 | SPECTRAL ANALYSIS FOR SYSTEM (7)

In this section, we investigate the distribution of spectrum for $A$ in the complex plane. Some analytic methods in Guo et al.$^{16,17}$ will be adopted here. Firstly, we consider the eigenvalue problem of (7). By $AZ = \lambda Z$, where $Z = (X, f, g) \in D(A)$, we have,
\[
\begin{align*}
AX + Bg(1) &= \lambda X, \\
g &= \lambda f, \quad f(0) = 0.
\end{align*}
\]
\[
(1 + d\lambda) \overline{f''} = \lambda^2 f, \quad f''(1) + dg(1) = -B'' C\overline{X}. \quad (13)
\]

So
\[
\begin{align*}
\begin{cases}
(\lambda - A)X = Bg(1), & g = \lambda f, \\
(1 + d\lambda) \overline{f''} = \lambda^2 f, & f(0) = 0, \quad (1 + d\lambda) f''(1) = -B'' C\overline{X}.
\end{cases}
\end{align*}
\]

The following Theorem 1 shows that $\sigma_r(A)$, the set of residual spectrum of $A$, is empty. As usual, $\sigma_p(A)$ and $\sigma_c(A)$ denote the set of point spectrum and continuous spectrum, respectively.

**Theorem 1.** Let $A$ be given by (5) and (6), then $\sigma_r(A) = \emptyset$. 

Proof. Since $\lambda \in \sigma_c(A)$ if and only if $\tilde{\lambda} \in \sigma_p(A^*)$, the proof will be accomplished if we show that $\sigma_p(A^*) = \sigma_c(A)$. From (8) and (9), the eigenvalue problem $A^*F = \lambda F$, where $F = (X, f, \bar{g}) \in D(A^*)$ reads:

$$\begin{cases}
-A\bar{X} - \bar{B}\bar{g}(1) = \lambda \bar{X}, \\
-\bar{g} = \lambda \bar{f}, \\
-(\tilde{f} - \bar{df})'' = \lambda \bar{g}, \\
\tilde{f}(0) = 0, \\
\tilde{f}''(1) - \bar{dg}(1) + B^T C\bar{X} = 0.
\end{cases} \tag{15}$$

Obviously, similar to the proof of Zhao and Wang, theorem 3.1 (15) is the same as (13) by setting $f = -\tilde{f}$, $g = \bar{g}$, $M_1 = \tilde{M}_1$, $M_2 = -\tilde{M}_2$. under assumptions (II) and (III), where $X^T = (M_1 M_2)$, $M_1 \in \mathbb{R}^{1 \times m}$, $M_2 \in \mathbb{R}^{1 \times k}$. Hence, $A^*$ has the same eigenvalues with $A$. Since the eigenvalues of $A$ are symmetric with the real axis, we have $\sigma_c(A) = \emptyset$. \qed

**Theorem 2.** Let $A$ be given by (5) and (6). Then $A^{-1}$ exists, and hence $0 \in \rho(A)$, the resolvent set of $A$.

**Proof.** For any given $Z_1 = (X_1, f_1, g_1) \in H$, solve

$$A(X, f, g) = (AX + Bg(1), g, (f + dg)'' = (X_1, f_1, g_1), \quad (X, f, g) \in D(A)$$

to get

$$\begin{cases}
AX + Bg(1) = X_1, \\
f(0) = 0, \\
f'' + dg''(1) + B^T C\bar{X} = 0.
\end{cases} \tag{16}$$

Then, we have

$$\begin{cases}
X = A^{-1}(X_1 - Bf(1)), \\
f(x) = (-df_1''(1) - B^T C\bar{X})x - \int_0^x (\int_s^1 (g_1(r) - df_1''(r))dr)ds \\
g(x) = f_1(x).
\end{cases} \tag{17}$$

Hence, $A^{-1}$ exists, and $0 \in \rho(A)$. \qed

**Lemma 3.** Let $A$ be given by (5) and (6). Then $-\frac{1}{d} \in \sigma_c(A)$.

**Proof.** By Theorem 1, it suffices to show $-\frac{1}{d} \notin \sigma_p(A) \cup \rho(A)$.

(i) Obviously, $0 \notin \sigma_p(A)$. Suppose $\lambda = -\frac{1}{d}$ is an eigenvalue of $A$, then by (14), we have $f = g = 0$, and hence, $X = 0$. This contradiction shows $-\frac{1}{d} \notin \sigma_p(A)$.

(ii) For any $G = (\bar{X}, \tilde{f}, \bar{g}) \in H$, solve $(\lambda I - A)F = G$, $F = (X, f, g) \in D(A)$, that is,

$$\begin{cases}
\lambda X - (AX + Bg(1)) = \bar{X}, \\
\lambda f - g = \tilde{f}, \\
\lambda g - (f + dg)'' = \bar{g}, \\
\tilde{f}(0) = 0, \\
\tilde{f}''(1) + dg''(1) + B^T C\bar{X} = 0.
\end{cases} \tag{18}$$

Substitute the second equation into the third equation, we have $\lambda^2 g - ((1 + d\lambda)g + \tilde{f})'' = \lambda \bar{g}$. If $\lambda = -\frac{1}{d}$, it has $\lambda^2 g - \tilde{f}'' = \lambda \bar{g}$, i.e,

$$\lambda^2 g = \tilde{f}'' + \lambda \bar{g}.$$ 

Since $g \in H^1_0(0, 1)$, the above equality holds true unless $\tilde{f}'' + \lambda \bar{g} \in H^1_0(0, 1)$. This shows that $-\frac{1}{d} \notin \rho(A)$. Combining the above two cases completes the proof. \qed
Theorem 3. Let $A$ be given by (5) and (6), and let
\[
\Delta(\lambda) = \det(\lambda - A + BB^H C \frac{1}{\sqrt{1 + d\lambda}} \tanh \sqrt{\frac{\lambda^2}{1 + d\lambda}}).
\] (19)

Then,
\[
\sigma_p(A) = \{ \lambda \in \mathbb{C} | \Delta(\lambda) = 0 \},
\] (20)
and each $\lambda \in \sigma_p(A)$ is geometrically simple.

Proof. The eigenvalue problem (13) is equivalent to
\[
\begin{cases}
AX + Bg(1) = \lambda X, \\
g = \lambda f,
\end{cases}
\] (21)
and $f$ satisfies
\[
\begin{cases}
f(x) = c \sinh \sqrt{\frac{\lambda^2}{1 + d\lambda}} x, \\
f'(1) = -\frac{\lambda c x}{1 + d\lambda},
\end{cases}
\] (22)
So we have
\[
c = -\frac{1}{\lambda} \frac{1}{\sqrt{1 + d\lambda}} \frac{1}{\cosh \sqrt{\frac{\lambda^2}{1 + d\lambda}}} B^T C \bar{X},
\] (23)
and then $X$ satisfies
\[
(\lambda - A + BB^H C \frac{1}{\sqrt{1 + d\lambda}} \tanh \sqrt{\frac{\lambda^2}{1 + d\lambda}})X = 0.
\] (24)
Therefore, $(X,f,g) \neq 0$ if and only if the equation
\[
\det(\lambda - A + BB^H C \frac{1}{\sqrt{1 + d\lambda}} \tanh \sqrt{\frac{\lambda^2}{1 + d\lambda}}) = 0
\]
have solutions. \qed

Theorem 4. Let $A$ be given by (5) and (6). Then $\sigma_c(A) = \{-\frac{1}{d}\}$.

Proof. \forall $\lambda \notin \sigma_p(A)$, and $\lambda \neq \frac{1}{d}$, it suffices to show that $\lambda \in \rho(A)$. In fact, for any $G = (\bar{X}, \bar{f}, \bar{g}) \in H$, solve $(\lambda I - A)F = G, F = (X,f,g) \in D(A)$, that is (18), we rewrite it here for convenience:
\[
\begin{cases}
\lambda X - (AX + Bg(1)) = \bar{X}, \\
\lambda f - g = \bar{f}, \\
\lambda g - (f + dg)'' = \bar{g}, \\
f(0) = 0, \\
f'(1) + dg'(1) + B^T C \bar{X} = 0.
\end{cases}
\]
By the second equation of (18),
\[
g = \lambda f - \bar{f}.
\] (25)
Substitute this into the third equation of (18), we can obtain the following problem:
\[
\begin{cases}
f'' - \frac{\lambda^2}{1 + d\lambda} f = \frac{df'' - \lambda f - \bar{g}}{1 + d\lambda}, \\
f(0) = 0.
\end{cases}
\] (26)
The solution of (26) is
\[
f(x) = c \sinh \sqrt{\frac{\lambda^2}{1 + d\lambda}} x + \sqrt{\frac{1 + d\lambda}{\lambda^2}} \int_0^x df''(s) - \lambda \bar{f}(s) - \bar{g}(s) \sinh \sqrt{\frac{\lambda^2}{1 + d\lambda}} (x - s) ds,
\] (27)
and then
\[ g(x) = \lambda c \sinh \sqrt{\frac{\lambda^2}{1+d\lambda}} x + \int_0^x d \hat{f}''(s) - \lambda \hat{f}'(s) - \bar{g}(s) \sinh \sqrt{\frac{\lambda^2}{1+d\lambda}} (x-s) ds - \bar{f}(x), \]
\[ g(1) = \lambda c \sinh \sqrt{\frac{\lambda^2}{1+d\lambda}} + \int_0^1 d \hat{f}''(s) - \lambda \hat{f}'(s) - \bar{g}(s) \sinh \sqrt{\frac{\lambda^2}{1+d\lambda}} (1-s) ds - \bar{f}(1), \]
where \( c \) is a constant which is to be uniquely determined by the boundary condition of (18). Substitute this into the fifth equation of (18), we can get
\[ c = -\frac{\int_0^1 (d \hat{f}''(s) - \lambda \hat{f}'(s) - \bar{g}(s)) \cosh \sqrt{\frac{\lambda^2}{1+d\lambda}} (1-s) ds - d \hat{f}'(1) + B^T C \bar{X}}{\lambda \sqrt{1+d\lambda} \cosh \sqrt{\frac{\lambda^2}{1+d\lambda}}}, \]
comparing with the first equality of (18), we have
\[ \lambda X - AX - B[\lambda - \lambda - \bar{g}(s) \sinh \sqrt{\frac{\lambda^2}{1+d\lambda}} (1-s) ds - \bar{f}(1)] = \bar{X}. \]
Hence,
\[ X = (\lambda - A + B B^T C \frac{1}{\sqrt{1+d\lambda}} \tanh \sqrt{\frac{\lambda^2}{1+d\lambda}})^{-1} [\bar{X} - B \bar{f}(1)] \]
\[ b + B \int_0^1 d \hat{f}''(s) - \lambda \hat{f}'(s) - \bar{g}(s) \sinh \sqrt{\frac{\lambda^2}{1+d\lambda}} (1-s) ds - B \frac{\tanh \sqrt{\frac{\lambda^2}{1+d\lambda}}}{\sqrt{1+d\lambda}} \] \[ \cdot \left( \int_0^1 (d \hat{f}''(s) - \lambda \hat{f}'(s) - \bar{g}(s)) \cosh \sqrt{\frac{\lambda^2}{1+d\lambda}} (1-s) ds - d \hat{f}'(1) \right). \]

Therefore, we can get the unique solution \((X, f, g)\) of (18) by a series of calculation. It shows that \((\lambda I - A)^{-1} \) exists and is bounded, that is \( \lambda \in \rho(A) \).

**Theorem 5.** Let \( A \) be given by (5) and (6). Then for each \( \lambda \in \sigma_p(A) \), we have \( \Re \lambda < 0 \).

**Proof.** By Lemma 2, we have \( \Re \lambda \leq 0 \), \( \forall \lambda \in \sigma_p(A) \). So we only need to show there is no eigenvalues on the imaginary axis. Let \( \lambda = ia \in \sigma_p(A), \ a \in \mathbb{R} \) and let \( Z = (X, f, g) \in D(A) \) be its associated eigenfunction of \( A \). Then by (11), we have
\[ \Re(AZ, Z) = -d \int_0^1 |g'(x)|^2 dx = 0, \]
and hence, \( g'(x) = 0 \). By \( g(0) = 0 \), we have \( g \equiv 0 \). Moreover, it follows from (13) that \( f = 0 \). By (24) and \( \lambda = ia \), we have
\[ (ia - (A - BB^T C \frac{1}{\sqrt{1+d\lambda}} \tanh \sqrt{\frac{\lambda^2}{1+d\lambda}}))X = 0. \]
Since \( A - BB^T C \frac{1}{\sqrt{1+d\lambda}} \tanh \sqrt{\frac{\lambda^2}{1+d\lambda}} \) is Hurwitz, the real parts of all eigenvalues of
\[ A - BB^T C \frac{1}{\sqrt{1+d\lambda}} \tanh \sqrt{\frac{\lambda^2}{1+d\lambda}} \]
is negative. Hence, \( \det(ia - (A - BB^T C \frac{1}{\sqrt{1+d\lambda}} \tanh \sqrt{\frac{\lambda^2}{1+d\lambda}})) \neq 0 \), and then \( Z \equiv 0 \). Therefore, there is no eigenvalues on the imaginary axis. \( \square \)
4 | EXPONENTIAL STABILITY FOR SYSTEM (7)

In this section, we will give the asymptotic expressions of eigenvalues and eigenfunctions and establish the exponential stability for system (7).

**Proposition 1.** Let $A$ be given by (5) and (6), and let $\Delta(\lambda)$ be given by (19). There are two branches of eigenvalues, the first branch is

$$\lambda_{k1} = \left(-\frac{1}{d}\right)^+ + \varepsilon_k, \quad \text{where} \quad \varepsilon_k = \frac{1}{2d - k^2\pi^2d^4} = \mathcal{O}(k^{-2}), \quad k > N,$$

which means $\lambda_{k1}$ tends to $-\frac{1}{d}$ from its left as $k \to \infty$; The other branch of eigenvalues have the following asymptotic expressions:

$$\lambda_{k2} = -(k - \frac{1}{2})^2d + \frac{1}{d} + \mathcal{O}(k^{-2}), \quad k > N,$$

where $N$ is a positive number. Therefore, $\text{Re}\lambda_{k2} \to -\infty$, as $k \to \infty$.

**Proof.** Let $h(\lambda) = \frac{1}{\sqrt{1+d\lambda}}\tanh\frac{\sqrt{\lambda^2}}{\sqrt{1+d\lambda}}$, $M = BB^HC$, we get

$$\Delta(\lambda) = \det(\lambda - A + h(\lambda)M) = h(\lambda)f_{n-1}(\lambda) + g_n(\lambda) = 0,$$

where $f_{n-1}(\lambda)$ and $g_n(\lambda)$ are polynomials of degree $n - 1$ and $n$, respectively. Let $p_{n-1}(\lambda) = f_{n-1}(\lambda)$, $q_n(\lambda) = g_n(\lambda)$, then by $\Delta(\lambda) = 0$, we have

$$\frac{1}{\sqrt{1+d\lambda}}\tanh\frac{\sqrt{\lambda^2}}{1+d\lambda}p_{n-1}(\lambda) + q_n(\lambda) = 0,$$

and then

$$\tanh\frac{\sqrt{\lambda^2}}{1+d\lambda}p_{n-1}(\lambda) + \sqrt{1+d\lambda}q_n(\lambda) = 0.$$

So we get

$$(e^{\sqrt{\frac{\lambda^2}{1+d\lambda}}} - e^{-\sqrt{\frac{\lambda^2}{1+d\lambda}}})p_{n-1}(\lambda) + \sqrt{1+d\lambda}(e^{\sqrt{\frac{\lambda^2}{1+d\lambda}}} + e^{-\sqrt{\frac{\lambda^2}{1+d\lambda}}})q_n(\lambda) = 0,$$

or

$$e^{2\sqrt{\frac{\lambda^2}{1+d\lambda}}}(p_{n-1}(\lambda) + \sqrt{1+d\lambda}q_n(\lambda)) + (-p_{n-1}(\lambda) + \sqrt{1+d\lambda}q_n(\lambda)) = 0,$$

which yields

$$e^{2\sqrt{\frac{\lambda^2}{1+d\lambda}}} = \frac{-p_{n-1}(\lambda) + \sqrt{1+d\lambda}q_n(\lambda)}{p_{n-1}(\lambda) + \sqrt{1+d\lambda}q_n(\lambda)} = -1 + \frac{2p_{n-1}(\lambda)}{\sqrt{1+d\lambda}q_n(\lambda) + p_{n-1}(\lambda)}$$

(36)

$$= -1 + \frac{2}{\sqrt{1+d\lambda} \cdot \frac{q_n(\lambda)}{p_{n-1}(\lambda)} + 1}.$$

It is obvious that $\lambda_{k1} = \left(-\frac{1}{d}\right)^+ + \frac{1}{2d - k^2\pi^2d^4}$, $k > N$, are asymptotic expressions of solutions of (36). In fact, if $1 + d\lambda$ is sufficiently small (for instance, we assume $\lambda = -\frac{1}{d} + \varepsilon$, where $\varepsilon$ is to be determined), the right hand side of Equation (36) tends to 1. Thus, solve

$$e^{2\sqrt{\frac{\lambda^2}{1+d\lambda}}} = 1,$$

we get that $2\sqrt{\frac{\lambda^2}{1+d\lambda}} = 2k\pi i$, i.e., $\sqrt{\frac{\lambda^2}{1+d\lambda}} = k\pi i$, where $i^2 = -1$. Then

$$\frac{\frac{1}{d} - \frac{2}{d\lambda} + \varepsilon^2}{d\varepsilon} = \frac{\lambda^2}{1+d\lambda} = -k^2\pi^2,$$

i.e.,

$$\frac{1}{d^3} \cdot \frac{1}{\varepsilon} - \frac{2}{d} + \frac{1}{d} \cdot \varepsilon = -k^2\pi^2.$$
If we neglect the argument $\frac{1}{d} \cdot \varepsilon$, we can obtain the asymptotic expression:

$$\varepsilon_k = \frac{1}{2d - k^2 \pi^2 d^2},$$

where $k$ is sufficiently large. Then we can get Equation (34).

On the other hand, (36) can be written as

$$e^{2\sqrt{\frac{\lambda}{\pi d}}} = -1 + O(\lambda^{-\frac{3}{2}}),$$

if $|\lambda| \to \infty$. A direct computation gives

$$\sqrt{\frac{\lambda^2}{1 + d\lambda}} = i \left(k - \frac{1}{2}\right) \pi + O(k^{-2}), k > N,$$

where $N$ is a positive number. Therefore,

$$\frac{\lambda^2}{1 + d\lambda} = -\left(k - \frac{1}{2}\right)^2 \pi^2 + O(k^{-2}), \quad k > N. \quad (39)$$

Hence,

$$\frac{1}{d} \lambda - \frac{1}{d} = -\left(k - \frac{1}{2}\right)^2 \pi^2 + O(k^{-2}), \quad k > N, \quad (40)$$

and then

$$\lambda = -\left(k - \frac{1}{2}\right)^2 \pi^2 d + \frac{1}{d} + O(k^{-2}), \quad k > N. \quad (41)$$

Remark 1. From the asymptotic expression (35), it is found that the principal parts of the spectrum of operator $A$ with large modulus have the same expression with comparison to the principal parts of the K-V damped wave equation with no coupling ODE, and it is like a fight, where the weak is suppressed by the strong.

**Proposition 2.** Let $\{\lambda_{k1}, k \in \mathbb{N}\}$ be the eigenvalues of $A$ with $\lambda_{k1}$ being given by (34). Then the corresponding eigenfunctions $\{(X_{ik1}, f_{k1}, g_{k1}), k \in \mathbb{N}\}$ have the following asymptotic expressions:

$$\begin{align*}
X_{ik1} &= O(k^{-2}), \quad i = 1, 2, \ldots, n, \\
f_{k1} &= \frac{k}{k^{\pi}} \sin k\pi x + O(k^{-2}), \\
g_{k1} &= O(k^{-1}),
\end{align*} \quad (42)$$

where $N$ is a positive number.

**Proof.** It is found from (34) that

$$\frac{\lambda_{k1}^2}{1 + d\lambda_{k1}} = \frac{1}{d^2} + 2\varepsilon_k (-\frac{1}{d}) + \varepsilon_k^2 = -k^2 \pi^2 + O(k^{-2}). \quad (43)$$

Therefore, combining with (22), we can choose:

$$f_{k1}(x) = -i \sinh ik\pi x + O(k^{-2}) = \sin k\pi x + O(k^{-2}), \quad k > N. \quad (44)$$

Moreover, by the second equality of (21), we get that

$$g_{k1}(x) = \lambda_{k1} f_{k1}(x) = -\frac{1}{d} \sin k\pi x + O(k^{-1}). \quad (45)$$

Since

$$f'_{k1}(1) = k\pi \cos k\pi + O(k^{-1}) = (-1)^k k\pi + O(k^{-1}); \quad (46)$$

thus, by the second equality of (22), we can get $X_{ik1}$ as following:

$$X_{ik1} = O(k^{-1}), \quad i = 1, 2, \ldots, n. \quad (47)$$
Moreover, normalize the above eigenfunctions in $\mathcal{H}$, we can get the approximate unit eigenfunctions (42). Here for convenience, we still use the function $X_{\lambda k_1}, f_{k_1}(x), g_{k_1}(x)$.

**Proposition 3.** Let $\{\lambda_{k_2}, k \in \mathbb{N}\}$ be the eigenvalues of $A$ with $\lambda_{k_2}$ being given by (35). Then the corresponding eigenfunctions $(X_{\lambda_{k_2}}, f_{k_2}, g_{k_2}, k \in \mathbb{N})$ have the following asymptotic expressions:

\[
\begin{cases}
X_{\lambda_{k_2}} = \mathcal{O}(k^{-2}), & i = 1, 2, \ldots, n, \\
f_{k_2} = \mathcal{O}(k^{-1}), & k > N, \\
g_{k_2} = \sin \left( k - \frac{1}{2} \right) \pi x + \mathcal{O}(k^{-3}),
\end{cases}
\]

where $N$ is a positive number.

**Proof.** It is found from (22) and (38) that for $\lambda_{k_2}$, its corresponding eigenfunction has the asymptotic form:

\[
f_{k_2}(x) = -i \sinh \sqrt{\frac{\lambda_{k_2}^2}{1 + d\lambda_{k_2}}} x = \sin \left( k - \frac{1}{2} \right) \pi x + \mathcal{O}(k^{-3}), \quad k > N.
\]

Moreover, by the second equality of (21), we get that

\[
g_{k_2}(x) = \lambda_{k_2} f_{k_2}(x) = \left[ -\left( k - \frac{1}{2} \right)^2 \pi^2 d + \frac{1}{d} \right] \sin \left( k - \frac{1}{2} \right) \pi x + \mathcal{O}(k^{-1}).
\]

Since

\[
f'_{k_2}(1) = \left( k - \frac{1}{2} \right) \pi \cos \left( k - \frac{1}{2} \right) \pi + \mathcal{O}(k^{-2}) = \mathcal{O}(k^{-2}),
\]

thus, by the second equality of (22),

\[
X_{\lambda_{k_2}} = \mathcal{O}(1), \quad i = 1, 2, \ldots, n.
\]

Moreover, normalize the above eigenfunctions in $\mathcal{H}$, we can get the approximate unit eigenfunctions (48). Here, for convenience, we still use the function $X_{\lambda_{k_2}}, f_{k_2}(x), g_{k_2}(x)$.

Now, we show the Riesz basis generation and exponential stability.

**Theorem 6.** Let $A$ be given by (5) and (6). Then there is a set of generalized eigenfunctions of $A$ which forms a Riesz basis for $\mathcal{H}$ and each eigenvalue of $A$ with large modulus is algebraically simple. Moreover, $e^{At}$, generated by $A$, is an analytic semigroup for $\mathcal{H}$.

**Proof.** Let $(n + 2)$-dimensional vector groups $e_i = (0, \ldots, 0, 1, 0, \ldots, 0), i = 1, 2, \ldots, n$, where only the $i$th component is 1, and

\[
F_{k_1} = \left( 0, \ldots, 0, \frac{1}{k \pi} \sin k \pi x, 0 \right), \quad F_{k_2} = \left( 0, \ldots, 0, 0, \sin(k - \frac{1}{2}) \pi x \right), \quad k \in \mathbb{N}.
\]

Then $\{e_1, \ldots, e_n, F_{k_1}, F_{k_2}, k \in \mathbb{N}\}$ form an orthogonal basis for $\mathcal{H}$.

Let $G_{k_1} = (X_{\lambda_{k_1}}, \ldots, X_{\lambda_{k_1}}, f_{k_1}, g_{k_1}), G_{k_2} = (X_{\lambda_{k_2}}, \ldots, X_{\lambda_{k_2}}, f_{k_2}, g_{k_2}), k > N$, where $X_{\lambda_{k}}, f_{k}, g_{k}$ are given by (42) and (48), then

\[
\sum_{k=N+1}^{\infty} \|F_{k_1} - G_{k_1}\|^2_H + \|F_{k_2} - G_{k_2}\|^2_H \leq \sum_{k=N+1}^{\infty} \mathcal{O}(k^{-2}) < \infty.
\]

By Guo,19, theorem 6.3, a modified classical Bari’s theorem, there is a set of generalized eigenfunctions of $A$ which forms a Riesz basis for $\mathcal{H}$ and each eigenvalue of the system operator $A$ with large modulus is algebraically simple. Finally, by Opmeer,20, theorem 13, $A$ generates an analytic semigroup $e^{At}$.

**Theorem 7.** Let $A$ be given by (5) and (6). Then

1. **The spectrum-determined growth condition holds true for $e^{At}$, that is, $s(A) = \omega(A)$, where $s(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}$ is the spectral bound of $A$, and $\omega(A) := \inf\{|\omega| : \exists M > 0 \text{ such that } ||e^{At}|| \leq Me^{\omega t}\}$ denotes the growth bound of $e^{At}$.
2. **The system (7) is exponentially stable, that is, there exist two positive constants $M$ and $\sigma$ such that the $C_0$-semigroup $e^{At}$ generated by $A$ satisfies $||e^{At}|| \leq Me^{-\sigma t}$.
Proof. By Theorem 7, we have that $A$ generates an analytical semigroup. Hence, as a conclusion, the spectrum-determined growth condition holds. By Theorem 1, Theorem 4, and Theorem 6, we have
\[ \text{Re} \lambda < 0, \quad \forall \lambda \in \sigma(A). \]
Hence, $e^{At}$ is exponentially stable.

5 | CONCLUSIONS

In this paper, the unstable ODE system was stabilized by the wave equation through a direct feedback connection. It just like a confrontation between two systems, where the strong one takes a leading role. Moreover, the delicate spectral analysis are aimed to verify the exponential stability of the coupled system, which is an interesting result and gives theoretical proof for numerical simulations.

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