On the 2-factor index of a graph

Liming Xionga,1, MingChu Lib

1Department of Mathematics, Beijing Institute of Technology, Beijing 100081, PR China
2School of Software, Dalian University of Technology, Dalian 116024, PR China

Received 11 November 2004; received in revised form 27 September 2006; accepted 9 November 2006
Available online 24 January 2007

Abstract

The 2-factor index of a graph $G$, denoted by $f(G)$, is the smallest integer $m$ such that the $m$-iterated line graph $L^m(G)$ of $G$ contains a 2-factor. In this paper, we provide a formula for $f(G)$, and point out that there is a polynomial time algorithm to determine $f(G)$.

© 2007 Elsevier B.V. All rights reserved.

MSC: 05C45; 05C38
Keywords: 2-factor; 2-factor index; Branch-bond; Iterated line graph

1. Introduction

We use [1] for terminology and notation not defined here and consider only loopless finite graphs. Let $G$ be a graph. For each integer $0 \leq i \leq \Delta(G)$, let $V_i(G)$ denote the set of vertices of $G$ having degree $i$. A branch in $G$ is a nontrivial path with end vertices that do not lie in $V_2(G)$ and with internal vertices of degree 2 (if existing). If a branch has length 1, then it has no internal vertices of degree 2. Let $B(G)$ denote the set of branches of $G$ and $B_1(G)$ the subset of $B(G)$ in which every branch has exactly one end vertex in $V_1(G)$. A 2-factor in $G$ is a spanning subgraph of $G$ such that its vertices have degree 2. For any two subgraphs $H_1$ and $H_2$ of $G$, the distance $d_G(H_1, H_2)$ between $H_1$ and $H_2$ is defined to be $\min\{d_G(v_1, v_2) | v_1 \in V(H_1) \text{ and } v_2 \in V(H_2)\}$.

The line graph of $G = (V(G), E(G))$ has $E(G)$ as its vertex set, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges are incident with a common vertex in $G$. The $m$-iterated line graph $L^m(G)$ is defined recursively by $L^0(G) = G$ and $L^m(G) = L(L^{m-1}(G))$. The hamiltonian index of a graph $G$, denoted by $h(G)$, is the smallest integer $m$ such that $L^m(G)$ is hamiltonian, and the 2-factor index of a graph, denoted by $f(G)$, is the minimum integer $m$ such that the $m$-iterated line graph contains a 2-factor.

Chartrand [2] showed that if a connected graph $G$ is not a path, then the hamiltonian index of $G$ exists. Lai [7] obtained a bound of $h(G)$. Because a hamiltonian cycle of $G$ is a connected 2-factor of $G$, $f(G)$ exists for any connected graph.
that is not a path. A circuit of a graph $G$ is a connected nontrivial subgraph of $G$ whose vertices have only even degrees. Harary and Nash-Williams characterized these graphs whose line graphs are hamiltonian.

**Theorem 1** (Harary and Nash-Williams [6]). Let $G$ be a graph with at least three edges. Then $h(G) \leq 1$ if and only if $G \cong K_{1,n}$, or $G$ has a circuit $H$ such that $d_G(e, H) = 0$ for any edge $e \in E(G)$.

Gould and Hynds gave a characterization of graphs whose line graphs contain a 2-factor. A star is the bipartite graph $K_{1,m}$ ($m \geq 3$), and the vertex of degree $m$ in $K_{1,m}$ is called the center of the star. A k-system that dominates is a collection $\Gamma$ of $k$ edge-disjoint circuits and stars in $G$ such that each edge $e$ of $G$ is either in one of the circuits or stars of $\Gamma$, or $e$ is adjacent to the center of a star of $\Gamma$.

**Theorem 2** (Gould and Hynds [5]). Let $G$ be a connected simple graph containing at least three edges. Then $f(G) \leq 1$ if and only if $G$ has a k-system that dominates for some $k$.

Xiong and Liu characterized the graphs for which the $n$-iterated line graph is hamiltonian, for any integer $n \geq 2$.

**Theorem 3** (Xiong and Liu [11]). Let $G$ be a connected graph that is not a 2-cycle and let $n \geq 2$ be an integer. Then $h(G) \leq n$ if and only if $E_{U_n}(G) \neq \emptyset$ where $E_{U_n}(G)$ denotes the set of those subgraphs $H$ of $G$ which satisfy the following conditions:

(i) any vertex of $H$ has even degree in $H$;
(ii) $V_0(H) \subseteq \bigcup_{i=3}^{L(G)/V_1(G)} \subseteq V(H)$;
(iii) $d_G(H_1, H - H_1) \leq n - 1$ for any subgraph $H_1$ of $H$;
(iv) $|E(b)| \leq n + 1$ for any branch $b$ in $B(G) \setminus B_H(G)$;
(v) $|E(b)| \leq n$ for any branch in $B_1(G)$.

Very recently, Ferrara and Gould proved the following result.

**Theorem 4** (Ferrara and Gould [3]). Let $G$ be a connected graph with at least three edges. Then for any $n \geq 2$, $L^n(G)$ has a 2-factor if and only if $F_n(G) \neq \emptyset$ where $F_n(G)$ denotes the set of those subgraphs $H$ of $G$ that satisfy the following five conditions:

(i') any vertex of $H$ has even degree in $H$;
(ii') $V_0(H) \subseteq \bigcup_{i=3}^{L(G)/V_1(G)} \subseteq V(H)$;
(iii') $d_G(H_1, H - H_1) \leq n + 1$ for any subgraph $H_1$ of $H$;
(iv') $|E(b)| \leq n + 1$ for any branch $b$ in $B(G) \setminus B_H(G)$;
(v') $|E(b)| \leq n$ for any branch in $B_1(G)$.

We observe that Theorem 4 does not hold for $n = 0$ or 1. To see this, let $C = u_1u_2 \cdots u_{3t} \cdots u_t$ be a cycle of length $t$, $t \geq 3s \geq 6$, and $x$ be a vertex outside $C$. Now let $G_1$ be the graph with $V(G_1) = V(C) \cup \{x\}$ and $E(G_1) = E(C) \cup \{xu_1, xu_2, xu_3\}$. It is easy to see that $C \cup \{x\} \not\subset F_0(G_1)$ but $G_1$ has no 2-factor. To see that Theorem 4 does not hold for $n = 1$, let $G_2$ be the unique tree on $2n$ vertices with degree sequence $(x_1, x_2, \ldots, x_{n+1}, x_{n+2}, \ldots, x_{2n})$ where $x_i = 1$ for $i = 1, 2, \ldots, n$ and $x_i = 1$ for $i = n + 1, \ldots, 2n$. It is easy to see that $G_2$ has no k-system that dominates for any $k$ and the empty subgraph with the set of vertices of degree three in $G_2$ is in $F_1(G_2)$. This implies that $f(G_2) \geq 2$ and $F_1(G_2) \neq \emptyset$.

Note that the conditions on the subgraphs in $E_{U_k}(G)$ of Theorem 3 and the subgraphs in $F_k(G)$ of Theorem 4 are the same except conditions (iii) and (iii'). The following natural result follows from the fact that all subgraphs $F$ in $F_{f(G)+2}(G)$ are in $E_{h(G)}(G)$ and all subgraphs $H$ in $E_{h(G)}(G)$ are in $F_{f(G)}(G)$.

**Theorem 5.** Let $G$ be a connected graph that is not a path. Then

$$h(G) - 2 \leq f(G) \leq h(G).$$
We will give a formula for \( f(G) \) \( \text{(Theorem 7)} \). Let \( G \) be a connected graph with at least three edges. Then for any \( f(G) \leq h(G) \). If \( h(G) = 0, 1, 2, \) then obviously \( f(G) \geq h(G) - 2 \). If \( h(G) \geq 3 \), then \( h(G) \leq f(G) + 2 \) by Theorem 3 and since subgraphs \( F \) in \( F_{f(G)+2}(G) \) are all in \( EU_{h(G)}(G) \). \( \Box \)

Observing that conditions (ii') and (iv') in the definition of \( F_k(G) \) imply condition (iii') in the definition of \( F_k(G) \), we obtain an equivalent version of Theorem 4 as follows.

**Theorem 6.** Let \( G \) be a connected graph with at least three edges. Then for any \( n \geq 2, L^n(G) \) has a 2-factor if and only if \( F_h(G) \neq \emptyset \) where \( F_h(G) \) denotes the set of those subgraphs \( H \) of \( G \) that satisfy the following four conditions:

1. any vertex of \( H \) has even degree in \( H \);
2. \( V_0(H) \subseteq \bigcup_{i=3}^h V_i(G) \subseteq V(H) \);
3. \( |E(b)| \leq n + 1 \) for any branch \( b \) in \( B(G) \setminus B_H(G) \);
4. \( |E(P)| \leq n \) for any branch in \( B_1(G) \).

**Proof.** Since the “only if” part is trivial, we only need to prove the “if” part of the theorem. It suffices to prove that the subgraph \( H \) satisfying the conditions (I)–(IV) also satisfies the conditions (i')–(v'). We will prove this by contradiction. If possible, suppose that \( H \) is a subgraph satisfying (I)–(IV) but \( d_G(H_1, H - H_1) \geq n + 2 \) for some subgraph \( H_1 \) of \( H \), we claim that the shortest path \( P \) between \( H_1 \) and \( H - H_1 \) is a branch in \( B(G) \setminus B_H(G) \), by (ii'). Hence by (iv'), \( |E(P)| \leq n + 1 \), a contradiction. This implies that (iii') holds for \( H \). Thus we have completed the proof of Theorem 6. \( \Box \)

The main purpose of this paper is to establish a formula for \( f(G) \).

2. **Branch-bonds**

In this section, we will introduce some notation and terminology about branch-bonds [10], which will be used in next section. For any subset \( S \) of \( B(G) \), \( G - S \) denotes the subgraph obtained from \( G[E(G) \setminus E(S)] \) by deleting all internal vertices of degree 2 in any branch of \( S \). A subset \( S \) of \( B(G) \) is called a branch cut if \( G - S \) has more components than \( G \). A branch-bond is a minimal branch cut. If \( G \) is connected, then a branch cut \( S \) of \( G \) is a minimal subset of \( B(G) \) such that \( G - S \) is disconnected. It is easily shown that, for a connected graph \( G \), a subset \( S \) of \( B(G) \) is a branch-bond if and only if \( G - S \) has exactly two components. We denote by \( BB(G) \) the set of branch-bonds of \( G \). Given \( S, T \subseteq V(G) \), let \( [S, T] = \{uv \in E(G): u \in S \text{ and } v \in T\} \). An edge cut is an edge set of the form \([S, S]\), where \( S \) is a nonempty proper subset of \( V(G) \) and \( S = V(G) \setminus S \). A minimal edge cut of \( G \) is called a bond. Note that a branch-bond of \( G \) is also a bond of \( G \) when every branch in the branch-bond is an edge.

McKee gave the following characterization of eulerian graphs.

**Theorem 7 (McKee [8]).** A connected graph is eulerian if and only if each bond contains an even number of edges.

The following characterization of eulerian graphs involves branch-bonds.

**Theorem 8 (Xiong et al. [10]).** A connected graph is eulerian if and only if each branch-bond contains an even number of branches.

3. **A formula for \( f(G) \)**

In this section we will establish a formula for \( f(G) \), which relates to the concept of odd branch-bonds. A branch-bond is called odd if it consists of an odd number of branches. The length of a branch-bond \( S \in BB(G) \), denoted by \( l(S) \), is the length of a shortest branch in it. Let \( BB_2(G) = \{S \in BB(G) \setminus BB_1(G) : S \text{ is odd}\} \) where \( BB_1(G) = B_1(G) \), and, for \( i = 1, 2, \)

\[
h_i(G) = \begin{cases} 
\max\{l(S) : S \in BB_i(G)\} & \text{if } BB_i(G) \neq \emptyset, \\
0 & \text{if } BB_i(G) = \emptyset.
\end{cases}
\]

We will give a formula for \( f(G) \) involving \( h_i(G) \). First we present a lower bound for it.
Theorem 9. Let $G$ be a connected graph that is not a path. Then
\[ f(G) \geq \max\{h_1(G), h_2(G) - 1\}. \]

Proof. If $f(G) = 0$, then the definition of a 2-factor implies that $h_1(G) = 0$, i.e., $BB_1(G) = \emptyset$. Obviously $l(S) \leq 1$ for any branch-bond $S$ with $|S| = 1$.

We further claim that $h_2(G) \leq 1$, which implies that Theorem 9 holds. We will prove this by contradiction. If possible, suppose that $h_2(G) > 2$, then there exists an odd branch-bond $S_0$ with $|S_0| \geq 3$ and $l(S_0) \geq 2$. Let $F$ be a 2-factor of $G$. By the definition of a branch-bond, each cycle of $F$ contains an even number of branches of $S_0$. Hence there exists a branch $b_0$ in the odd branch-bond $S_0$ such that $b_0$ is not in any cycle of $F$. However $|E(b_0)| \geq l(S_0) \geq 2$ implies that there exists a vertex $u$, of degree 2, such that $u$ is in $b_0$ but $u$ is not in any cycle of $F$, a contradiction. This settles the case that $f(G) = 0$.

If $f(G) = 1$, then, by Theorem 2, there exists a $k$-system $\Gamma$ that dominates. Obviously $h_1(G) \leq 1$ and $l(S) \leq 2$ for any branch-bond $S \notin BB_1(G)$ with $|S| = 1$. We furthermore claim that $h_2(G) \leq 2$, which implies that Theorem 9 holds. We will prove this by contradiction. If possible, suppose that $h_2(G) > 3$, then there exists an odd branch-bond $S_0$ with $|S_0| \geq 3$ and $l(S_0) \geq 3$. By the definition of a branch-bond, any circuit of $\Gamma$ contains an even number of branches of $S_0$. Hence there exists a branch $b_0$ in the odd branch-bond $S_0$ such that $b_0$ is not in any circuit of $\Gamma$. However, $|E(b_0)| \geq l(S_0) \geq 3$ implies that there is an edge $uv$, with $d(u) = d(v) = 2$, such that $u$ and $v$ in $b_0$ but $uv$ is neither in one of stars of $\Gamma$ nor has a vertex in one of the circuits of $\Gamma$, a contradiction. This settles the case that $f(G) = 1$.

It remains to consider the case that $f(G) \geq 2$. We can take an $S_i \in BB_i(G)$ such that $h_i(G) = l(S_i)$ for every $i \in \{1, 2\}$. For any subgraph $H \in F_f(G)$, it is obvious that $E(b) \cap E(H) = \emptyset$ for any $b \in S_1$. The definitions of $S_2$ and $H$ imply that there exists at least one branch $b \in S_2$ such that $E(b) \cap E(H) = \emptyset$. Hence by Theorem 6, we obtain $f(G) \geq h_1(G)$ by (IV) and $f(G) \geq h_2(G) - 1$ by (III). So $f(G) \geq \max\{h_1(G), h_2(G) - 1\}$, which completes the proof of Theorem 9. \qed

Now we state a formula for $f(G)$. Let
\[ \beta(G) = \max\{h_1(G), h_2(G) - 1\}. \]

Theorem 10. Let $G$ be a connected graph that is not a path such that $\beta(G) \geq 2$. Then $f(G) = \beta(G)$.

Proof. It suffices to prove that $f(G) \leq \beta(G)$ by Theorem 9. This theorem also implies $f(G) \geq \beta(G) \geq 2$. Hence by Theorem 6 we can assume that $H \in F_f(G)$ is a subgraph with a maximal number of branches $b \in B_H(G)$ such that $|E(b)| \geq \beta(G) + 2$. Then we obtain the following fact.

Claim 1. If $S$ is a branch-bond in $BB(G)$ which contains at least three branches, then $|E(b)| \leq \beta(G) + 1$ for any branch $b \in S \setminus B_H(G)$.

Proof of Claim 1. We will prove this by contradiction. If possible, suppose that there is a branch-bond $S$ with $|S| \geq 3$ and $b_0 \in S \setminus B_H(G)$ such that $|E(b_0)| \geq \beta(G) + 2$. Obviously $b_0$ is not a cycle. Let $u$ and $v$ be two end vertices of $b_0$. Let $S(u, b_0)$ be a branch-bond containing $b_0$ such that any branch of $S(u, b_0)$ has $u$ as an end vertex. Obviously $|S(u, b_0)| \geq 2$.

By the following algorithm, we will first find a cycle of $G$ that contains $b_0$ and then obtain a contradiction.

**Algorithm $b_0$.**

1. If $|S(u, b_0)|$ is even, then select a branch $b_1 \in S(u, b_0) \setminus (B_H(G) \cup \{b_0\})$ by Theorem 8. Otherwise, since $|E(b_0)| \geq \beta(G) + 2$, select a branch $b_1 \in S(u, b_0)$ with
\[ |E(b_1)| = l(S(u, b_0)) \leq h_2(G) \leq \beta(G) + 1 \]
(Obviously $b_1 \neq b_0$) and let $u_1(\neq u)$ be the other end vertex of $b_1$. If $u_1 = v$, then set $t := 1$ and stop. Otherwise $i := 1$. 

2. Select a branch-bond $S(u, u_i, b_0)$ in $G$ which contains $b_0$ but not $b_1, b_2, \ldots, b_i$, such that any branch in $S(u, u_i, b_0)$ has exactly one end vertex in $\{u, u_1, u_2, \ldots, u_i\}$. If $|S(u, u_i, b_0)|$ is even, then, by Theorem 8, select a branch $b_{i+1} \in S(u, u_i, b_0) \setminus (B_H(G) \cup \{b_0\})$.

Otherwise, since $|E(b_0)| \geq \beta(G) + 2$, select a branch $b_{i+1} \in S(u, u_i, b_0)$ such that

$$|E(b_{i+1})| = I(S(u, u_i, b_0)) \leq h_2(G) \leq \beta(G) + 1$$

(Obviously $b_{i+1} \neq b_0$), and let $u_{i+1}$ be the end-vertex of $b_{i+1}$ that is not in $\{u, u_1, u_2, \ldots, u_i\}$.

3. If $u_{i+1} = v$, then set $t := i + 1$ and stop. Otherwise replace $i$ by $i + 1$ and return to step 2.

Note that $|B(G)|$ is finite, and $d_G(v) \geq 2$ implies that the Algorithm $b_0$ will stop after a finite number of steps. It is easy to see that $G[\bigcup_{i=0}^{I-1} E(b_i)]$ is connected. Furthermore, since $u_i = v$ and $|S(u, u_i, b_0)| \geq 2$, $G[\bigcup_{i=0}^{I-1} E(b_i)]$ has a cycle of $G$ which contains $b_0$. Hence we have established the following fact.

**Claim 1.1.** $b_0$ is in a cycle $C_0$ of $G[\bigcup_{i=0}^{I-1} E(b_i)]$.

Let $H'$ be the subgraph of $G$ obtained from

$$G[(E(H) \cup (E(C_0) \setminus E(H))) \setminus (E(H) \cap E(C_0))]$$

by adding the remaining vertices of $\bigcup_{i=3}^{d_G(G)} V_i(G)$ as isolated vertices in $H'$.

Obviously $|E(b)| \leq h_2(G) \leq \beta(G) + 1$ for $b \in B_H(G) \cap \{b_1, b_2, \ldots, b_i\}$. Hence, by Claim 1.1, $H'$ satisfies (III). Obviously $H'$ satisfies (I), (II) and (IV), and this implies that $H'$ is also in $F_f(G)(G)$. But $H'$ contains $b_0$ which contradicts the maximality of $H$. Thus Claim 1 is true.

Now we will complete the proof of Theorem 10. By the definition of $\beta(G)$, $|E(b)| \leq h_1(G) \leq \beta(G)$ for any branch $b \in B_1(G)$ and $|E(b)| \leq h_2(G) \leq \beta(G) + 1$ for the branch $b$ in a branch-bond $S \not\in BB_1(G)$ such that $|S| = 1$. The last fact and Claim 1 implies that $|E(b)| \leq \beta(G) + 1$ for any branch $b \in B(G) \setminus B_H(G)$. It follows that $H \in F_{\beta(G)}(G)$, and so $f(G) \leq \beta(G)$. Therefore we have completed the proof of Theorem 10. □

**Remark 11.** Note that Theorem 10 does not hold for a graph $G$ with $\beta(G) < 1$. To see this, let $G_0$ be the graph depicted in Fig. 1. It is easy to see that $h_1(G_0) = 0$ and $h_2(G_0) = 2$, hence $\beta(G_0) = 1$. By Theorem 12, $f(G_0) \leq 2$. We claim that $f(G_0) = 2$. To see this, it suffices to show that $G_0$ has no $k$-system that dominates for any $k$. We will prove this by contradiction. If possible, suppose that $G_0$ has a $k$-system that dominates. It is easy to see that the unique cycle with all branches of length 4 of $G_0$ should be contained in $\Gamma$. Hence none of the vertices $u_i$ is a center of some star since $u_i$.

![Fig. 1. A graph $G_0$ with $f(G_0) = 2$ and $\beta(G_0) = 1$.](image-url)
The following result deals with these graphs $G$ with small $\beta(G)$.

**Theorem 12.** Let $G$ be a graph that is not a path such that $\beta(G) \leq 1$. Then $f(G) \leq 2$.

**Proof.** By Theorem 6, we only need to prove that $F_2(G) \neq \emptyset$. Let $H$ be a subgraph of $G$ with (I) and (II) and with a maximal number of branches $b \in B_H(G)$ such that $|E(b)| \geq 3$. Then, in a way similar to the one in Claim 1 in the proof of Theorem 10, we obtain the following claim.

**Claim 12.1.** If $S$ is a branch-bond in $BB(G)$ which contains at least three branches, then $|E(b)| \leq 2$ for any branch $b \in S \setminus B_H(G)$.

For any branch $b$ of $G$, if $G[E(b)]$ is not a cycle of $G$ then there exists a branch-bond $S \in BB(G)$ with $b \in S$. By $\beta(G) \leq 1$, we have $|E(b)| \leq 1$ for $b \in B_1(G)$, which implies that $H$ satisfies (IV). By Claim 12.1, $H$ satisfies (III). Hence $H \in F_2(G)$, and so $f(G) \leq 2$. Thus we have completed the proof of Theorem 12. □

A result in [4] implies the following.

**Theorem 13 (Fujisawa et al. [4]).** Let $G$ be a graph that is not a path such that $\beta(G) = 0$. Then $f(G) \leq 1$. It would be interesting to consider the following question.

**Question 14.** Which graph $G$ satisfies $f(G) = \beta(G) \leq 1$.

**Remark 15.** Note that the graph $G_0$ shown in Remark 11 is 2-connected and $F_1(G_0) \neq \emptyset$ since $C_0 \cup \{x_1, x_2, x_3, x_4, w\}$ is a subgraph in $F_1(G_0)$ where $C_0$ is the unique cycle with all branches of length 4. However $f(G_0) = 2$, this shows that Theorem 6 does not hold for $n = 1$ even for a 2-connected graph.

**Remark 16.** Woeginger [9] pointed out that there is a polynomial algorithm to determine $h_i(G)$ of $G$. Hence there is a polynomial algorithm to determine $\beta(G)$. So if $\beta(G) \geq 2$ then there is a polynomial algorithm to determine $f(G)$ by Theorem 10.

**Acknowledgement**

The authors would like to thank Ronald Gould for providing his papers and are grateful to the referees for their valuable comments and careful reading.

**References**