Subpancyclicity of line graphs and degree sums along paths

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Abstract

A graph is called subpancyclic if it contains a cycle of length \( \ell \) for each \( \ell \) between 3 and the circumference of the graph. We show that if \( G \) is a connected graph on \( n \geq 146 \) vertices such that \( d(u) + d(v) + d(x) + d(y) > (n + 10/2) \) for all four vertices \( u, v, x, y \) of any path \( P = uvxy \) in \( G \), then the line graph \( L(G) \) is subpancyclic, unless \( G \) is isomorphic to an exceptional graph. Moreover, we show that this result is best possible, even under the assumption that \( L(G) \) is hamiltonian. This improves earlier sufficient conditions by a multiplicative factor rather than an additive constant.

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1. Introduction

During the last 50 years, a key issue in the vast literature on cyclic properties of graphs has been the development of sufficient conditions to guarantee the existence of certain cycles in graphs. Most of these conditions involve the neighborhood of single vertices, pairs of (nonadjacent) vertices, or larger sets of (mutually nonadjacent) vertices. Typical examples (see e.g. [2]) are the well-known minimum degree condition \( \delta(G) \geq n/2 \) (where \( n \) is the number of vertices of the graph \( G \)), due to Dirac, for the existence of a hamiltonian cycle in \( G \), and its subsequent generalizations to all pairs of nonadjacent vertices having degree sum at least \( n \) (Ore), and other sets of vertices meeting some degree condition. These conditions have in common that the average degree in all conditions is bounded from below by (roughly) the same function of \( n \), namely \( n/2 \). Even if one adds certain conditions to the graph, e.g. that the graph is highly connected or 1-tough, these bounds cannot be improved considerably, but at most by a small constant. A similar phenomenon is often encountered in other chains of subsequently improved results guaranteeing certain cyclic properties of graphs. We omit the details, but stress the common feature of the results in this area of graph theory. In contrast to this, the main result of this paper involves a degree condition on sets of four vertices along a path on four
vertices, which improves earlier results on smaller sets along shorter paths considerably. The best previously known result guarantees the same conclusion with a lower bound on the degree sum of sets of three vertices along a path on three vertices which is roughly the same as our lower bound for four vertices. Hence the lower bound on the average degree in the condition is decreased by a factor rather than a small constant. A small drawback is that we have to exclude some exceptional graphs from the conclusion, but this is unavoidable since the previously known result is best possible (in the sense that the lower bound cannot be decreased without excluding some exceptional graphs). We refer to the next sections for the details, and start with some useful definitions, notation and related results.

2. Preliminaries

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only. Let $G$ be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. Let $H$ be a subgraph of $G$. If $S$ is a subgraph of $H$, then the degree of $S$ in $H$, denoted by $d_H(S)$ (or just $d(S)$ if $H = G$), is defined to be the degree sum of the vertices of $S$, i.e., $d_H(S) = \sum_{u \in V(S)} d_H(u)$. With $c(G)$ we denote the circumference of $G$, i.e., the length of a longest cycle of $G$, and with $\lambda(G)$ the set of all different cycle lengths of cycles in $G$. $G$ is called pancyclic if $\lambda(G) = [3, |V(G)|] = [3, 4, \ldots, |V(G)|]$. $G$ is said to be subpancyclic if $\lambda(G) = [3, c(G)] = [3, 4, \ldots, c(G)]$. By the length of a path we refer to the number of edges on this path.

Define

$$\rho_i(G) = \min\{d(P) : P \text{ is a path of length } i - 1 \text{ in } G\}.$$  

Obviously $\delta(G) = \rho_1(G)$. As introduced in [1], let $f_i(n)$ be the smallest integer such that for any graph $G$ of order $n$ with $\rho_i(G) > f_i(n)$, the line graph $L(G)$ of $G$ is pancyclic whenever $L(G)$ is hamiltonian. Van Blanken et al. [1] proved that $f_i(n)$ has the order of magnitude $O(n^{1/3})$. If we do not impose the condition that $L(G)$ is hamiltonian, the following degree condition results guaranteeing subpancyclicity have been obtained. We note here that pancyclicity cannot be guaranteed by these conditions.

**Theorem 1.** Let $G$ be a connected graph of order $n$. If $G$ satisfies one of the following conditions:

(i) $[10]$ $\rho_2(G) > (\sqrt{8n + 1} + 1)/2$ and $n \geq 600$;
(ii) $[11]$ $\rho_3(G) > (n + 6)/2$ and $n \geq 76$;
(iii) $[11]$ $\rho_4(G) > (2n + 16)/3$ and $n \geq 76$,

then $L(G)$ is subpancyclic and the bounds on $\rho_i(G)$ are all sharp.

Here sharpness means that the bounds cannot be lowered without excluding some exceptional graphs from the conclusion.

Concentrating on sufficient degree conditions for the line graph itself in order to guarantee subpancyclicity, Trommel, et al. showed a consequence of Theorem 1(i) in this respect for large line graphs.

**Corollary 2 (Trommel et al. [8]).** Let $G$ be a line graph with order $n > 100577$. If 

$$\delta(G) = \rho_1(G) > (\sqrt{8n + 1} - 3)/2,$$

then $G$ is subpancyclic.

As a first exercise, we show that we can obtain the following analogous consequences of Theorem 1(ii) and (iii) by using similar arguments as in the proof of Corollary 2.

**Corollary 3.** Let $G$ be a line graph with order $n > 2775$. If $G$ satisfies one of the following conditions:

(i) $\rho_2(G) > (n - 2)/2$;
(ii) $\rho_3(G) > (2n - 2)/3$;

then $G$ is subpancyclic.
Proof. Let $G = L(H)$ be a line graph with order $n > 2775$ and satisfying one of the conditions (i) and (ii) of Corollary 3. Since
\[
n > \left( \frac{75}{2} \right),
\]
$|V(H)| \geq 76$. If $H$ is a tree, then $G$ is subpancyclic. If $H$ is not a tree, then $n = |E(H)| \geq |V(H)| \geq 76$. Note that
\[
\rho_i(H) = \rho_{i-1}(G) + 2(i - 1) \text{ for } i = 3, 4.
\]
We obtain that either $\rho_3(H) = \rho_2(G) + 4 \geq (n - 2)/2 + 4 \geq (|V(H)| + 6)/2$ or $\rho_4(H) = \rho_3(G) + 5 \geq (2n - 2)/3 + 6 \geq (2|V(H)| + 16)/3$. Hence $H$ satisfies the conditions of Theorem 1 and so $G = L(H)$ is subpancyclic. \[\square\]

Theorem 1 shows that the graphs in [4,7,9] are pancyclic.

Rather than looking at line graphs, one could consider the larger class of so-called claw-free graph and ask for similar degree conditions guaranteeing subpancyclicity. One such result related to Theorem 1 has appeared in [8].

Theorem 4 (Trommel et al. [8]). Let $G$ be a claw-free graph on at least 5 vertices. If $\delta(G) > \sqrt{3n + 1} - 2$, then $G$ is subpancyclic and the bound is sharp.

More recently, Gould and Pfender improved Theorem 4 by considering a degree sum condition on pairs of nonadjacent vertices instead of a minimum degree condition, as follows.

Theorem 5 (Gould et al. [5]). Let $G$ be a claw-free graph on at least 5 vertices. If $d(u) + d(v) > 2\sqrt{3n + 1} - 4$ for any pair of vertices $u, v$ with $uv \notin E(G)$, then $G$ is subpancyclic and the bound is sharp.

3. Main result of this paper

We turn back to degree conditions on the graph $G$ for guaranteeing the subpancyclicity of its line graph $L(G)$. We will show that we can considerably decrease the lower bound in the best known degree condition for this by excluding one class of exceptional graphs. In fact, Theorem 6 below shows that when we exclude an exceptional graph, the lower bound on the degree sums of the vertices along 4-paths which ensures that its line graph is subpancyclic can be improved roughly from $2n/3$ to $n/2$. Moreover, this bound is then almost the same as the bound on degree sums of the vertices along 3-paths (comparing the statements in Theorem 1).

Theorem 6. Let $G$ be a connected graph of order $n \geq 146$. If
\[
\rho_4(G) > (n + 10)/2,
\]
then $L(G)$ is subpancyclic unless $G$ is isomorphic to the exceptional graph $F$ defined below. Moreover, the bound on $\rho_4(G)$ is sharp, even under the assumption that $L(G)$ is hamiltonian.

The exceptional graph $F$ is defined as follows: let $n \equiv 1 \pmod{3}$, and let $C_1, C_2, \ldots, C_{(n-1)/3}$ be $(n - 1)/3$ edge-disjoint cycles of length 4. Now $F$ is obtained from those cycles such that $C_1, C_2, \ldots, C_{(n-1)/3}$ have exactly one common vertex in $F$ and $E(F) = E(C_1) \cup E(C_2) \cup \cdots \cup E(C_{(n-1)/3})$. Obviously $|V(F)| = n$ and $\rho_4(F) = (2n + 16)/3 = f_4(n)$, by Theorem 1.

Using similar arguments as in the proof of Corollary 3, we obtain the following consequence of Theorem 6.

Corollary 7. Let $G$ be a line graph with order $n > 10440$. If $\rho_3(G) > (n - 2)/2$, then $G$ is subpancyclic unless $G$ is isomorphic to the line graph of $F$.

4. Proof of Theorem 6

Before we present our proof of the main result, we introduce some additional terminology and notation, and state a number of preliminary results.
By a circuit of a graph \( G \) we will mean an eulerian subgraph of \( G \), i.e., a connected subgraph in which every vertex has even degree. Note that by this definition (the trivial subgraph induced by) a single vertex is also a circuit. If \( C \) is a circuit of \( G \), then \( \overline{E}(C) \) denotes the set of edges of \( G \) incident with at least one vertex of \( C \) and we let \( UP_4(C) = \{ P : P \text{ is a path of length 3 in } C \} \). We write \( \varepsilon(C) \) for \( |E(C)| \) and \( \overline{\varepsilon}(C) \) for \( |\overline{E}(C)| \). The distance \( d_H(G_1, G_2) \) between two subgraphs \( G_1 \) and \( G_2 \) of \( H \) is defined to be \( \min\{d_H(v_1, v_2) : v_1 \in V(G_1) \text{ and } v_2 \in V(G_2)\} \). The diameter of a connected subgraph \( H \), denoted by \( \text{dia}(H) \), is defined to be \( \max\{d_H(u, v) : u, v \in V(H)\} \). By \( C_k \) we denote a cycle of length \( k \). For any subgraph \( H \) of \( G \), let \( N(H) = \bigcup_{u \in V(H)} N(u) \).

Harary and Nash-Williams [6] characterized those graphs with line graphs that are hamiltonian. One can easily prove a more general result (see, e.g., [3]).

**Theorem 8** (Broersma [3]). The line graph \( L(G) \) of a graph \( G \) contains a cycle of length \( k \geq 3 \) if and only if \( G \) contains a circuit \( C \) such that \( \varepsilon(C) \leq k \leq \overline{\varepsilon}(C) \).

Before we present our proof of the main result, we start with a useful technical lemma that will avoid ending up in too many subcases.

**Lemma 9.** If \( G \) is a graph of order \( n \) which satisfies the conditions of Theorem 6 but whose line graph \( L(G) \) is not subpancyclic, then \( G \) contains neither a circuit \( C_0 \) with \( \varepsilon(C_0) \leq k \leq \overline{\varepsilon}(C_0) \) nor a cycle of length \( k + 1 \), where \( k = \max\{i : i \in [3, \varepsilon(L(G))] \} \).

**Proof.** Assume \( G \) is a graph satisfying the conditions of Lemma 9. It follows from Theorem 8 that \( G \) does not contain a circuit \( C_0 \) with \( \varepsilon(C_0) \leq k \leq \overline{\varepsilon}(C_0) \). We claim that \( G \) does not contain a cycle of length \( k + 1 \). Suppose to the contrary that \( G \) has such a cycle \( C \). Note that \( [3, \text{L}(G)] \leq \lambda(L(G)) \). Hence \( k \geq \lambda(G) + 1 \). Since \( \rho_4(G) > (n + 10)/2 \geq 78 \),

\[
\varepsilon(C) = k + 1 \geq \lambda(G) + 2 \geq \rho_4(G)/4 + 2 > (n + 26)/8 \geq 21.
\]

(4.1)

We are going to derive a contradiction, and start with the following claim.

**Claim.** \( G \) does not contain a cycle \( C' \) with \( \varepsilon(C')/2 < \varepsilon(C') \leq k \).

**Proof of Claim.** Suppose otherwise that \( C' \) is a cycle of \( G \) such that \( \varepsilon(C')/2 < \varepsilon(C') \leq k \). Note that in \( \sum_{P \in UP_4(C')} d(P) \), every edge in \( \overline{E}(C') \) is counted at most 8 times. Hence, by (4.1) and \( \rho_4(G) \geq (n + 10)/2 \geq 78 \),

\[
\overline{\varepsilon}(C') \geq \sum_{P \in UP_4(C')} (d(P) - 8)/8 + \varepsilon(C') \\
\geq (\rho_4 - 8) \varepsilon(C')/8 + \varepsilon(C') = \rho_4 \varepsilon(C')/8 \\
\geq \rho_4 \varepsilon(C)/16 \\
\geq k + 1.
\]

On the other hand, \( \varepsilon(C') \leq k \). Thus \( L(G) \) contains a \( C_k \), a contradiction. This completes the proof of Claim. \( \Box \)

By Claim, \( C \) has no chord. Since \( \rho_4 \geq 78 \), \( C \) cannot be a hamiltonian cycle of \( G \). Let \( u \) be a vertex in \( V(G) \setminus V(C) \). Since \( G \) does not contain a circuit \( C_0 \) with \( \varepsilon(C_0) \leq k \leq \overline{\varepsilon}(C_0) \), \( u \) is adjacent to at most three vertices of \( C \). By (4.1), we obtain the following:

\[
\overline{\varepsilon}(C) \leq 3|V(G) \setminus V(C)| + \varepsilon(C) = 3(n - \varepsilon(C)) + \varepsilon(C) < (11n - 26)/4.
\]

(4.2)
On the other hand, since $C$ has no chord,
\[
\bar{v}(C) \geq \sum_{P \in UP_4(C)} (d(P) - 8)/4 + \bar{v}(C)
\]
\[
\geq (\rho_4 - 8)\bar{v}(C)/4 + \bar{v}(C)
\]
\[
= (\rho_4 - 4)\bar{v}(C)/4
\]
\[
\geq (n^2 + 28n + 52)/64,
\]
which contradicts (4.2) and $n \geq 146$. This completes the proof of Lemma 9. □

We now present the proof of Theorem 6.

**Proof of Theorem 6.** We will prove the theorem by contradiction.

Assuming $G$ is a graph of order $n$ which satisfies the conditions of Theorem 6 but whose line graph $L(G)$ is not subpancyclic, we define

\[
k = \max\{i : i \in [3, c(L(G))] \setminus \lambda(L(G))\},
\]
so this $k$ is the same as in Lemma 9.

By the definition of $k$, $L(G)$ contains a cycle $C_k + 1$ of length $k + 1$. Hence, by Theorem 8, we obtain that $G$ contains a circuit $C$ with $\bar{v}(C) \leq k + 1 \leq \bar{v}(C)$. By Lemma 9, $\bar{v}(C) = k + 1$. Since $C$ is a circuit, there exist edge-disjoint cycles $D_1, D_2, \ldots, D_r$ such that $C = \bigcup_{i=1}^r D_i$ and we assume that these cycles are chosen in such a way that $r$ is maximized.

By Lemma 9, it suffices to consider the case that $r \geq 2$. Hence,
\[
|V(D_i) \cap V(D_j)| \leq 2 \quad \text{for } \{i, j\} \subseteq \{1, 2, \ldots, r\}. \tag{4.3}
\]

It is easy to verify that (4.1) also holds here.

Let $H$ be the graph with $V(H) = \{D_1, D_2, \ldots, D_r\}$ and $D_i D_j \in E(H)$ if and only if $V(D_i) \cap V(D_j) \neq \emptyset$. Since $C$ is a circuit, $H$ is connected. Without loss of generality, we assume that $D_1$ and $D_r$ are two vertices of $H$ such that
\[
d_H(D_1, D_r) = \text{dia}(H). \tag{4.4}
\]

Hence, any element of $\{D_1, D_r\}$ is not a cut vertex of $H$, so $C_1 = \bigcup_{i=2}^r D_i$ and $C' = \bigcup_{i=1}^{r-1} D_i$ are two circuits of $G$. Let
\[
E_1(D_i) = E(D_i) \cap \bar{E}(C_i) \quad \text{and} \quad E_2(D_i) = E(D_i) \setminus E_1(D_i)
\]
and
\[
V_1(D_i) = V(D_i) \cap V(C_i) \quad \text{and} \quad V_2(D_i) = \{u, v : uv \in E_2(D_i)\},
\]
where $i \in \{1, r\}$.

For any path $P$ of $C$, let $d_2(P) = d(P) - d_C(P)$. Since $\bar{v}(C) \geq \bar{v}(C) - |E_2(D_i)| = k + 1 - |E_2(D_i)|$,
\[
|V_2(D_i)| - 1 \geq |E_2(D_i)| \geq 2, \tag{4.5}
\]
where $i \in \{1, r\}$. Otherwise $\bar{v}(C') \leq k \leq \bar{v}(C_i)$ which contradicts Lemma 9.

Since $\bar{v}(C') \geq \bar{v}(C) - |E_2(D_i)| + |\bar{E}(D_i) \setminus E(C)|$,
\[
|\bar{E}(D_i) \setminus E(C)| \leq |E_2(D_i)| - 2, \tag{4.6}
\]
where $\{s, t\} = \{1, r\}$. Otherwise $\bar{v}(C') \leq k \leq \bar{v}(C')$ which contradicts Lemma 9.

Before we turn to a case distinction, we need one more claim.

**Claim 1.** Let $P$ be a path of length 3 in $D_s$. Then
\[
d_C(P) > (n + 14)/2 - |E_2(D_i)| \tag{4.7}
\]
and
\[ |E_2(D_t)| \leq 2|V_2(D_t)|/3 \quad \text{and} \quad d_C(P) > (n + 14)/2 - 2|V_2(D_t)|/3, \]
where \( \{s, t\} = \{1, r\} \).

**Proof of Claim 1.** Let \( P \) be a path of length 3 in \( D_t \). Then
\[ |E(D_t)| \setminus E(C) \geq d(P) - d_C(P). \]
Hence be (4.6) and \( p_3(G) > (n + 10)/2, \)
\[ d_C(P) > (n + 10)/2 - (|E_2(D_t)| - 2), \]
i.e., (4.7) is true.
By (4.5) and (4.7),
\[ d_C(P) > (n + 16)/2 - |V_2(D_t)|. \]
In order to obtain (4.8), it suffices to prove the following assertion.

Each component of \( C[E_2(D_1) \cup E_2(D_r)] \) is a path of length at most two. (4.10)

Suppose to the contrary that there exists a \( t \in \{1, r\} \) and a path \( P_0 = u_0v_0x_0y_0 \) of \( D_t \) such that \( \{u_0, v_0, x_0, y_0\} \subseteq V_2(D_t) \). By (4.9), \( d_C(P_0) > (n + 16)/2 - |V_2(D_t)| \) where \( \{s, t\} = \{1, r\} \). Since \( d_C(P_0) = 8 \),
\[ |V_2(D_t)| > n/2 \geq 78. \]
Hence there exists a path \( P_0' = u_0'y_0'x_0'y_0' \) in \( D_t \) such that \( u_0'y_0' \in E_2(D_t) \) and \( \{x_0', y_0'\} \cap V_1(D_t) = \emptyset \). By (4.9),
\[ d_C(P') > (n + 16)/2 - |V_2(D_t)|. \]
For any \( x \in N_C(x_0') \cap N_C(y_0') \), \( C - x \) has at least a nontrivial component, denoted by \( Q_x \), which does not contain any vertex of \( D_t \). (otherwise \( \varepsilon(C') \leq k \leq \overline{\varepsilon}(C') \), where \( C' = C - \{xx_0', xy_0', x_0'y_0', \} \), a contradiction).

It is easy to see that
\[ |V(Q_x)| \geq 3. \]
Otherwise \( \varepsilon(C') \leq k \leq \overline{\varepsilon}(C') \), where \( C' = C - Q_x \), a contradiction.

Let \( B \) denote the cut-vertex set of \( N_C(x_0') \cap N_C(y_0') \) such that for any \( x \in B \), \( C - x \) has a nontrivial component, denoted by \( Q_x \), which does not contain any vertex of \( V(D_1) \cup V(D_r) \). Set
\[ \beta = |N_C(x_0') \cap N_C(y_0')|. \]
Obviously,
\[ |\{Q_x : x \in B\}| = |B| \geq \beta - 1. \]
Using (4.11), (4.13) and (4.14), we obtain
\[
d_C(x_0') + d_C(y_0') = |N_C(x_0') \cup N_C(y_0')| + |N_C(x_0') \cap N_C(y_0')|
\leq \begin{cases} n - (|V_2(D_1)| + |V_2(D_r)| - 4) + 1 & \text{if } \beta \leq 1, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 2 + 3(\beta - 1) + \beta & \text{if } \beta \geq 2, \\ \leq n - (|V_2(D_1)| + |V_2(D_r)|) + 5 \\ \leq n - n/2 - |V_2(D_r)| + 5 \\ = (n + 10)/2 - |V_2(D_r)|, \end{cases}
\]
which contradicts (4.12) and \( d_C(u_0') = d_C(v_0') = 2 \). This implies that assertion (4.10) is true. Hence (4.8) also holds. This completes the proof of Claim 1. \( \square \)
For convenience, we now introduce the following notation.

\[ S = \{x, y, x', y'\}, \]
\[ N_i = \{u \in V(C) : |N_C(u) \cap S| = i\}, \]
\[ M_1 = ((N_C(x) \cap N_C(y)) \cup (N_C(x') \cap N_C(y'))) \cap N_2, \]
\[ M_2 = N_2 \setminus M_1, \]
\[ n_i = |N_i| \quad \text{and} \quad m_i = |M_i|. \]

Obviously,
\[ n_2 = m_1 + m_2. \quad (4.15) \]

We will complete the proof by deriving contradictions in the following three cases.

**Case 1.** \( \text{dia}(H) \geq 2. \)

This implies that \( V(D_1) \cap V(D_r) = \emptyset. \)

We can take two paths \( P = uvx \) and \( P' = u'vx'y' \) of length 3 in \( D_1 \) and \( D_r \) respectively with \( \{uv, u'v'\} \subseteq E_2(D_1) \cup E_2(D_r) \) and \( \{x, x'\} \subseteq V_1(D_1) \cup V_1(D_r) \) such that \( V(P) \cap V(P') = \emptyset. \)

We now prove three claims in order to get contradictions.

**Claim 2.** \( |N_3 \cup N_4| \leq 1, \) i.e., \( n_3 + n_4 \leq 1. \)

**Proof of Claim 2.** Suppose otherwise that \( |N_3 \cup N_4| \geq 2, \) say, \( w, w' \in N_3 \cup N_4. \) Obviously,
\[ w, w' \in (N_C(x) \cap N_C(y)) \cup (N_C(x') \cap N_C(y')). \]

Without loss of generality, we assume that \( wx, wy \in E(C). \) Hence \( C' = C - \{wx, wy, xy\} \) is a circuit with \( \varepsilon(C') = \varepsilon(C) - 3 \leq k \leq \bar{\varepsilon}(C'), \) a contradiction. This completes the proof of Claim 2. \( \square \)

**Claim 3.** Each element of \( M_1 \) is a cut vertex of \( C. \)

**Proof of Claim 3.** Suppose otherwise that there exists a vertex \( w \in M_1, \) say, \( w \in N_C(x) \cap N_C(y) \cap N_2, \) which is not a cut vertex of \( C. \) Hence \( C' = C - \{wx, wy, xy\} \) is a circuit with \( \varepsilon(C) - 3 = \varepsilon(C') \leq k \leq \bar{\varepsilon}(C'), \) a contradiction. This completes the proof of Claim 3. \( \square \)

Let \( W_1 \) denote the cut vertex set of \( C \) in \( M_1 \) such that for any \( z \in W_1, \) \( C - z \) has a nontrivial component which does not contain any element of \( S. \) Every vertex of \( M_1 \) is in \( W_1 \) except when it is a cut vertex such that deleting it creates two components which contain \( \{x, y\} \) and \( \{x', y'\}, \) respectively; in the exceptional case, \( \text{dia}(H) \geq 3. \) Hence at most two vertices of \( M_1 \) are not in \( W_1 \) (in case of two they are in \( N_C(x) \cap N_C(y) \cap N_2 \) and \( N_C(x') \cap N_C(y') \cap N_2, \) respectively, note that in this case, \( \text{dia}(H) \geq 4), \) i.e.,
\[ |W_1| \geq m_1 - 2. \quad (4.16) \]

Similarly, we obtain
\[ \text{if } \emptyset \neq N_3 \cup N_4 \cup M_2 \quad \text{then } |W_1| = m_1. \quad (4.17) \]

**Claim 4.** If \( m_2 \geq 3, \) then for any pair of vertices \( \{w, w'\} \) of \( M_2, \) a cycle of \( C[\{w, w'\} \cup ((N_C(w) \cup N_C(w')) \cap S)] \) which does not contain the possible edge \( ww', \) is a nontrivial cut set of \( C, \) i.e., deleting it creates at least two nontrivial components of \( C. \)

**Proof of Claim 4.** Otherwise, \( C[\{w, w'\} \cup ((N_C(w) \cup N_C(w')) \cap S)] \) has a cycle \( C' \) which does not contain \( wv, \) such that \( \varepsilon(C'') \leq k \leq \bar{\varepsilon}(C'') \) where \( C'' = C - C', \) a contradiction. This completes the proof of Claim 4. \( \square \)

Let \( W_2 \) denote the cut vertex set of \( C \) in \( M_2 \) such that for any vertex \( y \in W_2, \) \( C - y \) has a nontrivial component which does not contain any element of \( S. \) By Claim 4, every vertex in \( M_2 \) is in \( W_2 \) except at most two vertices, \( w_0, w_{00} \)
(say); in this exceptional case, deleting the edges of \((\cup_{u \in S} \{w_0u, w_00u\}) \cap E(C)\) we will get exactly one component of \(C\) that does not contain any element of \(S\) (In fact, the component is nontrivial if \(m_2 \geq 3\)). Hence we obtain

\[ |W_2| \geq m_2 - 2. \]  

(4.18)

In a similar way, we obtain

if \(n_3 = 1\) then \(|W_2| \geq m_2 - 1\),

(4.19)

and

if \(n_4 = 1\) then \(|W_2| = m_2\).

(4.20)

For \(y \in W_1 \cup W_2\), take one nontrivial component of \(C - y\) which does not contain any element of \(S\), denoted by \(Q_y\). Then it is easy to see that

\[ |V(Q_y)| \geq 3. \]  

(4.21)

Otherwise \(\varepsilon(C') \leq k \leq \varepsilon(C')\), where \(C'_y = C - Q_y\). We also obtain

\[ |\{Q_y : y \in W_1 \cup W_2\}| = |W_1 \cup W_2|. \]  

(4.22)

If \(N_3 \cup N_4 \neq \emptyset\) then, using Claims 1–4 and (4.17) up to (4.22), we obtain

\[
d_C(S) = \left\{ \begin{array}{ll}
4 \sum_{i=1}^{4} N_i + n_2 + 2n_3 + 3n_4 & \\
\leq n - (|V_2(D_1)| + |V_2(D_r)| - 4 + 3n_2) + n_2 + 3 & \text{if } n_4 = 1,
\end{array} \right.
\]

(4.23)

\[
\leq n - (|V_2(D_1)| + |V_2(D_r)| - 5 + 3(n_2 - 1)) + n_2 + 2 & \text{if } n_2 \geq 2, \ n_3 = 1,
\]

\[
\leq n - (|V_2(D_1)| + |V_2(D_r)| - 5) + 3 & \text{if } n_2 \leq 1, \ n_3 = 1,
\]

\[
\leq n - (|V_2(D_1)| + |V_2(D_r)|) + 8.
\]

If \(N_3 \cup N_4 = \emptyset\) then, using Claims 3, 4 and (4.15), (4.16), (4.18), (4.21), (4.22), we obtain

\[
d_C(S) = \left\{ \begin{array}{ll}
4 \sum_{i=1}^{4} N_i + n_2 + 2n_3 + 3n_4 = \left\{ \begin{array}{ll}
4 \sum_{i=1}^{4} N_i + n_2 & \\
\leq n - (|V_2(D_1)| + |V_2(D_r)| - 5 + 3(m_1 - 2) + 3(m_2 - 2)) + m_1 + m_2 & \text{if } m_1 \geq 2, \ m_2 \geq 3,
\end{array} \right.
\end{array} \right.
\]

(4.23)

\[
\leq n - (|V_2(D_1)| + |V_2(D_r)| - 5 + 3(m_1 - 2)) + m_1 + 2 & \text{if } m_1 \geq 2, \ m_2 \leq 2,
\]

\[
\leq n - (|V_2(D_1)| + |V_2(D_r)| - 5) + 3 & \text{if } m_1 = 1, \ m_2 \leq 2,
\]

\[
\leq n - (|V_2(D_1)| + |V_2(D_r)|) + 8 & \text{if } m_1 = 0, \ m_2 \leq 2.
\]

Hence we conclude that

\[ d_C(S) \leq n - |V_2(D_1)| + |V_2(D_r)| + 8. \]  

(4.23)

On the other hand, by (4.8), \(d_C(P) + d_C(P') > n + 14 - 2(|V_2(D_1)| + |V_2(D_r)|)/3\).

Hence

\[ d_C(S) > n + 6 - 2(|V_2(D_1)| + |V_2(D_r)|)/3. \]  

(4.24)

Using (4.23) and (4.24), we obtain

\[ |V_2(D_1)| + |V_2(D_r)| < 6, \]

which contradicts (4.5).
Proof of Claim 5. Suppose otherwise that $|V(D_1)| \geq 5$. Hence by (4.10), we can take a path $P' = u'u'v'y'$ in $D_r$ such that $u'u'v' \in E_2(D_r)$, $y \notin \{x', y'\}$ and $|\{x', y'\} \cap V_2(D_r)| \leq 1$. By (4.5), we can take a path $P = uvxy$ in $D_1$ such that $\{uv, vx\} = E_2(D_1)$.

In a way similar to Claims 2, 3, we obtain
\[ n_3 = n_4 = 0 \quad \text{and} \quad n_2 \leq 1. \]  

By (4.7),
\[ d_C(P) + d_C(P') > n + 14 - |E_2(D_1)| - |E_2(D_r)|. \]

If $|E_2(D_r)| = 2$ or 3, then by (4.26),
\[ d_C(S) > n + 6 - |E_2(D_1)| - |E_2(D_r)| \geq n + 1. \]

On the other hand, by (4.25),
\[ d_C(S) = \sum_{i=1}^{4} N_i + n_2 + 2n_3 + 3n_4 \leq n - (|V_2(D_1)| + |V_2(D_r)| - 8) + 1 \]
\[ = n - |V_2(D_1)| - |V_2(D_r)| + 9, \]
which contradicts (4.27).

If $|E_2(D_r)| \geq 4$, then using (4.8), we obtain $|V_2(D_r)| \geq 3|E_2(D_r)|/2 \geq 6$. This implies that by (4.5),
\[ |V_2(D_1)| + |V_2(D_r)| \geq 9. \]  

On the other hand, using (4.25), we obtain
\[ d_C(S) = \sum_{i=1}^{4} N_i + n_2 + 2n_3 + 3n_4 \]
\[ \leq n - (|V_2(D_1)| + |V_2(D_r)| - 8) + 1 \]
\[ = n - |V_2(D_1)| - |V_2(D_r)| + 9, \]
implying that by (4.24),
\[ |V_2(D_1)| + |V_2(D_r)| < 3(9 - 6) = 9, \]
which contradicts (4.28). This completes the proof of Claim 5. □

Claim 5. $|V(D_r)| = 4$ for $i \in \{1, 2, \ldots, r\}$.

Proof of Claim 6. Suppose otherwise that there exists a $D_j$ ($i \in \{2, 3, \ldots, r - 1\}$), say $D_2$, such that $|V(D_2)| \neq 4$. If $|V(D_2)| = 3$, then $C' = C - D_2$ is a circuit with $\varepsilon(C') \leq k \leq \varepsilon(C)$, a contradiction. Hence $|V(D_2)| \geq 5$, and $D_2$ plays the same role as $D_r$ in this subcase since $|V(D_1) \cap V(D_2)| = 1$. In a way similar to the proof of Claim 5, we obtain a contradiction which completes the proof of Claim 6. □
Thus, \( H_{1462} \) is a complete graph. Hence by (4.10), we obtain the following:

In order to derive contradictions, we first prove the following two claims.

\textbf{Proof of Claim 7.} Suppose otherwise that there exists a pair of \( \{i, j\} \) such that \(|V(D_i) \cap V(D_j)| = 2\), then by Claim 6 there exists a triangle \( C_3 \) in \( D_i \cup D_j \). Thus, \( C' = C - C_3 \) is a circuit with \( \varepsilon(C') \leq k \leq \varepsilon(C) \), a contradiction. \( \square \)

By Claims 6 and 7, \( C \cong F \). We claim that \( d_C(u) = d(u) \) for any \( u \in V(C) \). Suppose otherwise that there exists a vertex \( x \in V(C) \) such that \( d_C(x) < d(x) \), then \( C_i = C - D_i \) (\( x \notin V(D_i) \) unless \( x = y \)) is a circuit with \( \varepsilon(C_i) \leq k \leq \varepsilon(C) \), a contradiction. This settles our claim. Since \( G \) is connected, \( V(G) = V(C) \) and \( G \cong C \).

\textbf{Case 3:} \( \text{dia}(H) = 1 \) and \(|V(D_i) \cap V(D_j)| = 2\).

Then again \( H \) is a complete graph.

Let \( V(D_1) \cap V(D_r) = \{u, v\} \). Hence there exist four paths \( P_1, P_2, P_3, P_4 \) such that \( D_1 = P_1 \cup P_2, D_r = P_3 \cup P_4 \), and \( V(P_i) \cap V(P_j) = \{u, v\} \) for \( \{i, j\} \subseteq \{1, 2, 3, 4\} \).

In order to derive contradictions, we first prove the following two claims.

\textbf{Proof of Claim 8.} Suppose otherwise that \(|(V(P_s) \cup V(P_t)) \cap V(D_i)| = 2\) for any pair of \( \{s, t\} \subseteq \{1, 2, 3, 4\} \) and \( i \in \{2, 3, \ldots, r - 1\} \).

\textbf{Proof of Claim 9.} Suppose otherwise that there exists a \( D_i \), say \( D_2 \), such that \(|\{u, v\} \cap V(D_2)| \leq 1\).

It follows from Claim 9 that there exist \( 2r = d_C(u) = d_C(v) \) edge-disjoint paths \( P_1, P_2, \ldots, P_{2r} \) such that \( C = \bigcup_{i=1}^{2r} P_i \) and \( V(P_i) \cap V(P_j) = \{u, v\} \) (\( i \neq j \)).

Hence by (4.10), we obtain the following:

\[ |V(P_i)| \leq 5 \quad \text{for} \quad i \in \{1, 2, \ldots, 2r\}. \quad (4.29) \]

Now it is easy to verify that

\[ |E_2(D_1)| + |E_2(D_r)| \leq 4 + \min\{4, \{i \in \{1, 2, \ldots, 2r\} : |V(P_i)| = 5\}\}. \quad (4.30) \]

By (4.5), there exist two paths \( P_1' \) and \( P_2' \) of length 3 in \( D_1 \) and \( D_r \), respectively such that each path contains exactly one vertex in \( V_1(D_1) \cup V_1(D_r) \). Let \( u, v \) denote two such vertices in \( P_1' \) and \( P_2' \), respectively, i.e., \( u \in V(P_1') \cap V_1(D_1) \) and \( v \in V(P_2') \cap V_1(D_r) \). Then by (4.7), \( d_C(P_1') + d_C(P_2') \geq n + 14 - |E_2(D_1)| - |E_2(D_r)| \). Hence

\[ d_C(u) + d_C(v) > n + 2 - |E_2(D_1)| - |E_2(D_r)|. \quad (4.31) \]
If there exists a pair \( \{i, j\} \) such that \(|V(P_i)| \leq 3\) and \(|V(P_j)| \leq 4\), then \( C' = C - (P_i \cup P_j) \) is a circuit with \( \omega(C') \leq k \leq \pi(C') \), which contradicts Lemma 9. Hence we obtain
\[
\{(i, j) : |V(P_i)| \leq 3 \quad \text{and} \quad |V(P_j)| \leq 4\} = \emptyset.
\] (4.32)

Without loss of generality, we can assume that \(|V(P_1)|, |V(P_2)|, \ldots, |V(P_{2r})|\) is an increasing sequence and \( D_i = P_{2i-1} \cup P_{2i} \) for \( i \in \{1, 2, \ldots, r\} \).

We need one more claim.

**Claim 10.** If \(|V(P_1)| = 2\), i.e., \( uv \in E(C) \), then \( r \geq 4 \).

**Proof of Claim 10.** Suppose otherwise that \( r = 2 \) or \( 3 \). By (4.29) and (4.32), \(|V(P_i)| = 5\) for \( i \in \{2, 3, \ldots, 2r\} \). Since \( n \geq 146\), there exists a vertex \( x \) of \( C \) with \( d_C(x) \leq d(x) - 1 \). Hence there exists a circuit \( C' \) such that \( \omega(C') \leq k \leq \pi(C') \), where \( C' = C - (P_i \cup P_j) \) (\( P_i \neq P_j \)), and \( x \notin V(P_i) \) unless \( x \in \{u, v\} \), which contradicts Lemma 9. This completes the proof of Claim 10. □

Finally, we will obtain some inequalities which contradict (4.30).

If \( 3 \leq |V(P_i)| \leq 5 \) then, using (4.32), we obtain
\[
d_C(u) + d_C(v) \leq n - 2 - |\{i \in \{1, 2, \ldots, 2r\} : |V(P_i)| = 5\}|.
\] (4.33)

Combining (4.31) and (4.33) yields an inequality that contradicts (4.30).

If \(|V(P_1)| = 2\), i.e., \( uv \in E(C) \), then using (4.29) and (4.32), we obtain that \(|V(P_i)| = 5\) for \( i \geq 2 \), and
\[
d_C(u) + d_C(v) \leq n - 2 - |\{i \in \{1, 2, \ldots, 2r\} : |V(P_i)| = 5\}|.
\] (4.34)

Using (4.31), (4.32), (4.34) and Claim 10, we obtain
\[
|E_2(D_1)| + |E_2(D_r)| \geq 2 + |\{i \in \{1, 2, \ldots, 2r\} : |V(P_i)| = 5\}| = 2 + (2r - 1) \geq 9,
\]
which contradicts (4.30).

This completes the proof of the first conclusion of the theorem. We now prove that the results in Theorem 6 are best possible in the sense that the condition \( \rho_4(G) > (n + 10)/2 \) cannot be relaxed, even under the condition that \( L(G) \) is hamiltonian. To see this, we construct a graph \( G_0 \) as follows:

Let \( s = (n - 2)/2 \) (\( n \equiv 2 \) (mod 4)) and let \( V(G_0) = \{u_1, v_1, u_2, v_2, \ldots, u_s, v_s, x, y\} \) and \( E(G_0) = \bigcup_{i=1}^{s} \{xu_i, u_iv_i, v_iy\} \). Clearly \( G_0 \) is a graph with \( \rho_4(G_0) = s + 6 = (n + 10)/2 \). Theorem 8 implies that \( L(G_0) \) is hamiltonian and \( 3s - 1 \in [3, \omega(G_0) \backslash \lambda(L(G_0))] \), which implies that \( L(G_0) \) is not (sub)pancyclic. This completes the proof of Theorem 6. □

**References**