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Exact controllability of a micro beam with boundary bending moment

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ABSTRACT
In this paper, we show the exact controllability of a micro beam with boundary bending moment at time \( T > 0 \), which gives a confirmative answer to the open problem proposed by P. Guzmán and J. Zhu. The semi-group approach is adopted in investigation. The asymptotic spectral expression of the micro beam system is presented to verify an observability inequality for the dual system. By the duality principle, the exact controllability of the system is then established.

1. Introduction
Micro beams are widely applied in micro-devices such as micro-actuators (De Boer, Luck, Ashurst, & Maboudian, 2004), micro-switches (Joglekar & Pawaskar, 2011), micro-sensors (Moser & Gijs, 2007) and atomic force microscopes (Giessibl, 2003). Micro-beams are typically on the order of microns and sub-microns. When the materials are in micron-scale, the size dependence of material deformation behaviour has been found by some experimental results (Fleck, Muller, Ashby, & Hutchinson, 1994; Ma & Clarke, 1995; Stolken & Evans, 1998). This size dependence can be successfully modelled by using higher-order continuum theories. One of the most successful higher-order continuum theories is the modified strain gradient theory proposed in Lam, Yang, Chong, Wang, and Tong (2003). Using this theory, some dynamic models of micro beams have been established in Kong, Zhou, Nie, and Wang (2009) and Zhao, Zhou, Wang, and Wang (2012). In this paper, we consider a micro beam model (see Guzmán & Zhu (2015); Kong et al. (2009); Zhao et al. (2012)):

\[
\begin{align*}
&z_{tt}(x, t) = \frac{K}{\rho A} z_{xxxx}(x, t), \\
&z(x, 0) = z_0(x), \quad t > 0,
\end{align*}
\]

where \( z(x, t) \) and \( u(t) \), respectively, represent the lateral deflection and boundary bending moment, and \( K, S, \rho \) and \( A \) are positive physical constants. In Guzmán and Zhu (2015), using the multiplier method together with the controllability–observability duality, system (1.1) is proved to be exactly controllable in time \( T > T' \), where \( T' = 2L \max \{1, \sqrt{\frac{\rho A}{K}}\} \).

In this paper, by a spectral analysis, we show that the gap of the system eigenvalues approaches \( +\infty \), which yields that system (1.1) is exactly controllable in any time \( T > 0 \). This gives a positive answer to the open problem in Guzmán and Zhu (2015). We present our main result as follows.

Theorem 1.1: For \( T > 0 \), system (1.1) in \( L^2(0, L) \times H^{-3}(0, L) \) is exactly controllable on \([0, T]\), i.e. for any given \((z_0, z_1)\), \((\tilde{z}_0, \tilde{z}_1) \in L^2(0, L) \times H^{-3}(0, L)\), there exists control \( u(t) \in L^2(0, T) \) such that the unique solution of Equation (1.1) \( z \in C([0, T]; L^2(0, L)) \cap C^1([0, T]; H^{-3}(0, L)) \) satisfies \( z(\cdot, T) = \tilde{z}_0(\cdot) \) in \( L^2(0, L) \) and \( z_t(\cdot, T) = \tilde{z}_1(\cdot) \) in \( H^{-3}(0, L) \).

To prove Theorem 1.1, we adopt the semi-group approach (Pazy, 1983) to study system (1.1). First, we write the system into an abstract evolution equation. Second, by giving a detailed spectral analysis, we present the asymptotic expansion for both the eigenvalues and the eigenfunctions of the system operator. Then, according to the duality principle, we derive the exact observability of the adjoint system. Finally, we can deduce the exact controllability of system (1.1) directly.

The rest of the paper is organised as follows. In Section 2, we present the exact observability of the adjoint system, which is equivalent to the exact controllability of Equation (1.1) according to the duality principle. In Section 3, we analyse the spectral behaviour of the adjoint system. Section 4 gives the proof for Theorem 1.1. Finally, we give the conclusion in Section 5.

2. Preliminaries and observability inequality
For simplicity, throughout the paper, we consider system (1.1) with the unit length \( L = 1 \):

\[
\begin{align*}
&z_{tt}(x, t) = \frac{K}{\rho A} z_{xxxx}(x, t), \\
&z(0, t) = z_t(1, t) = 0, \\
&z_x(0, t) = z_x(1, t) = 0, \\
&z(0, 0) = z_0(x), \quad z_t(0, 0) = z_1(x), \\
&0 < x < 1, t > 0.
\end{align*}
\]

Now, we formulate system (2.1) into an abstract evolution equation in the state space \( \mathcal{H} = L^2(0, 1) \times H^{-3}(0, 1) \), where
$H^{-3}(0,1)$ is the dual space of $H^3_0(0,1)$ with respect to the pivot space $L^2(0,1)$. The inner product induced norm in $H^3_0(0,1)$ is given by

$$
\|f\|_{H^3_0(0,1)}^2 = \int_0^1 \left[ \frac{K}{\rho A} |f'''(x)|^2 + \frac{S}{\rho A} |f''(x)|^2 \right] dx, \\
\forall f \in H^3_0(0,1).
$$

Define an unbounded linear operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ as follows:

$$
A\phi := \frac{S}{\rho A} \frac{d^4 \phi}{dx^4} - \frac{K}{\rho A} \frac{d^6 \phi}{dx^6}, \quad D(A) = H^6(0,1) \cap H^3_0(0,1),
$$

where $H^6(0,1)$ and $H^3_0(0,1)$ are the usual Sobolev spaces. Obviously, $A$ is a self-adjoint and positive operator.

Let $a(\cdot, \cdot)$ be a bilinear form on $H^3_0(0,1)$ defined by

$$
a(f, g) := \int_0^1 \left[ \frac{K}{\rho A} f'''(x)g'''(x) + \frac{S}{\rho A} f''(x)g''(x) \right] dx, \\
\forall f, g \in H^3_0(0,1). \quad (2.2)
$$

It is found that $a(\cdot, \cdot)$ is continuous and coercive on $H^3_0(0,1)$. According to the Lax–Milgram theorem, for any $\varphi \in H^{-3}(0,1)$, there exists a unique $f \in H^3_0(0,1)$ such that

$$
a(f, g) = \langle \varphi, g \rangle_{H^{-3}(0,1) \times H^3_0(0,1)}, \quad \forall g \in H^3_0(0,1).
$$

Setting

$$
\langle Af, g \rangle_{H^{-3}(0,1) \times H^3_0(0,1)} = \langle \varphi, g \rangle_{H^{-3}(0,1) \times H^3_0(0,1)} = a(f, g), \quad \forall f, g \in H^3_0(0,1),
$$

we have that $A$ is a canonical isomorphism from $D(A) = H^3_0(0,1)$ onto $H^{-3}(0,1)$. Therefore, $A$ can be extended to the space $H^{-3}(0,1)$ and still denoted by $A$.

It is easy to see that $D(A^{1/2}) = L^2(0,1)$ and $A^{1/2}$ is an isomorphism from $L^2(0,1)$ onto $H^{-3/2}(0,1)$. The following Gelfands triple inclusions are valid:

$$
D(A^{1/2}) \hookrightarrow H^{-3/2}(0,1) = (H^{-3}(0,1))^* \hookrightarrow (D(A^{1/2}))^*, \quad (2.3)
$$

where $(D(A^{1/2}))^*$ is the dual space of $D(A^{1/2})$ considering $H^{-3/2}(0,1)$ as the pivot space.

An extension $\hat{A} \in \mathcal{L} \left( (D(A^{1/2})), (D(A^{1/2}))^* \right)$ of $A$ is defined by

$$
\langle \hat{A} f, g \rangle_{(D(A^{1/2}))^* \times (D(A^{1/2}))} = \langle A^{1/2} f, A^{1/2} g \rangle_{H^{-3/2}(0,1)} = \langle f, g \rangle_{L^2(0,1)}, \quad \forall f, g \in D(A^{1/2}). \quad (2.4)
$$

Define the map $\Upsilon : L^2(0,1) \to (C, L^2(0,1))$, i.e. $\Upsilon u = v$ if and only if

$$
\begin{cases}
S_0^{(4)}(x) - K^{(6)}(x) = 0, & 0 < x < 1, \\
v(0) = v(1) = v'(0) = v'(1) = v''(0) = v''(1) = 0, & v'''(1) = u.
\end{cases}
$$

Using this map $\Upsilon$, we can write system (2.1) in $(D(A^{1/2}))^*$ as

$$
\ddot{z} + \ddot{\hat{A}}(z - \Upsilon u) = 0, \quad (2.6)
$$

which is further written as

$$
\ddot{z} + \ddot{\hat{A}}z + Bu = 0, \quad (2.7)
$$

where $B \in \mathcal{L} (C, (D(A^{1/2}))^*)$ is given by

$$
Bu = -\hat{A}\Upsilon u, \quad \forall u \in C. \quad (2.8)
$$

Define $B^* \in \mathcal{L} (D(A^{1/2}), C)$ by

$$
\{B^* f, u\}_{(D(A^{1/2}))^* \times (D(A^{1/2}))} = \{f, Bu\}_{D(A^{1/2}) \times (D(A^{1/2}))}, \quad \forall f \in D(A^{1/2}), u \in C. \quad (2.9)
$$

Then for any $f \in D(A^{1/2})$ and $u \in C$,

$$
\{f, Bu\}_{D(A^{1/2}) \times (D(A^{1/2}))} = \{f, \hat{A}\hat{A}^{-1}Bu\}_{D(A^{1/2}) \times (D(A^{1/2}))},
$$

$$
= \{A^{1/2} f, A^{1/2} \hat{A}^{-1}Bu\}_{H^{-3/2}(0,1)},
$$

$$
= -\{f, \Upsilon u\}_{L^2(0,1)} = -\{A\hat{A}^{-1} f, \Upsilon u\}_{L^2(0,1)},
$$

$$
= -\frac{K}{\rho A} (A^{-1} f(x))^\prime\prime\prime |_{x=1}. \quad (2.10)
$$

Finally, we obtain

$$
B^* f = -\frac{K}{\rho A} (A^{-1} f(x))^\prime\prime\prime |_{x=1}, \quad \forall f \in D(A^{1/2}).
$$

Now, we can write system (2.1) in $\mathcal{H}$ as

$$
\begin{cases}
\frac{d}{dt} \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} = \hat{A} \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} + Bu(t), \\
\hat{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad D(\hat{A}) = H^3_0(0,1) \times L^2(0,1), \\
B = \begin{pmatrix} 0 \\ -B \end{pmatrix}.
\end{cases}
$$

(2.12)

Hence, the dual system of (2.12) is

$$
\begin{cases}
\frac{d}{dt} \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} = -\hat{A}^* \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix}, \\
y(t) = B^* \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix},
\end{cases}
$$

(2.13)

where $\hat{A}^* = -\hat{A}$ and $B^* = (0, B^*)$.

In order to consider the exact observability, we transform Equation (2.13) into an equivalent system in the space

$$
X := H^3_0(0,1) \times L^2(0,1).
$$

(2.14)
with the inner product induced norm given by
\[
\| (f, g) \|_X^2 := \int_0^1 \left[ \frac{K}{\rho A} |f'''(x)|^2 + \frac{S}{\rho A} |f''(x)|^2 + |g(x)|^2 \right] dx, \\
\forall (f, g) \in X.
\]

Define a bounded linear operator \( \mathcal{D} : \mathcal{R}(A^{-1}) = H^3_0(0, 1) \rightarrow H^{-3}(0, 1) \) by
\[
\mathcal{D} f = Af, \quad \forall f \in H^3_0(0, 1),
\]
where \( \mathcal{R}(A^{-1}) \) denotes the range of \( A^{-1} \).

Then \( \mathcal{D}^{-1} \) is a bounded linear operator from \( H^{-1}(0, 1) \) onto \( H^3_0(0, 1) \) given by
\[
\mathcal{D}^{-1} = A^{-1}.
\]

We now define an isometric transformation \( T : \mathcal{H} \rightarrow X \) via
\[
T \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & -A^{-1} \\ I & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -A^{-1}g \\ f \end{pmatrix} \in X, \quad \forall (f, g) \in \mathcal{H}.
\]  
(2.15)

Then, it is easy to see that
\[
T^{-1} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & I \\ -D & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ -Af \end{pmatrix} \in \mathcal{H}, \quad \forall (f, g) \in \mathcal{H}.
\]

Define linear operators \( A_0 : D(A_0) \subset X \rightarrow X \) and \( B_0 \), respectively, by
\[
A_0 = TA_0 T^{-1}, \quad B_0 = B^* T^{-1}.
\]

Since for any \( f \in H^3(0, 1) \cap H^3_0(0, 1) \), we have
\[
-B^* Af = \frac{K}{\rho A} f'''(1),
\]
then \( A_0 \) and \( B_0 \) are explicitly given by
\[
A_0 = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \quad \text{i.e.} \quad A_0 (f, g) = (g, -Af)
\]
and
\[
D(A_0) = \{ f \in H^3(0, 1) \cap H^3_0(0, 1) \} \times H^3_0(0, 1),
\]
\[
B_0 = (-B^* A, 0).
\]

Thus, with transformation \( T \), we convert Equation (2.13) into the following system in \( X \):
\[
\begin{cases}
\frac{d}{dt} \left( \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} \right) = A_0 \left( \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} \right), \\
y_w(t) = B_0 \left( \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} \right).
\end{cases}
\]  
(2.18)

and the abstract evolution equation is given as follows:
\[
\begin{cases}
\frac{d}{dt} \left( \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix} \right) = A_0 \left( \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix} \right), \\
y_\psi(t) = B_0 \left( \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix} \right) \in X.
\end{cases}
\]

where \((\psi(\cdot, t), \dot{\psi}(\cdot, t))^T = T (z(\cdot, t), \dot{z}(\cdot, t))^T \in X \) and \( y_\psi(t) \) is the collocated observation.

By means of the duality principle (Tucsnak & Weiss, 2009), we have that the exact controllability for system (2.1) is equivalent to the exact observability of the adjoint system (2.18). Now, we show that system (2.18) is exactly observable at any time \( T > 0 \). First, we have the well-posedness of system (2.18).

**Theorem 2.1:** Let \( T > 0 \) and \((w_0, w_1) \in X = H^3_0(0, 1) \times L^2(0, 1) \). Then, system (2.18) has a unique mild solution \((w, \dot{w}) \in C([0, T]; X) \), which satisfies \( w(\cdot, T) = w_0 \) and \( w_1(\cdot, T) = w_1 \).

Then, we present our second main result which is the exact observability of the adjoint system (2.18) at any time \( T > 0 \).

**Theorem 2.2:** Let \( T > 0 \) and \((w_0, w_1) \in X = H^3_0(0, 1) \times L^2(0, 1) \). System (2.18) is exactly observable in \( X \) on \([0, T]\), i.e. there exists \( D_T > 0 \) such that
\[
\| w_{xx}(1, \cdot) \|_{L^2(0, T)}^2 \geq D_T \| (w_0, w_1) \|_X^2, \quad \forall (z_0, z_1) \in D(A_0).
\]  
(2.19)

### 3. Spectral analysis

In this section, we consider the spectrum of operator \( A_0 \). First, we have the following lemma.

**Lemma 3.1:** Let \( A_0 \) be given by Equation (2.16), \( A_0 \) is skew-adjoint and has compact resolvent on \( X \). Therefore, the spectrum \( \sigma(A_0) \) consists of isolated eigenvalues and the corresponding eigenfunctions form an orthonormal basis of \( X \). Moreover, \( A_0 \) generates a \( C_0 \)-group \( e^{at} \) on \( X \).

Now we are in a position to consider the eigenvalue problem of \( A_0 \). Let \( \lambda \in \sigma(A_0) \) and \( \Psi = (\phi, \psi) \) be an eigenfunction of \( A_0 \) corresponding to \( \lambda \). Then we have \( \psi = \lambda \phi \) and \( \phi \) satisfies the following equation:
\[
\begin{cases}
\phi^{(6)}(x) - \frac{S}{K} \phi^{(4)}(x) - \lambda^2 \frac{\rho A}{K} \phi(x) = 0, \quad 0 < x < 1, \\
\phi(0) = \phi(1) = \phi'(0) = \phi'(1) = \phi''(0) = \phi''(1) = 0.
\end{cases}
\]  
(3.1)
Due to the fact that all the eigenvalues lie on the imaginary axis and are symmetric about the real axis, we only need to consider those $\lambda$ located in the upper half imaginary axis:

$$
\lambda := \sqrt{K \rho A} \mu^3, \quad \mu \in \mathcal{S} := \left\{ \mu \in \mathbb{C} : 0 \leq \arg \mu \leq \frac{\pi}{6} \right\}.
$$

(3.2)

Let $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$ be the roots of equation $\theta^6 + 1 = 0$ that are arranged so that

$$
\text{Re}(\mu \omega_1) \leq \text{Re}(\mu \omega_2) \leq \text{Re}(\mu \omega_3) \leq \text{Re}(\mu \omega_4)
\leq \text{Re}(\mu \omega_5) \leq \text{Re}(\mu \omega_6), \quad \forall \mu \in \mathcal{S},
$$

(3.3)

where

$$
\begin{align*}
\omega_1 &= \exp \left( \frac{5}{6} i \pi \right) = -\frac{\sqrt{3}}{2} + \frac{1}{2} i, \\
\omega_2 &= \exp \left( \frac{7}{6} i \pi \right) = -\frac{\sqrt{3}}{2} - \frac{1}{2} i, \\
\omega_3 &= \exp \left( \frac{1}{2} i \pi \right) = i, \quad \omega_4 = \exp \left( \frac{3}{2} i \pi \right) = -i, \\
\omega_5 &= \exp \left( \frac{1}{6} i \pi \right) = \frac{\sqrt{3}}{2} + \frac{1}{2} i, \\
\omega_6 &= \exp \left( \frac{11}{6} i \pi \right) = \frac{\sqrt{3}}{2} - \frac{1}{2} i.
\end{align*}
$$

(3.4)

Proposition 3.1: For $\mu \in \mathcal{S}$, if we set $\delta := \sin \frac{\pi}{6} = \frac{1}{2}$, then we have

$$
\begin{align*}
\text{Re}(\mu \omega_i) &\leq -|\mu| \delta, \quad i = 1, 2, \\
\text{Re}(\mu \omega_i) &\geq |\mu| \delta, \quad i = 5, 6.
\end{align*}
$$

We have the following result about the asymptotic fundamental solutions of system (3.1) from Birkhoff (1908) (see also Naimark 1967).

Lemma 3.2: For $\mu \in \mathcal{S}$ with $|\mu|$ large enough, the equation

$$
\phi^{(6)}(x) - \frac{S}{K} \phi^{(4)}(x) + \mu^2 \phi(x) = 0, \quad x \in (0, 1),
$$

(3.5)

has six linearly independent asymptotic fundamental solutions,

$$
\Phi_i(x, \mu) = e^{i \omega_i x} \left( 1 + \frac{1}{6 \mu \omega_i} \frac{S}{K} x + \frac{1}{72 \mu^2 \omega_i^2} \frac{S^2}{K^2} x^2 + O(\mu^{-3}) \right),
$$

(i = 1, 2, \ldots, 6)

(3.6)

and hence their derivatives for $i = 1, 2, \ldots, 6$ and $j = 1, 2, 3$ are given by

$$
\frac{d^j}{dx^j} \Phi_i(x, \mu) = (\mu \omega_i)^j e^{i \omega_i x} \left( 1 + \frac{1}{6 \mu \omega_i} \frac{S}{K} x + \frac{j}{6 \mu^2 \omega_i^2} \frac{S^2}{K^2} x^2 + O(\mu^{-3}) \right).
$$

(3.7)

The next two lemmas give the approximative asymptotic about the eigenvalues and the eigenfunctions of $A_0$.

Lemma 3.3: Let $\sigma(A_0) = \{\lambda_n, \overline{\lambda}_n, n \in \mathbb{N}\}$ be the eigenvalues of $A_0$. Then the following asymptotic expansions hold:

$$
\begin{align*}
\mu_n &= n \pi + \frac{1}{6n \pi} \frac{S}{K} - \frac{\sqrt{3}}{3n^2 \pi} \frac{S}{K} + O(n^{-3}) \\
\lambda_n &= i \sqrt{\frac{K}{\rho A}} \mu_n^3 \\
&= i \sqrt{\frac{K}{\rho A}} \left[ n^3 \pi^3 - \sqrt{3} \frac{S}{K} + \frac{1}{2} n \pi \frac{S}{K} + O(n^{-1}) \right]
\end{align*}
$$

(3.8)

for sufficiently large positive integers $n$ and $\mu_n \in \mathcal{S}$.

Lemma 3.4: Let $\sigma(A_0) = \{\lambda_n, \overline{\lambda}_n, n \in \mathbb{N}\}$ be the eigenvalues of $A_0$ and let $\lambda_n := \sqrt[3]{\frac{S}{K}}$ with $\mu_n$ given by Equation (3.8). Then for sufficiently large positive integer $n$, the corresponding eigenfunctions $\{\Psi_n = (\phi_n, \lambda_n \phi_n), \Psi_n = (\overline{\phi}_n, \overline{\lambda}_n \overline{\phi}_n), n \in \mathbb{N}\}$ have the following asymptotic expansion:

$$
\begin{align*}
\lambda_n \phi_n(x) &= (3 + \sqrt{3}i) e^{i \omega_1 x} + (-3 + \sqrt{3}i) e^{i \omega_2 x} \\
&- \sqrt{3} e^{i \omega_3 x} - \sqrt{3} e^{i \omega_4 x} \\
&+ 3 e^{i \omega_1 x} e^{i \omega_2 x} + (\sqrt{3} + 3i) e^{i \omega_3 x} e^{i \omega_4 x} + O(n^{-1}), \\
\phi_n''(x) &= -i \sqrt{\frac{K}{\rho A}} \left[ \left(-3 + \sqrt{3}i\right) e^{i \omega_1 x} \omega_1 + \sqrt{3} e^{i \omega_1 x} \right] + O(n^{-1}).
\end{align*}
$$

(3.9)

Moreover, there exist positive constants $\delta_1$ and $\delta_2$ independent of $n$, such that for sufficiently large $n$,

$$
\delta_1 \leq \|\phi_n''\|_{L^2(0, 1)} \cdot \|\lambda_n \phi_n\|_{L^2(0, 1)} \leq \delta_2.
$$

(3.10)

4. Proof of main results

In this section, we give the proof of Theorem 1.1. For this purpose, we begin by proving Lemma 3.1.

Proof of Lemma 3.1: Our proof is similar to Brezis’ method of proving Theorem 8.22 in Brezis (2010). First, it is easy to check that $A_0$ is skew-adjoint. Next, we show that $A_0^{-1}$ exists. For any given $(\phi, \psi) \in X$, solving

$$
A_0(f, g) = (g, -Af) = (\phi, \psi)
$$

gives $g(x) = \phi(x)$ with $f$ satisfying

$$
\begin{align*}
f^{(6)}(x) - \frac{S}{K} f^{(4)}(x) + f''(x) &= 0, \\
f(0) &= f(1) = f'(0) = f'(1) = f''(0) = f''(1) = 0.
\end{align*}
$$

(4.1)
There exists a unique solution of Equation (4.1) if and only if there is only a trivial solution of the homogeneous equation

\[
\begin{aligned}
f^{(4)}(x) - \frac{8}{S^2} f^{(6)}(x) &= 0, \\
f(0) = f(1) = f'(0) = f'(1) = f''(0) = f''(1) = 0.
\end{aligned}
\]

(4.2)

Then the general solutions of Equation (4.2) are

\[ f(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{\sqrt{\frac{S}{K}} x} + c_6 e^{-\sqrt{\frac{S}{K}} x}. \]

Substituting these solutions into the boundary conditions in Equation (4.2), we have

\[
\begin{align*}
&c_1 + c_3 + c_6 = 0 \\
&c_1 + c_2 + c_3 + c_4 + e^{\sqrt{\frac{S}{K}} c_5} + e^{-\sqrt{\frac{S}{K}} c_6} = 0 \\
&c_2 + \sqrt{\frac{S}{K}} c_5 - \sqrt{\frac{S}{K}} c_6 = 0 \\
&c_2 + 2c_3 + 3c_4 + \sqrt{\frac{S}{K}} e^{\sqrt{\frac{S}{K}} c_5} - \sqrt{\frac{S}{K}} e^{-\sqrt{\frac{S}{K}} c_6} = 0 \\
&2c_3 + \frac{S}{K} c_5 + \frac{S}{K} c_6 = 0 \\
&2c_3 + 6c_4 + \frac{S}{K} e^{\sqrt{\frac{S}{K}} c_5} + \frac{S}{K} e^{-\sqrt{\frac{S}{K}} c_6} = 0.
\end{align*}
\]

(4.3)

The characteristic determinant of Equation (4.3) is

\[
\begin{vmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 2 & 6
\end{vmatrix}
\begin{vmatrix}
e^{\sqrt{\frac{S}{K}} x} & e^{-\sqrt{\frac{S}{K}} x} \\
e^{\sqrt{\frac{S}{K}} x} & e^{-\sqrt{\frac{S}{K}} x} \\
\frac{S}{K} e^{\sqrt{\frac{S}{K}} x} & \frac{S}{K} e^{-\sqrt{\frac{S}{K}} x} \\
\frac{S}{K} e^{\sqrt{\frac{S}{K}} x} & \frac{S}{K} e^{-\sqrt{\frac{S}{K}} x}
\end{vmatrix}
\]

\[
\Delta = -8 \sqrt{\frac{S}{K}} \sqrt{\frac{S}{K}} + 48 \sqrt{\frac{S}{K}} \\
+ \left( \frac{S^2}{K^2} - 8 \frac{S}{K} \sqrt{\frac{S}{K}} + 24 \frac{S}{K} - 24 \sqrt{\frac{S}{K}} \right) e^{\sqrt{\frac{S}{K}} x} \\
+ \left( -\frac{S^2}{K^2} - 8 \frac{S}{K} \sqrt{\frac{S}{K}} + 24 \frac{S}{K} - 24 \sqrt{\frac{S}{K}} \right) e^{-\sqrt{\frac{S}{K}} x}.
\]

(4.4)

Multiplying \( e^{\sqrt{\frac{S}{K}} x} \) by both sides of Equation (4.4), we have

\[
e^{\sqrt{\frac{S}{K}} x} \Delta = \left( \frac{S^2}{K^2} - 8 \frac{S}{K} \sqrt{\frac{S}{K}} + 24 \frac{S}{K} - 24 \sqrt{\frac{S}{K}} \right) e^{\sqrt{\frac{S}{K}} x} \\
+ \left( -8 \frac{S}{K} \sqrt{\frac{S}{K}} + 48 \sqrt{\frac{S}{K}} \right) e^{\sqrt{\frac{S}{K}} x} \\
- \frac{S^2}{K^2} - 8 \frac{S}{K} \sqrt{\frac{S}{K}} + 24 \frac{S}{K} - 24 \sqrt{\frac{S}{K}}.
\]

(4.5)

When \( \frac{S^2}{K} - 8 \frac{S}{K} \sqrt{\frac{S}{K}} + 24 \frac{S}{K} - 24 \sqrt{\frac{S}{K}} = 0 \), we get \( \sqrt{\frac{S}{K}} = 2 \). Then Equation (4.5) becomes

\[
e^{\sqrt{\frac{S}{K}} x} = \left( -8 \frac{S}{K} \sqrt{\frac{S}{K}} + 48 \sqrt{\frac{S}{K}} \right) e^{\sqrt{\frac{S}{K}} x} - \frac{S^2}{K^2} \\
= 16e^{\sqrt{\frac{S}{K}}} - 112 \approx 6.224 \neq 0.
\]

If \( \sqrt{\frac{S}{K}} \neq 2 \), we suppose \( \Delta = 0 \). Then we have

\[
e^{\sqrt{\frac{S}{K}} x} = \frac{\pm \frac{S}{K} \sqrt{\frac{S}{K}} + 4 \frac{S}{K} - 24}{\frac{S}{K} \sqrt{\frac{S}{K}} - 8 \frac{S}{K} + 24 \sqrt{\frac{S}{K}} - 24}.
\]

(4.6)

Since function \( e^{\frac{S}{K}} \) has no intersections with \( \frac{\pm x^2 + 4x^2 - 24}{x^2 + 24x - 24} \) when \( x > 0 \), we can deduce that Equation (4.6) is false which means, on the contrary, \( \Delta \neq 0 \). This yields that Equation (4.1) has a unique solution, which means we get the unique \( (f, g) \in D(A_0) \).

Now we are in a position to show that \( A_0^{-1} \) is compact. Multiplying both sides of the first equation of (4.1) by \( f(x) \) and then integrating by parts over \([0, 1]\), we obtain

\[
\int_0^1 K \cdot f''''(x) + \frac{S}{\rho A} |f''(x)|^2 \, dx = -\int_0^1 \psi(x) f(x) \, dx
\]

and thus \( \| f \|_{H^2_0} \leq C \| \psi \|_{L^2} \| f \|_{L^2} \), where \( C \) is a positive constant. By Poincaré’s inequality, we have \( \| f \|_{H^2_0} \leq D \| \psi \|_{L^2} \), where \( D \) is also a positive constant. This can be written as

\[
\| A^{-1} \psi \|_{H^2_0} \leq D \| \psi \|_{L^2}, \quad \forall \psi \in L^2(0, 1).
\]

Since the injection of \( H^2_0(0, 1) \) into \( L^2(0, 1) \) is compact, we deduce that \( A^{-1} \) is compact on \( L^2(0, 1) \). Therefore, \( A_0^{-1} \) is compact on \( X \) and \( \sigma(A_0) \) consists of isolated eigenvalues.

We now claim that the eigenfunctions of \( A_0 \) form an orthonormal basis of \( X \). By Proposition 3.2.12 in Tucsnak and Weiss (2009), \( A \) is diagonalisable with an orthonormal basis \( (\psi_k)_{k \in \mathbb{N}} \) of eigenvectors and the corresponding sequence of eigenvalues \( (\lambda_k) \) satisfies \( \lambda_k > 0 \) and \( \lambda_k \to \infty \). Letting \( \mu_k = \sqrt{\lambda_k} \), for all \( k \in \mathbb{N} \) we define \( \phi_{-k} = -\phi_k \) and \( \mu_{-k} = -\mu_k \). Then \( A_0 \) is diagonalisable, with the eigenvalues \( i\mu_k \) corresponding to the orthonormal basis of eigenvectors

\[
\phi_k = \frac{1}{2} \begin{bmatrix} 1 \\ i\mu_k \phi_k \end{bmatrix}, \quad \forall k \in \mathbb{Z}^+.
\]

where \( \mathbb{Z}^+ \) denotes the set of all the non-zero integers.

Finally, due to the fact that \( A_0 \) is skew-adjoint, \( A_0 \) generates a \( C_0 \)-group on \( X \) by the Stone theorem (see Paży, 1983, Theorem 10.8, p. 41). Then the proof is complete. \( \square \)

We are now in a position to prove the asymptotic behaviour of the eigenvalues and the eigenfunctions.
Proof of Lemma 3.3: Substituting Equations (3.6) and (3.7) into the boundary conditions in Equation (3.1), we obtain asymptotic expressions for the boundary conditions for large enough $|\mu|$: 

\[
U_i(\Phi_i, \mu) = \Phi_i(0, \mu) = 1 + O(\mu^{-3}) := [1], \quad i = 1, 2, 3, 4,
\]

\[
U_5(\Phi_5, \mu) = \Phi_5(0, \mu) = \mu \omega_i \left( 1 + \frac{1}{6 \mu^2 \omega_i^2} \right) + O(\mu^{-3})
\]

\[
:= \mu \omega_i \left[ 1 + \frac{1}{6 \mu^2 \omega_i^2} \right], \quad i = 1, 2, 3, 4,
\]

\[
U_4(\Phi_4, \mu) = \Phi_4''(0, \mu) = (\mu \omega_i)^2 \left( 1 + \frac{1}{3 \mu^2 \omega_i^2} \right) + O(\mu^{-3})
\]

\[
:= (\mu \omega_i)^2 \left[ 1 + \frac{1}{3 \mu^2 \omega_i^2} \right], \quad i = 1, 2, 3, 4,
\]

\[
U_3(\Phi_3, \mu) = \Phi_3(1, \mu)
\]

\[
= \exp(\mu \omega_i) \left( 1 + \frac{1}{6 \mu \omega_i} \right) + \frac{1}{12 \mu \omega_i^2} + O(\mu^{-3})
\]

\[
:= \exp(\mu \omega_i) \left[ 1 + \frac{1}{6 \mu \omega_i} \right], \quad i = 3, 4, 5, 6.
\]

\[
U_2(\Phi_2, \mu) = \Phi_2''(1, \mu)
\]

\[
= \mu \omega_i \exp(\mu \omega_i) \left( 1 + \frac{1}{6 \mu \omega_i} \right) + \frac{1}{6 \mu \omega_i^2} \left( 1 + \frac{S}{12 \mu \omega_i} \right) + O(\mu^{-3})
\]

\[
:= \mu \omega_i \exp(\mu \omega_i) \left[ 1 + \frac{1}{6 \mu \omega_i} \right] + \frac{1}{6 \mu \omega_i^2} \left( 1 + \frac{S}{12 \mu \omega_i} \right), \quad i = 3, 4, 5, 6.
\]

\[
U_1(\Phi_1, \mu) = \Phi_1''(1, \mu)
\]

\[
= (\mu \omega_i)^2 \exp(\mu \omega_i) \left( 1 + \frac{1}{6 \mu \omega_i} \right) + \frac{1}{3 \mu \omega_i^2} \left( 1 + \frac{S}{24 \mu \omega_i} \right) + O(\mu^{-3})
\]

\[
:= (\mu \omega_i)^2 \exp(\mu \omega_i) \left[ 1 + \frac{1}{6 \mu \omega_i} \right] + \frac{1}{3 \mu \omega_i^2} \left( 1 + \frac{S}{24 \mu \omega_i} \right), \quad i = 3, 4, 5, 6.
\]

and

\[
U_6(\Phi_6, \mu) = U_6(\Phi_5, \mu) = U_5(\Phi_6, \mu) = U_5(\Phi_5, \mu)
\]

\[
= U_4(\Phi_6, \mu) = U_4(\Phi_5, \mu) = U_3(\Phi_2, \mu)
\]

\[
= U_3(\Phi_1, \mu) = U_2(\Phi_2, \mu) = U_2(\Phi_1, \mu)
\]

\[
= U_1(\Phi_1, \mu) = U_1(\Phi_2, \mu) = U_1(\Phi_2, \mu) = U_1(\Phi_1, \mu) = 0.
\]

Obviously, $\lambda = \sqrt{-\frac{i}{\rho \omega}} \mu^3 \in \sigma(\omega)$ if and only if the characteristic determinant $\Delta(\mu) = 0$, where

\[
\Delta(\mu) := [U_6(\Phi_6, \mu) U_6(\Phi_5, \mu) U_5(\Phi_6, \mu) U_5(\Phi_5, \mu) U_4(\Phi_6, \mu) U_4(\Phi_5, \mu) U_3(\Phi_6, \mu) U_3(\Phi_5, \mu) U_2(\Phi_6, \mu) U_2(\Phi_5, \mu) U_1(\Phi_6, \mu) U_1(\Phi_5, \mu) U_1(\Phi_5, \mu)]
\]

\[
M_1(\mu) = \begin{bmatrix}
[1]_3 \\
\mu \omega_i \left[ 1 + \frac{1}{6 \mu \omega_i} \right] \\
(\mu \omega_i)^2 \left[ 1 + \frac{1}{3 \mu \omega_i^2} \right] \\
\end{bmatrix}
\]

\[
M_2(\mu) = \begin{bmatrix}
[1]_3 \\
\mu \omega_i \left[ 1 + \frac{1}{6 \mu \omega_i} \right] \\
(\mu \omega_i)^2 \left[ 1 + \frac{1}{3 \mu \omega_i^2} \right] \\
\end{bmatrix}
\]

\[
M_3(\mu) = \begin{bmatrix}
[1]_3 \\
\mu \omega_i \left[ 1 + \frac{1}{6 \mu \omega_i} \right] \\
(\mu \omega_i)^2 \left[ 1 + \frac{1}{3 \mu \omega_i^2} \right] \\
\end{bmatrix}
\]

\[
\exp(\mu \omega_i) \left[ 1 + \frac{1}{6 \mu \omega_i} \right] + \frac{1}{6 \mu \omega_i^2} \left( 1 + \frac{S}{12 \mu \omega_i} \right) + O(\mu^{-3})
\]

\[
:= \mu \omega_i \exp(\mu \omega_i) \left[ 1 + \frac{1}{6 \mu \omega_i} \right] + \frac{1}{6 \mu \omega_i^2} \left( 1 + \frac{S}{12 \mu \omega_i} \right), \quad i = 3, 4, 5, 6.
\]

\[
(\mu \omega_i)^2 \exp(\mu \omega_i) \left[ 1 + \frac{1}{6 \mu \omega_i} \right] + \frac{1}{3 \mu \omega_i^2} \left( 1 + \frac{S}{24 \mu \omega_i} \right) + O(\mu^{-3})
\]

\[
:= (\mu \omega_i)^2 \exp(\mu \omega_i) \left[ 1 + \frac{1}{6 \mu \omega_i} \right] + \frac{1}{3 \mu \omega_i^2} \left( 1 + \frac{S}{24 \mu \omega_i} \right), \quad i = 3, 4, 5, 6.
\]

\[
(\mu \omega_i)^2 \exp(\mu \omega_i) \left[ 1 + \frac{1}{6 \mu \omega_i} \right] + \frac{1}{3 \mu \omega_i^2} \left( 1 + \frac{S}{24 \mu \omega_i} \right) + O(\mu^{-3})
\]

\[
:= (\mu \omega_i)^2 \exp(\mu \omega_i) \left[ 1 + \frac{1}{6 \mu \omega_i} \right] + \frac{1}{3 \mu \omega_i^2} \left( 1 + \frac{S}{24 \mu \omega_i} \right), \quad i = 3, 4, 5, 6.
\]

\[
(\mu \omega_i)^2 \exp(\mu \omega_i) \left[ 1 + \frac{1}{6 \mu \omega_i} \right] + \frac{1}{3 \mu \omega_i^2} \left( 1 + \frac{S}{24 \mu \omega_i} \right) + O(\mu^{-3})
\]

\[
:= (\mu \omega_i)^2 \exp(\mu \omega_i) \left[ 1 + \frac{1}{6 \mu \omega_i} \right] + \frac{1}{3 \mu \omega_i^2} \left( 1 + \frac{S}{24 \mu \omega_i} \right), \quad i = 3, 4, 5, 6.
\]
Substituting Equation (3.4) into Equation (4.15), we have
\[
\begin{align*}
M_4(\mu) &= \\
&= \begin{bmatrix}
\mu \omega_4 \\
\mu \omega_4 \exp(\mu \omega_4) \\
(\mu \omega_4)^2 \exp(\mu \omega_4)
\end{bmatrix} \\
&= \begin{bmatrix}
\mu \omega_4 \\
\mu \omega_4 \exp(\mu \omega_4) \\
(\mu \omega_4)^2 \exp(\mu \omega_4)
\end{bmatrix} \\
&= \begin{bmatrix}
\mu \omega_4 \\
\mu \omega_4 \exp(\mu \omega_4) \\
(\mu \omega_4)^2 \exp(\mu \omega_4)
\end{bmatrix}
\end{align*}
\]

Expanding the above determinant, we obtain
\[
\mu^6 e^{-\mu(\omega_0 + \omega_1)} \Delta(\mu) = \\
\begin{bmatrix}
3 + \sqrt{3} + \frac{i S}{2 \mu K} + \frac{1}{\mu^2 K} \left(1 - \sqrt{3} i\right) + \\
\frac{1}{12 K} \left(1 + \sqrt{3} i\right) e^{\mu \omega_1} + \\
-3 + \frac{\sqrt{3} + i S}{2 \mu K} + \frac{1}{\mu^2 K} \times \left(1 - \sqrt{3} i\right) + \\
\frac{1}{12 K} \left(1 + \sqrt{3} i\right) e^{\mu \omega_1} + \\
O(\mu^{-3})
\end{bmatrix}
\]

Substituting Equation (3.4) into Equation (4.15), we have
\[
\mu^6 e^{-\mu(\omega_0 + \omega_1)} \Delta(\mu) = \\
= -3e^{\alpha \omega_0} + 3e^{-\alpha \omega_0} + \\
\frac{\sqrt{3} + i S}{2 \mu K} e^{\alpha \omega_0} + \\
\frac{\sqrt{3} + i S}{2 \mu K} e^{-\alpha \omega_0} \\
+ \frac{1}{\mu^2 K} \left[1 - \sqrt{3} i + \frac{1}{12 K} \left(1 + \sqrt{3} i\right) e^{\mu \omega_1} + \\
\frac{1}{\mu^2 K} \left[1 - \sqrt{3} i + \frac{1}{12 K} \left(1 + \sqrt{3} i\right) e^{\mu \omega_1} + \\
O(\mu^{-3})
\end{align*}
\]

The equation \( \Delta(\mu) = 0 \) implies that
\[
\begin{align*}
3e^{\alpha \omega_0} - 3e^{-\alpha \omega_0} + \frac{\sqrt{3} - i S}{2 \mu K} e^{\alpha \omega_0} - \frac{\sqrt{3} + i S}{2 \mu K} e^{-\alpha \omega_0} \\
+ \frac{1}{\mu^2 K} \left[1 - \sqrt{3} i + \frac{1}{12 K} \left(1 + \sqrt{3} i\right) e^{\mu \omega_1} + \\
\frac{1}{\mu^2 K} \left[1 - \sqrt{3} i + \frac{1}{12 K} \left(1 + \sqrt{3} i\right) e^{\mu \omega_1} + \\
O(\mu^{-3}) = 0.
\end{align*}
\]

which can be written as
\[
\begin{align*}
e^{\alpha \omega_0} - e^{-\alpha \omega_0} = 0.
\end{align*}
\]

Note that the following equation:
\[
\begin{align*}
e^{\alpha \omega_0} - e^{-\alpha \omega_0} = 0
\end{align*}
\]

has solutions
\[
\begin{align*}
\mu_k = \frac{1}{\omega_3} \kappa \pi i, \quad k = 1, 2, \ldots
\end{align*}
\]

Let \( \mu_k \) be the solutions of Equation (4.18). Applying Rouche’s theorem (see Naimark, 1967, p. 70) to Equation (4.18), we can get the following expression:
\[
\begin{align*}
\mu_k = \mu_k + \alpha_k = \frac{1}{\omega_3} \kappa \pi i + \alpha_k, \\
\alpha_k = O(k^{-1}), \quad k = N, N + 1, \ldots
\end{align*}
\]

where \( N \) is a positive integer. Substituting \( \mu_k \) into Equation (4.17), and using the fact that \( \exp(\mu_k \omega_3) = \exp(-\mu_k \omega_3) \), we have
\[
\begin{align*}
e^{\alpha \omega_0} - e^{-\alpha \omega_0} + \frac{\sqrt{3} - i S}{2 \mu K} e^{\alpha \omega_0} - \frac{\sqrt{3} + i S}{2 \mu K} e^{-\alpha \omega_0} \\
- \frac{1}{\mu^2 K} \left[1 - \sqrt{3} i + \frac{1}{12 K} \left(1 + \sqrt{3} i\right) e^{\mu \omega_1} + \\
\frac{1}{\mu^2 K} \left[1 - \sqrt{3} i + \frac{1}{12 K} \left(1 + \sqrt{3} i\right) e^{\mu \omega_1} + \\
O(\mu_k^{-3}) = 0.
\end{align*}
\]

Expanding the exponential function according its Taylor series, we get
\[
\begin{align*}
\alpha_k = \frac{1}{6 \mu_k} \frac{S}{3 \mu_k^3} + \frac{\sqrt{3} S}{3 \mu_k^3} + O(k^{-3}).
\end{align*}
\]
Therefore, we have
\[
\mu_k = \frac{1}{\omega_0} k \pi i + \frac{1}{6 \mu_k} S - \frac{\sqrt{3} S}{3 \mu_k^2 K} + \mathcal{O}(k^{-3})
\]
(4.22)
\[
= kp + \frac{1}{6 \kappa K} S - \frac{\sqrt{3} S}{3 \kappa^2 \pi^2 K} + \mathcal{O}(k^{-3}), \quad k = N, N + 1, \ldots,
\]
and
\[
\lambda_k = i \sqrt{\frac{K}{\rho A}} \mu^k = i \sqrt{\frac{K}{\rho A}} \left[ kp + \frac{1}{6 \kappa K} S - \frac{\sqrt{3} S}{3 \kappa^2 \pi^2 K} + \mathcal{O}(k^{-3}) \right]^3
\]
(4.23)
\[
= i \sqrt{\frac{K}{\rho A}} \left[ k^3 \pi^3 - \frac{\sqrt{3} S}{2 K} + \frac{1}{2} k \pi S + \mathcal{O}(k^{-1}) \right],
\]
\[k = N, N + 1, \ldots.\]

The proof is complete. \(\square\)

**Proof of Lemma 3.4:** From Proposition 3.1, Equation (3.6) and the characteristic determinant \(\Delta(\mu)\), for each \(\lambda = i \sqrt{\frac{K}{\rho A}} \mu_3\), the corresponding eigenfunction \(\phi(x)\) satisfies
\[
\mu^{-4} e^{(\omega_0 + \omega_0)} \phi(x) = \det \left[ N_1(\mu, x), N_2(\mu, x), N_3(\mu, x), N_4(\mu, x), N_5(\mu, x), N_6(\mu, x) \right],
\]
(4.24)
where \(N_i(\mu, x)\) are six vector-valued functions for \(\mu, x\) and \(i = 1, 2, \ldots, 6\) given by
\[
N_1(\mu, x) = \begin{bmatrix} [1]_1 \\ \mu_1 [1]_1 \\ (\mu_1)^2 [1]_1 \\ O_{2 \times 1} \\ e^{\omega_0 x} [1]_1 \end{bmatrix}, \quad N_2(\mu, x) = \begin{bmatrix} [1]_1 \\ \mu_2 [1]_1 \\ (\mu_2)^2 [1]_1 \\ O_{2 \times 1} \\ e^{\omega_0 x} [1]_1 \end{bmatrix},
\]
\[
N_3(\mu, x) = \begin{bmatrix} [1]_1 \\ \mu_3 [1]_1 \\ (\mu_3)^2 [1]_1 \\ e^{\omega_0 x} [1]_1 \\ e^{\omega_0 x} [1]_1 \end{bmatrix}, \quad N_4(\mu, x) = \begin{bmatrix} [1]_1 \\ \mu_4 [1]_1 \\ (\mu_4)^2 [1]_1 \\ e^{\omega_0 x} [1]_1 \\ e^{\omega_0 x} [1]_1 \end{bmatrix},
\]
\[
N_5(\mu, x) = \begin{bmatrix} [1]_1 \\ \mu_5 [1]_1 \\ (\mu_5)^2 [1]_1 \\ e^{\omega_0 x} [1]_1 \\ e^{\omega_0 x} [1]_1 \end{bmatrix}, \quad N_6(\mu, x) = \begin{bmatrix} [1]_1 \\ \mu_6 [1]_1 \\ (\mu_6)^2 [1]_1 \\ e^{\omega_0 x} [1]_1 \\ e^{\omega_0 x} [1]_1 \end{bmatrix}.
\]

Then Equation (4.24) becomes
\[
\mu^{-4} e^{(\omega_0 + \omega_0)} \phi(x) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & e^{\omega_0 x} & e^{\omega_0 x} & 1 & 1 \\ 0 & 0 & e^{\omega_0 x} & e^{\omega_0 x} & e^{\omega_0 x} & e^{\omega_0 x} \end{bmatrix} + \mathcal{O}(\mu^{-1})
\]
(4.24)
\[
= (3 + \sqrt{3}) e^{-\omega_0 x} + (-3 + \sqrt{3}) e^{\omega_0 x} - \sqrt{3} e^{\omega_0 x}
\]
\[
- \sqrt{3} e^{-\omega_0 x} + (-3 + \sqrt{3}) e^{-\omega_0 x} + (3 + \sqrt{3}) e^{\omega_0 x} + \mathcal{O}(\mu^{-1}).
\]

Then, taking the third-order derivative on the above equation, we have
\[
\mu^{-7} e^{-\mu(\omega_0 + \omega_0)} \phi''(x) = (-\sqrt{3} + 3) e^{-\omega_0 x} + (\sqrt{3} + 3) e^{\omega_0 x} - \sqrt{3} e^{\omega_0 x}
\]
\[
+ \sqrt{3} e^{-\omega_0 x} + (-3 - \sqrt{3}) e^{\omega_0 x} + (3 - \sqrt{3}) e^{-\omega_0 x} + \mathcal{O}(\mu^{-1}).
\]

Now, let \(\phi_n(x) = -i \sqrt{\frac{K}{\rho A}} \mu_n^{-7} e^{(\omega_0 + \omega_0)} \phi(x)\). Hence, for eigenvalue \(\lambda_n\), the corresponding eigenfunction \(\phi_n\) has the form in Equation (3.9).

On the other hand, we note that Equations (3.4) and (3.8) yield
\[
\mu_1 \omega_1 = \left( -\frac{\sqrt{3}}{2} + \frac{3}{2} \right) n \pi + \mathcal{O}(n^{-1}),
\]
\[
\mu_2 \omega_2 = \left( -\frac{\sqrt{3}}{2} - \frac{1}{2} \right) n \pi + \mathcal{O}(n^{-1}),
\]
\[
\mu_3 \omega_3 = in \pi + \mathcal{O}(n^{-1}),
\]
\[
\mu_4 \omega_4 = -in \pi + \mathcal{O}(n^{-1}),
\]
\[
\mu_5 \omega_5 = \left( \frac{3}{2} + \frac{1}{2} \right) n \pi + \mathcal{O}(n^{-1}),
\]
\[
\mu_6 \omega_6 = \left( \frac{3}{2} - \frac{1}{2} \right) n \pi + \mathcal{O}(n^{-1}).
\]

Thus, we have
\[
\left\| e^{\omega_0 x} \right\|_{L^2(0,1)}^2 = \mathcal{O}(n^{-1}), \quad \left\| e^{\omega_0 x} \right\|_{L^2(0,1)}^2 = 1 + \mathcal{O}(n^{-1}),
\]
\[
\left\| e^{\omega_0 x} \right\|_{L^2(0,1)}^2 = 1 + \mathcal{O}(n^{-1}), \quad \left\| e^{\omega_0 x} \right\|_{L^2(0,1)}^2 = \mathcal{O}(n^{-1}),
\]
\[
\left\| e^{\omega_0 x} \right\|_{L^2(0,1)}^2 = \mathcal{O}(n^{-1}), \quad \left\| e^{\omega_0 x} \right\|_{L^2(0,1)}^2 = \mathcal{O}(n^{-1}).
\]

These, together with (3.9), yield the desired result (3.10). The proof is complete. \(\square\)

According to Lemma 3.1, we can directly obtain the existence of the unique mild solution for system (2.18). Thus, we omit the proof of Theorem 2.1.

**Proof of Theorem 2.2:** Since \(\{\Psi_n, \bar{\Psi}_n, n \in \mathbb{N}\}\) form an orthogonal basis of \(X\), then for every \(F \in X\),
\[ F = \sum_{n \in \mathbb{N}} \left( a_n \psi_n + b_n \overline{\psi_n} \right), \]
\[ e^{\lambda t}F = \sum_{n \in \mathbb{N}} \left( a_n e^{\lambda t} \psi_n + b_n e^{\lambda t} \overline{\psi_n} \right), \quad a_n, b_n \in \mathbb{R}. \]

Therefore, for every \((w_0, w_1) \in X,\)
\[ B_0 \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} = B_0 e^{\lambda t} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = -\sum_{n \in \mathbb{N}} \left( a_n e^{\lambda t} \phi''_n(1) + b_n e^{\lambda t} \overline{\phi''_n(1)} \right). \]

According to Equation (3.4), Theorem 3.3 and Theorem 3.4, we have
\[ \lim_{n \to +\infty} \left( \sqrt{\frac{K}{\rho A}} \mu_{n+1} - \sqrt{\frac{K}{\rho A}} \mu_n \right) = +\infty, \]

and
\[ \phi''_n(1) = \begin{cases} -2\sqrt{3} e^{-\frac{\pi^2}{2m^2}} + 6i + O(n^{-1}), & n \text{ odd}, \\ -6i e^{-\frac{\pi^2}{2m^2}} - 6i + O(n^{-1}), & n \text{ even}. \end{cases} \]

Then, applying Ingham theorem (Komornik & Loreti, 2005, Theorem 4.6, p. 67), for every \(T > 0,\) we have
\[ \|w_{xx}(t, \cdot)\|^2_{L^2(0, T)} = \int_0^T |y_0(t)|^2 \, dt = \int_0^T \left| \sum_{n \in \mathbb{N}} \left( a_n e^{\lambda t} \phi''(1) + b_n e^{\lambda t} \overline{\phi''(1)} \right) \right|^2 dt \geq D_T \sum_{n \in \mathbb{N}} (|a_n|^2 + |b_n|^2) \geq D_T \|w_0, w_1\|^2_X. \]

The proof is complete. \(\Box\)

We end this section with the proof of the main result.

**Proof of Theorem 1.1:** It is noting that the well-posedness results of system (1.1) have already been given in Guzmán and Zhu (2015). By the dual principle theorem (Tucsnak & Weiss, 2009, Theorem 11.2.1, p. 365), the exact observability presented in Theorem 2.2 yields the exact controllability of system (1.1). \(\Box\)

5. Conclusion

In this paper, we addressed the exact controllability of a micro beam with boundary bending moment at time \(T > 0.\) To adopt the semi-group approach, system (1.1) is written into an abstract evolution equation. A detailed spectral analysis for the system operator is given to show the asymptotic behaviour of the eigenvalues and the eigenfunctions. According to the duality principle, the exact observability of the dual system is obtained, with which we deduced the exact controllability of system (1.1).

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**References**


