The hamiltonian index of a 2-connected graph

Liming Xiong\textsuperscript{a,b,*}, Qiuxin Wu\textsuperscript{c}

\textsuperscript{a} Department of Mathematics, Beijing Institute of Technology, Beijing 100081, PR China
\textsuperscript{b} Department of Mathematics, Jiangxi Normal University, Nanchang, 330027, PR China
\textsuperscript{c} College of Science, Beijing Institute of Machinery, Beijing 100085, PR China

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Abstract

Let $G$ be a graph. Then the hamiltonian index $h(G)$ of $G$ is the smallest number of iterations of line graph operator that yield a hamiltonian graph. In this paper we show that $h(G) \leq \max\{1, \frac{|V(G)| - \Delta(G)}{3}\}$ for every 2-connected simple graph $G$ that is not isomorphic to the graph obtained from a dipole with three parallel edges by replacing every edge by a path of length $l \geq 3$. We also show that $\max\{h(G), h(G)\} \leq \frac{|V(G)| - 3}{6}$ for any two 2-connected nonhamiltonian graphs $G$ and $G'$ with at least 74 vertices. The upper bounds are all sharp.

Keywords: Hamiltonian index; Maximum degree; Complement graph; Connectivity

1. Introduction

We use [1] for terminology and notation not defined here and consider only simple finite graphs. Throughout the paper we use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degree of a graph $G$, respectively.

The line graph of $G = (V(G), E(G))$ has $E(G)$ as its vertex set, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges share a common endvertex in $G$. The $m$-iterated line graph $L^m(G)$ is defined recursively by $L^0(G) = G$, $L^1(G) = L(G)$ and $L^m(G) = L(L^{m-1}(G))$. The hamiltonian index of a graph $G$, denoted by $h(G)$, is the smallest integer $m$ such that $L^m(G)$ contains a hamiltonian cycle.

Chartrand [3] showed that the hamiltonian index of $G$ always exists for a connected graph $G$ that is not a path. There have already appeared many results on $h(G)$ in the literature (see [2,4,7–9,11,13,14]). A formula for determining $h(G)$ given in [12] shows that we only need to consider blocks and nontrivial paths whose edges are cut-edges when we want to determine the hamiltonian index of a graph. Note that every block is 2-connected. This motivates us to consider the hamiltonian index of 2-connected graphs.

Saražin gave an upper bound on $h(G)$ as follows.

\textbf{Theorem 1} (Saražin [8]). Let $G$ be a connected simple graph on $n$ vertices that is not a path. Then $h(G) \leq n - \Delta(G)$. 

* Corresponding author at: Department of Mathematics, Beijing Institute of Technology, Beijing 100081, PR China.
E-mail addresses: lmxiong@bit.edu.cn (L. Xiong), qxwu@263.net (Q. Wu).

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For 2-connected simple graphs, we improve Theorem 1 as follows.

**Theorem 2.** Let $G$ be a 2-connected simple graph of order $n$. If $\Delta(G) \leq n - 3$, then

$$h(G) \leq \frac{n - \Delta(G)}{3}$$

unless $G$ is isomorphic to the graph $G_1$ obtained from a dipole with three parallel edges by replacing every edge by a path of length $l = \frac{|V(G)|+1}{3} \geq 3$.

Xiong gave a relation of the hamiltonian index of a graph $G$ and its complement graph $\overline{G}$.

**Theorem 3** (Xiong [11]). Let $G$ and its complement $\overline{G}$ be connected graphs of order $n \geq 61$ that are not paths. Then either $L(G)$ or $L(\overline{G})$ is hamiltonian, and if neither $G$ nor $\overline{G}$ is hamiltonian, then

$$\max\{h(G), h(\overline{G})\} \leq \frac{n-1}{2}$$

and the above equality holds if and only if either $G$ or $\overline{G}$ is isomorphic to the graph of order $n = 2t - 1$ obtained by identifying one endvertex of a path of length $t - 1$ with exactly one vertex of a complete graph of order $t$.

In this paper we consider the same problem for 2-connected graphs. We improve the above theorem as follows.

**Theorem 4.** Let $G$ and $\overline{G}$ be 2-connected simple graphs of order $n \geq 74$. If neither $G$ nor $\overline{G}$ is hamiltonian, then

$$\max\{h(G), h(\overline{G})\} \leq \frac{n-3}{6}.$$ 

In Section 2, we will give some auxiliary results which are applied in Sections 3 and 4 to prove our main results. The sharpness of Theorems 2 and 4 is presented in the last section.

2. Preliminaries

Let $G$ be a graph. A subgraph of $G$ is called eulerian if it is connected and every vertex has even degree. For any two subgraphs $H_1$ and $H_2$ of $G$, define the distance $d_G(H_1, H_2)$ between $H_1$ and $H_2$ to be the minimum of the distances $d_G(v_1, v_2)$ over all pairs with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$. If $d_G(e, H) = 0$ for an edge $e$ of $G$ then we say that $H$ dominates $e$. A subgraph $H$ of $G$ is called dominating if it dominates all edges of $G$. There is a characterization of graphs $G$ with $h(G) \leq 1$ which involves the existence of a dominating eulerian subgraph in $G$.

**Theorem 5** (Harary and Nash-Williams [6]). Let $G$ be a graph with at least three edges. Then $h(G) \leq 1$ if and only if $G$ has a dominating eulerian subgraph.

A graph is called trivial if it has only one vertex and is called even if every vertex has even degree. For a nonnegative integer $k$, we define $V_k(G)$ by $V_k(G) = \{x \in V(G) : d_G(x) = k\}$, where $d_G(x)$ is the degree of $x$ in $G$. A branch in $G$ is a nontrivial path with endvertices that do not lie in $V_2(G)$ and with inner vertices of degree two (if existing). We denote by $B(G)$ the set of branches of $G$ and by $B_1(G)$ the subset of $B(G)$ in which at least one endvertex has degree one. For any subgraph $H$ of $G$, denote by $B_H(G)$ the set of branches of $G$ whose edges are all in $H$. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of $G$ induced by $S$. In this paper, we also consider the subgraph induced by a set of edges. For $F \subseteq E(G)$, the subgraph $H$ defined by $V(H) = V(F)$ and $E(H) = E(G) - F$ is said to be the subgraph induced by $F$, and is denoted by $G[F]$. When we simply say an “induced subgraph”, it means the subgraph induced by a set of vertices.

The following theorem can be considered as an analogue of Theorem 5 for the $k$-iterated line graph $L^k(G)$ of a graph $G$. 

Theorem 6 (Xiong and Liu [12]). Let $G$ be a connected graph that is not a path and let $k \geq 2$ be an integer. Then $h(G) \leq k$ if and only if $EU_k(G) \neq \emptyset$ where $EU_k(G)$ denotes the set of those subgraphs $H$ of $G$ which satisfy the following five conditions:

(I) $H$ is an even graph;

(II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

(III) $d_G(H_1, H - H_1) \leq k - 1$ for every induced subgraph $H_1$ of $H$ with $\emptyset \neq V(H_1) \subsetneq V(H)$;

(IV) $|E(B)| \leq k + 1$ for every branch $B \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$;

(V) $|E(B)| \leq k$ for every branch $B$ in $\mathcal{B}_1(G)$.

Note that if we only consider 2-connected graphs then Condition (V) in the definition of $EU_k(G)$ is superfluous.

For any subset $S$ of $\mathcal{B}(G)$, we denote by $G - S$ the subgraph obtained from $G$ by deleting all edges and internal vertices of branches of $S$. A subset $S$ of $\mathcal{B}(G)$ is called a branch cut if $G - S$ has more components than $G$. A minimal branch cut is called a branch-bond. Obviously, for a connected graph $G$, a subset $S$ of $\mathcal{B}(G)$ is a branch-bond if and only if $G - S$ has exactly two components. We denote by $\mathcal{BB}(G)$ the set of branch-bonds of $G$. A branch-bond is called odd if it consists of an odd number of branches. The length of a branch-bond $S \in \mathcal{BB}(G)$, denoted by $l(S)$, is the length of a shortest branch in $S$. Let $\mathcal{BB}_1(G) = \mathcal{B}_1(G)$, $\mathcal{BB}_2(G) = \{S \in \mathcal{BB}(G) \setminus \mathcal{BB}_1(G) : |S| = 1\}$, and let $\mathcal{BB}_3(G) = \{S \in \mathcal{BB}(G) : |S| \geq 3$ and $S$ is odd$\}$. For $i \in \{1, 2, 3\}$, define

$$h_i(G) = \begin{cases} \max\{l(S) : S \in \mathcal{BB}_i(G)\} & \text{if } \mathcal{BB}_i(G) \neq \emptyset, \\ 0 & \text{if } \mathcal{BB}_i(G) = \emptyset. \end{cases}$$

The following result is known.

Theorem 7 (Xiong, Broersma, Li and Li, [13]). Let $G$ be a connected graph. Then

$$h(G) \leq \max\{h_1(G), h_2(G) - 1, h_3(G) + 1\},$$

and if $h(G) \geq 1$, then

$$h(G) \geq \max\{h_1(G), h_2(G) - 1, h_3(G) - 1\}.$$

Since $h_i(G) = 0 (i = 1, 2)$ for a 2-connected graph $G$ and $h_3(G) \leq 1$ if $G$ is hamiltonian, Theorem 7 also holds for the graph $G$ with $h(G) = 0$ and one can obtain the following result from the above result.

Theorem 8. Let $G$ be a 2-connected graph that is not a path. Then

$$h_3(G) - 1 \leq h(G) \leq h_3(G) + 1.$$

The following characterization of eulerian graphs involves branch-bonds.

Theorem 9 (Xiong, Broersma, Li and Li, [13]). A connected graph is eulerian if and only if each branch-bond contains an even number of branches.

3. Proof of Theorem 2

In this section we present the proof of Theorem 2. By $S \triangle T$ we denote the symmetric difference $(S \setminus T) \cup (T \setminus S)$. Note that if $H$ is an even graph, then $G[E(H) \triangle E(C)]$ is also an even graph, but $G[E(H) \triangle E(C)]$ may have more components than $H$.

The following well-know result will be used in our proof.

Theorem 10 (Veldman, [10]; Xiong, [11]). Let $G$ be a graph with diameter at most two. Then $L(G)$ is hamiltonian.

Now we present the proof of Theorem 2.
Proof of Theorem 2. Suppose that $G$ is not isomorphic to $G_1$ for any $l \geq 3$. Throughout the proof, we let $u$ be a vertex with maximum degree in $G$. If $d(u) = \Delta(G) = 2$, then Theorem 2 holds trivially. Now assume that $d(u) \geq 3$. If there is no odd branch-bond in $G$, i.e., $h_3(G) = 0$, then $h(G) \leq 1$ by Theorem 8, we are done. So suppose that there is at least one odd branch-bond in $G$. Let $B_0$ be an odd branch-bond with $h_3(G) = \min |E(B)| : B \in B_0$, and let $I(B_0)$ be the set of inner vertices of branches in $B_0$. Then we have the following fact.

Claim 1. There are at least $3(h_3(G) - 1)$ vertices in $I(B_0)$ such that there are at most two vertices which are adjacent to the neighbors of $u$ among the $3(h_3(G) - 1)$ inner vertices.

Proof of Claim 1. It follows from the fact that $G$ is a 2-connected graph that is not isomorphic to $G_1$.

Claim 2. $h_3(G) \leq \frac{n-\Delta(G)}{3} + 1$.

Proof of Claim 2. By the definition of an odd branch-bond, besides $u$ there is at least one that does not belong to $I(B_0) \cup N(u)$. Hence $3(h_3(G) - 1) + (\Delta(G) - 2) + 2 \leq n$ by Claim 1, which completes the proof of Claim 2.

To prove that $h(G) \leq \frac{n-\Delta(G)}{3}$, we distinguish the following three cases.

Case 1. $3 \leq n - \Delta(G) \leq 5$.

We claim that $h(G) \leq 1$, i.e., $L(G)$ is hamiltonian. By $n - \Delta(G) \leq 5$, $|V(G) \setminus (N(u) \cup \{u\})| \leq 4$. Hence the hypothesis that $G$ is 2-connected implies that $d_G(u, x) \leq 3$ for any vertex $x \in V(G)$ and there exist at most two vertices for which the equality is achieved. If $V(G) \setminus (N(u) \cup \{u\}) = \emptyset$, then we are done by Theorem 10. Hence we only need to consider the case that $V(G) \setminus (N(u) \cup \{u\}) \neq \emptyset$.

If there is a vertex $x$ with $d_G(u, x) = 3$, then there is a cycle containing $u$ and $x$ since $G$ is 2-connected. Now we choose a cycle $C_1$ containing $u$ and $x$, such that

1. $C_1$ contains a maximum number of vertices of $V(G) \setminus N(u)$;
2. subject to (1), $C_1$ contains a maximum number of vertices of degree two.

Note that $C_1$ contains at least three vertices in $V(G) \setminus (N(u) \cup \{u\})$. Hence $V(G) \setminus (N(u) \cup V(C_1))$ has at most one element. Now let $H$ be an eulerian subgraph containing $V(C_1)$ with a maximum number of edges. Then we claim that $H$ is a dominating eulerian subgraph of $G$. We will prove this by contradiction. If possible, suppose that there is an edge $st$ such that $s, t \not\in V(H)$. First suppose that $s, t \in N(u)$. Then $G[\{E(H) \cup \{us, ut, st\}\}]$ is an eulerian subgraph containing $V(C_1)$ which has more edges than $H$, a contradiction. Now suppose that exactly one of $s, t$ is not in $N(H)$, say $t \not\in N(u)$. Then either $s$ or $t$ is adjacent to a vertex in $N(u) \setminus \{s\}$ since $G$ is a 2-connected graph that is not isomorphic to $G_2$, say $t$ is adjacent to $v \in N(u) \setminus \{s\}$. Then $G[E(H) \Delta E(stus)]$ is an eulerian subgraph containing $V(C_1)$ which has more edges than $H$, a contradiction. This settles our claim which implies that $L(G)$ is hamiltonian by Theorem 5.

It remains to consider the case that $d_G(u, x) = 2$ for any vertex $x \in V(G) \setminus (N(u) \cup \{u\})$. Note that there are at most four such vertices by the hypothesis that $G$ is a 2-connected graph with $n - \Delta(G) \leq 5$.

Let $H$ be an eulerian subgraph containing $u$, such that

1. $H$ contains a maximum number of vertices of $G$;
2. subject to (1), $H$ contains a maximum number of edges of $G$.

Then we claim that $H$ is a dominating eulerian subgraph of $G$, which implies that $L(G)$ is hamiltonian by Theorem 5. We will prove this by contradiction. If possible, suppose that there is an edge $st \in E(G)$ such that $\{s, t\} \cap V(H) = \emptyset$. First suppose that $s, t$ are both in $N(u)$, then $G[E(H) \cup E(stus)]$ is an eulerian subgraph containing $u$ that has more vertices than $H$, a contradiction. Next suppose that neither $s$ nor $t$ is in $N(u)$. If there are two distinct vertices in $N(u)$, say $s'$ and $t'$, which are adjacent to $s$ and $t$, respectively, then $s'u, t'u \in E(H)$, and either $d_H(u) = 2$ or $d_H(u) \geq 4$ and $G[E(H) \Delta E(st'us's')]$ is disconnected (otherwise $H' = G[E(H) \Delta E(st'us's')]$ is an eulerian subgraph containing $u$ that has more vertices than $H$, a contradiction). Hence since $G$ is 2-connected, there is a path $P$ of $G$ between $G[\{s, t, s', t'\}]$ and $G[N(u) \setminus \{s', t'\}]$ such that $P$ does not contain $u$. Without loss of generality, we assume that $P' = P(t', v)$ is a shortest path of $G$ between $G[\{s, t, s', t'\}]$ and $G[N(u) \setminus \{s', t'\}]$ such that $u \not\in V(P')$, endvertices $t'$ and $v$ in $N(u) \setminus \{s', t'\}$. Note that if $w_1, w_2$ are two vertices in $N(u)$ with $w_1w_2 \in E(H)$, then at least one of $\{uw_1, uw_2\}$ is in $E(H)$ (otherwise $G[E(H) \Delta E(uw_1w_2u)]$ is an eulerian subgraph containing $u$ that has more edges than $H$, a contradiction). Hence since there are at most two vertices in $V(G) \setminus \{u, s, t\}$, $G[E(H) \Delta E(us's't't')] \cup E(P') \cup \{uv\}$ is an eulerian subgraph containing $u$ that has more vertices than $H$, a contradiction. So $s, t$ are adjacent to the same vertex $w$ in $N(u)$. Hence $w \not\in V(H)$ since
otherwise we can add the triangle $stw$ to $H$ to obtain a new eulerian subgraph containing $u$ that has more vertices than $H$, a contradiction. Since $G$ is 2-connected, there is a vertex $w'$ of $V(G) \setminus (N(u) \cup \{u\})$ that is adjacent to one of $\{s, t\}$, say, $w' \in N(t)$. By the hypothesis that $d_G(u, x) = 2$ for any vertex $x \in V(G) \setminus (N(u) \cup \{u\})$, $w'$ is adjacent to $N(u)$, say, $w''' \in E(G)$ and $w'' \in N(u)$. Note that there is at most one vertex in $V(G) \setminus (N(u) \cup \{u, s, t, w'''\})$. Hence $H' = G[E(G) \cup E(\{w'''w'w''u\})]$ is an eulerian subgraph containing $u$ that has more vertices than $H$, a contradiction. Finally suppose that exactly one of $\{s, t\}$ is in $N(u)$, say $s \in N(u)$. If $t$ is adjacent to a vertex $t_1 \in N(u)$, then $G(E[H] \cup E(st_1uus)]$ is an eulerian subgraph containing $u$ that has more vertices than $H$, a contradiction. Hence $t$ is adjacent to a vertex $t_1 \notin N(u) \cup \{u\}$. By the previous arguments, we may assume $t_1 \in V(H)$. Then $G[E(H) \cup E(usttt_1tu)]$ is not connected for any vertex $t_2 \in N(u) \cap N(t_1)$ (otherwise $G[E(H) \cup E(usttt_1tu)]$ is an eulerian subgraph containing $u$ that has more vertices than $H$, a contradiction). Hence for each $t' \notin N(t_1) \cap (N(u) \setminus \{s\})$, we have that $t't'u \in E(H)$, $d_H(t') \geq 4$ and neither of $\{t't', t'u\}$ is in any triangle of $H$. Note that there are at most two vertices in $V(G) \setminus (N(u) \cup \{u, t, t_1\})$. Hence since $d_H(t_1)$ is a positive even integer, there is a vertex $t_2 \in \overline{N(t_1)} \cap N(H)(u)$ such that $N(t_1) \cap N(u) \neq \emptyset$, say $t_3 \in N(t_2) \cap N(u) \neq \emptyset$. Since $t_2u$ is not in any triangle of $H$, there is at least one of $\{t_2t_3, t_3u\}$ that is not in $E(H)$. Hence $G[E(H) \cup E(usttt_1tttt_3u)]$ is an eulerian subgraph containing $u$ that has more vertices than $H$, a contradiction. This settles the case.

**Case 2.** $6 \leq n - \Delta(G) \leq 8$.

We claim that $h(G) \leq 2$. By the hypothesis that $6 \leq n - \Delta(G) \leq 8$ and $G$ is 2-connected, $d_G(u, x) \leq 5$ for any vertex $x \in V(G) \setminus N(u)$ and there is at most one vertex for which the equality is achieved. If $V(G) \setminus (N(u) \cup \{u\}) = \emptyset$, then we are done by Theorem 10. Hence we distinguish the following three subcases.

**Subcase 2.1.** There is a vertex $x$ with $d_G(u, x) = 5$. Then there is a cycle $C$ containing $u$ and $x$ since $G$ is 2-connected. Note that $C$ contains exactly 10 vertices. Hence $C$ is a cycle of $G$ containing all vertices of $V(G) \setminus N(u)$. Let $C'$ be an eulerian subgraph of $G - E(C)$ with a maximum number of edges. Then $C \cup C'$ is a dominating eulerian subgraph of $G$, which implies that $h(G) \leq 1$ by Theorem 5.

**Subcase 2.2.** $d_G(u, x) \leq 4$ for any vertex $x \in V(G)$ and there is at least a vertex $x_0$ with $d_G(u, x_0) = 4$. Since $G$ is 2-connected, there exists a cycle containing $u$ and $x_0$. We choose a cycle $C_1$ containing $u$ and $x_0$, such that

1. $C_1$ contains a maximum number of branches of length four;
2. $C_1$ subject to (1), $C_1$ contains a maximum number of vertices in $V(G)$.

Let $C_2$ be an eulerian subgraph containing $V(C_1)$ with a maximum number of edges of $G$ and $H$ the graph obtained from $C_2$ by adding the remaining vertices of $\bigcup_{i=3}^{2\Delta(G)} V_i(G)$ as isolated vertices in $H$. Then we claim that $H \in EU_2(G)$, which implies that $h(G) \leq 2$ by Theorem 6. It suffices to prove that $H$ satisfies the conditions (I)–(IV) since $G$ is 2-connected. By the choice of $H$, $H$ satisfies (I) and (II). Note that there are at most two vertices in $V(G) \setminus (V(C_1) \cup N(u))$. Hence since any vertex $w$ in $V(H) \setminus V(C_2)$ has degree at least three, if $w$ has no neighbor in $V(C_1)$, then $w$ has at least two neighbors $w_1, w_2$ in $N(u)$ and $w_1, w_2$ are both in $V(C_2)$ (otherwise $C_2 = G[E(C_2) \cup E(w,w_1w_2w)]$ is an eulerian subgraph containing $V(C_1)$ that has more edges than $C_2$, a contradiction). This implies that $H$ satisfies (III). It remains to prove that $H$ satisfies (IV). We will prove this by contradiction. If possible, suppose that there is a branch $B_0$ of length at least four such that $E(B_0) \cap E(H) = \emptyset$. Since $|V(G) \setminus (V(C_1) \cup N(u))| \leq 2$, $B_0$ has length four and $u$ is an endvertex of $B_0$. Let $u'$ be the other endvertex of $B_0$. Then $u' \notin N(u) \cup V(C_1)$. First suppose that $u'$ is in $N(u) \setminus V(C_1)$, then $G[E(C_2) \cup (E(B_0) \cup \{u'u\})]$ is an eulerian subgraph containing $V(C_1)$ that has more edges than $C_2$, a contradiction. Next suppose that $u'$ is in $V(C_1) \setminus \{x_0\}$, then $G[E(C_1) \cup (E(B_0) \cup E(P(u, w)))$, where $P(u, w)$ is the section of $C_1$ from $u$ to $w$ that does not contain $x_0$, is a cycle that has more vertices than $C_1$, which contradicts (2). Finally suppose that $u' = x_0$. Hence since $G$ is not isomorphic to $G_4$, $C_1$ contains at most one branch of $G$ with length four. Then $G[E(C_1) \cup (E(B_0) \cup E(P(u, x_0)))$, where $P(u, x_0)$ is the section of $C_1$ from $u$ to $x_0$ that contains no branch of length four, is a cycle containing more branches of length four than $C_1$, this contradicts (1). This proves that $H$ satisfies (IV). Hence $H \in EU_2(G)$ implies that $h(G) \leq 2$. This settles the subcase.

**Subcase 2.3.** $2 \leq d_G(u, x) \leq 3$ for any vertex $x \in V(G) \setminus (N(u) \cup \{u\})$. Since the empty subgraph of $G$ with the vertex set $\bigcup_{i \geq 3} V_i(G)$ satisfies (I) and (II), we can choose a subgraph $H$ of $G$ with (I) and (II), such that

1. $H$ contains a maximum number of branches of length at least four;
2. $H$ subject to (1), $\max_{\theta \in V(H)} d_H(H_1, H - H_1)$ is minimized;
(3) subject to the above, $H$ contains a minimum number of induced subgraphs $F$ for which $d_G(F, H - F) = \max_{\emptyset \neq V(H) \subseteq V(H)} d_G(H_1, H - H_1)$. 

Then we claim that $H \in EU_2(G)$. We will prove that $H$ satisfies (IV) by contradiction. If possible, suppose that $B_0$ is a branch of length at least four with endvertices $w_1$ and $w_2$, such that $E(B_0) \cap E(H) = \emptyset$. Let $P(u, w_1)$ and $P(u, w_2)$ be two shortest paths from $u$ to $w_1$ and $w_2$, respectively. Then $|E(P(u, w_1))| \leq 3$. Let $H'$ be the subgraph obtained from $H_1 = G[E(H) \triangle E(B_0) \cup E(P(u, w_1)) \cup E(P(u, w_2))]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices in $H'$. Then $H'$ is a subgraph with (I) and (II) that contains more branches of length at least four than $H$, a contradiction which implies that $H$ satisfies (IV).

It remains to prove that $d_G(H_1, H - H_1) \leq 1$ for every induced subgraph $H_1$ of $H$ with $\emptyset \neq V(H_1) \subseteq V(H)$. We will prove this by contradiction. If possible, suppose that there is an induced subgraph $F$ of $H$ with $\emptyset \neq V(F) \subseteq V(H)$, such that $d_G(F, H - F) = \max_{\emptyset \neq V(H) \subseteq V(H)} d_G(H_1, H - H_1) \geq 2$, then every shortest path between $F$ and $H - F$ is a branch of $G$, and there are at least two branches between $F$ and $H - F$ since $G$ is 2-connected. Without loss of generality, suppose $u \in V(F)$. First suppose that $H - F$ is nontrivial. Take any two branches $B_1, B_2$ between $F$ and $H - F$. Let $V(B_i) \cap V(F) = \{u_i\}$ and $V(B_i) \cap V(H - F) = \{v_i\}$. Then $d_G(u_i, u_i) (i \in \{1, 2\})$ and $d_G(v_1, v_2)$ are at most two (otherwise, since $G$ is 2-connected, there is a vertex $w$ such that $d_G(u, w) \geq 4$, a contradiction). Let $P_i = P(u_i, u_i)$ ($P_3 = P(v_1, v_2)$, respectively) be a shortest path between $u_i$ and $u_i$ for $i \in \{1, 2\}$ (between $v_1$ and $v_2$, respectively). Then $|E(P_i)| \leq 2$ for $i \in \{1, 2, 3\}$. Note that if $P_i$ has an inner vertex of degree two which is in $V(H)$, then there is another path of $H$ between the endvertices of $P_i$. Let $H'$ be the subgraph obtained from $G[E(H) \triangle (\bigcup_{i=1}^{3} E(P_i) \cup E(B_i))]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices in $H'$. Then $H'$ is a subgraph with (I) and (II) that has less induced subgraphs $F$ for which $d_G(F, H' - F) = \max_{\emptyset \neq V(H) \subseteq V(H')} d_G(H_1, H' - H_1)$ than $H$, contradicting (3). Now suppose that $H - F$ is trivial, then there are at least three branches between $F$ and $H - F$. Then there are two branches $B_1, B_2$ between $F$ and $H - F$, such that, there is a path $P(u_1, u_2)$ between $u_1 \in V(B_1) \cap V(F)$ and $u_2 \in V(B_2) \cap V(F)$ such that $|E(B) \cap E(P(u_1, u_2))| \leq 2$ for any branch $B \in B(G)$ (otherwise, since $G$ is 2-connected, there is a vertex $w$ such that $d_G(u, w) \geq 4$, a contradiction). Let $H'$ be the subgraph obtained from $G[E(H) \triangle (E(B_1) \cup E(B_2) \cup E(P(u_1, u_2)))]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices in $H'$. Then $H'$ is a subgraph with (I) and (II) that has less induced subgraphs $F$ for which $d_G(F, H' - F) = \max_{\emptyset \neq V(H) \subseteq V(H')} d_G(H_1, H' - H_1)$ than $H$, contradicting (3). This implies that $H$ satisfies (III). Hence $H \in EU_2(G)$. This settles the subcase.

Case 3. $n - \Delta(G) \geq 9$. 

If $h_3(G) \leq \frac{n - \Delta(G)}{3} - 1$, then $h(G) \leq h_3(G) + 1 \leq \frac{n - \Delta(G)}{3}$ by Theorem 8 and we are done. It remains to consider the case that $\frac{n - \Delta(G)}{3} - 1 < h_3(G) \leq \frac{n - \Delta(G)}{3} + 1$ by Claim 2. Hence, $h_3(G) \geq 3$. We will prove that $h(G) \leq h_3(G) - 1$. Let $H$ be a subgraph of $G$ with (I) and (II) such that 

(1) $H$ contains a maximum number of branches of length at least $h_3(G) + 1$; 

(2) subject to (1), $\max_{\emptyset \neq V(H) \subseteq V(H)} d_G(H_1, H - H_1)$ is minimized; 

(3) subject to the above, $H$ contains a minimum number of induced subgraphs $F$ for which $d_G(F, H - F) = \max_{\emptyset \neq V(H) \subseteq V(H)} d_G(H_1, H - H_1)$.

We claim that $H \in EU_{h_3(G) - 1}(G)$. It suffices to prove that $H$ satisfies the conditions (III) and (IV). We have the following fact.

Claim 3. There are at most six vertices in $V(G) \setminus (I(B_0) \cup N(u) \cup \{u\})$.

Proof of Claim 3. We prove Claim 3 by contradiction. If possible, suppose that there are at least seven vertices in $V(G) \setminus (I(B_0) \cup N(u) \cup \{u\})$, then $3(h_3(G) - 1) + (\Delta(G) - 2) + 1 + 7 \leq n$ by Claim 1, i.e., $h_3(G) \leq \frac{n - \Delta(G)}{3} - 1$, a contradiction. This completes the proof of Claim 3.

We will prove that $H$ satisfies (III) by contradiction. If possible, suppose that there is an induced subgraph $F$ of $H$ such that $d_G(F, H - F) = \max_{\emptyset \neq V(H) \subseteq V(H)} d_G(H_1, H - H_1) \geq h_3(G) - 1$, and hence there is a shortest path $P$ between $F$ and $H - F$ such that $|E(P)| \geq h_3(G) - 1$. Then $P$ is a branch of $G$ and $\bigcup_{B \in B_0} E(B)$ is a subset of either of $\{E(F), E(H - F)\}$. Since $G$ is 2-connected, there are at least two branches between $F$ and $H - F$ and hence there is a cycle containing the two branches. We choose a cycle $C'$ containing at least one vertex of $V(G) \setminus (V(F) \cup V(H - F))$ such that the length of a longest branch of $G$ whose edges belong to $E(C') \cap (E(F) \cup E(H - F))$ is minimum. Then $C'$ has at least one branch of $G$ whose inner vertices are in $V(F) \cup V(H - F)$ with length at least
If $h_3(G) - 1$ (otherwise $H' = G[E(H) \triangle E(C')]$ is a subgraph with (I) and (II) that has less induced subgraphs $F$ for which $d_G(F, H' - F) = \max_{\emptyset \neq V(H') \subseteq V(H')} d_G(H_1, H' - H_1)$ than $H$, a contradiction). Hence $B_0$ contains exactly three branches (otherwise, since there are at least $5(h_3(G) - 1)$ inner vertices of branches in $B_0$ such that there are at most four vertices which are adjacent to the neighbors of $u$ among the $5(h_3(G) - 1)$ inner vertices, $5(h_3(G) - 1) + (\Delta(G) - 5) + 2(h_3(G) - 2) + 5 \leq n$, i.e., $h_3(G) \leq \frac{n - \Delta(G)}{3}$ which implies that $h_3(G) \leq \frac{n - \Delta(G)}{3} - 1$ since $h_3(G)$ is an integer, a contradiction). Note that $C'$ contains an even number of branches in $B_0$ by the definition of odd branch-bonds. First suppose that $C'$ contains no branches of $B_0$, then there are at least $2(h_3(G) - 2) + 3 + 2 \geq 7$ vertices in $V(G) \setminus (I(B_0) \cup N(u) \cup \{u\})$, which contradicts Claim 3. Now suppose that $C'$ contains exactly two branches $B_1, B_2$ in $B_0$. Then there are at least one of $B_1, B_2$ with length at least $h_3(G) + 1$ (otherwise, if the inner vertices of both of $B_1, B_2$ belong to $V(F) \cup V(H - F)$, then by Claim 1, $3(h_3(G) - 1) + (\Delta(G) - 2) + 2(h_3(G) - 2) + 1 + 5 \leq n$ which implies that $h_3(G) \leq \frac{n - \Delta(G)}{3}$, a contradiction). Hence there are at least $h_3(G) - 2 + 4$ vertices in $V(G) \setminus (I(B_0) \cup N(u) \cup \{u\})$, $2h_3(G) + (h_3(G) - 1) + (\Delta(G) - 2) + 1 + (h_3(G) - 2) + 4 \leq n$ by Claim 1. So $h_3(G) \leq \frac{n - \Delta(G)}{4}$, which implies that $h_3(G) \leq \frac{n - \Delta(G)}{3} - 1$ since $h_3(G)$ is an integer, a contradiction. This proves that $H$ satisfies (III).

It remains to prove that $H$ satisfies (IV). We will prove this by contradiction. If possible, suppose that there is a branch $B_0 \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$ with $|E(B_0)| \geq h_3(G) + 1$. Let $u$ and $v$ be two endvertices of $B_0$ and $S(u, B_0)$ be a branch-bond containing $B_0$ such that any branch of $S(u, B_0)$ has $u$ as an endvertex. Since $G$ is 2-connected, $|S(u, B_0)| \geq 2$.

By the following algorithm, we first find a cycle of $G$ that contains $B_0$ and then obtain a contradiction.

**Algorithm $B_0$.**

1. If $|S(u, B_0)|$ is even, then select a branch $B_1 \in S(u, B_0) \setminus (\mathcal{B}_H(G) \cup \{B_0\})$ by **Theorem 9**. Otherwise, since $|E(B_0)| \geq h_3(G) + 1$, select a branch $B_1 \in S(u, B_0)$ with $|E(B_1)| = l(S(u, B_0)) \leq h_3(G)$ (obviously $B_1 \neq B_0$) and let $u_1 (\neq u)$ be the other endvertex of $B_1$. If $u_1 = v$, then set $t := 1$ and stop. Otherwise $i := 1$.

2. Select a branch-bond $S(u_i, B_0)$ in $G$ which contains $B_0$ but not $B_1, B_2, \ldots, B_i$ such that any branch in $S(u_i, B_0)$ has exactly one endvertex in $\{u_1, u_2, \ldots, u_i\}$. If $|S(u_i, B_0)|$ is even, then, by **Theorem 9**, select a branch $B_{i+1} \in S(u_i, B_0) \setminus (\mathcal{B}_H(G) \cup \{B_0\})$.

   Otherwise, since $|E(B_0)| \geq h_3(G) + 1$, select a branch $B_{i+1} \in S(u_i, B_0)$ such that $|E(B_{i+1})| = l(S(u_i, B_0)) \leq h_3(G)$ (obviously $B_{i+1} \neq B_0$), and let $u_{i+1}$ be the endvertex of $B_{i+1}$ that is not in $\{u_1, u_2, \ldots, u_i\}$. If $u_{i+1} = v$, then set $t := i + 1$ and stop. Otherwise replace $i$ by $i + 1$ and return to step 2.

   Note that $|\mathcal{B}(G)|$ is finite, and $d_G(v) \geq 2$ implies that Algorithm $B_0$ will stop after a finite number of steps. Note that $G[\bigcup_{i=0}^{L} E(B_i)]$ is connected. Furthermore, since $u_t = v$ and $|S(u_t, B_0)| \geq 2$, $G[\bigcup_{i=0}^{L} E(B_i)]$ has a cycle $C_0$ of $G$ which contains $B_0$. Let $H'$ be the subgraph of $G$ obtained from $G[E(H) \triangle E(C_0)]$ by adding the remaining vertices of $\bigcup_{i=0}^{L} V_i(G)$ as isolated vertices in $H'$. By the choice of $B_j$, $|E(B_j)| \leq h_3(G)$ for $B \in \mathcal{B}_H(G) \cap \{B_1, B_2, \ldots, B_j\}$. Hence, since $B_0$ is in $C_0$, $H'$ is a subgraph with (I) and (II) that has less branches of length $h_3(G) + 1$ than $H$, a contradiction. Hence $H$ satisfies (IV). This proves that $H \in EU_{h_3(G) - 1}(G)$, which implies that $h(G) \leq h_3(G) - 1 \leq \frac{n - \Delta(G)}{3}$. The proof of **Theorem 2** is completed. □

By the proof of **Theorem 2**, one can obtain the following result which answers partly a question proposed in [13].

**Theorem 11.** Let $G$ be a 2-connected simple graph of order $n$. Then if $h_3(G) \geq \frac{n - \Delta(G)}{3} - 2 \geq \frac{7}{3}$ then $h(G) = h_3(G) - 1$. 
4. Proof of Theorem 4

Before giving the proof of Theorem 4, we state the following well-known result involving the vertex degree sequence of a graph.

**Theorem 12** (Chvátal [5]). Let $G$ be a simple graph with vertex degrees $d_1 \leq d_2 \leq \cdots \leq d_n$, where $n \geq 3$. If $i < \frac{n}{2}$ implies that $d_i > i$ or $d_{n-i} \geq n - i$, then $G$ is hamiltonian.

We need its following consequence. By $[x]$ we denote the minimum integer not less than $x$.

**Theorem 13.** Let $G$ be a simple graph of order $n \geq 14$ with $h_3(G) \geq \frac{n+4}{6}$. Then $\overline{G}$ is hamiltonian.

**Proof of Theorem 13.** Let $B_0$ be an odd branch-bond in $BB_3(G)$ such that $h_3(G) = \min \{|E(B)| : B \in B_0\}$. Then there are at least $3(h_3(G) - 1) \geq \frac{n}{2} - 1$ inner vertices of branches of $B_0$ which have degree exactly $n - 3$ in $\overline{G}$ and there is at least one vertex $w$ of $G$ that is not adjacent to $[\frac{n}{2}] - 1$ vertices of the $3(h_3(G) - 1)$ inner vertices and the neighbor of one of the $[\frac{n}{2}] - 1$ vertices. Hence $w$ has at least $[\frac{n}{2}] - 1 + 1$ neighbors in $\overline{G}$ and so $d_{\overline{G}}(w) \geq \frac{n}{2}$. The other $n - 3(h_3(G) - 1) - 1$ vertices have degree at least $3(h_3(G) - 2) \geq \frac{n-8}{2} \geq 3$ in $\overline{G}$. Hence $\overline{G}$ satisfies the conditions of Theorem 12, which implies that $\overline{G}$ is hamiltonian. \(\Box\)

Now we present the proof of Theorem 4.

**Proof of Theorem 4.** Without loss of generality, we can assume that neither $G$ nor $\overline{G}$ is hamiltonian, and $L(\overline{G})$ is hamiltonian. Then $\Delta(G) \geq \frac{n-1}{4}$ since otherwise $\delta(\overline{G}) = n - 1 - \Delta(G) \geq \frac{3}{2}$ which implies that $\overline{G}$ is hamiltonian by Theorem 12, a contradiction.

We distinguish the following cases to finish our proof.

**Case 1.** $h_3(G) \leq \frac{n-9}{6}$.

By Theorem 8, $h(G) \leq h_3(G) + 1 \leq \frac{n-3}{6}$ and then we are done.

**Case 2.** $\frac{n-9}{6} < h_3(G) < \frac{n+4}{6}$.

Let $B_0$ be an odd branch-bond in $BB_3(G)$ such that $h_3(G) = \min \{|E(B)| : B \in B_0\}$. Then $B_0$ has exactly three branches. We will prove this by contradiction. If possible, suppose that $B_0$ has at least five branches, then $B_0$ has at least $5(h_3(G) - 1) \geq \frac{n}{2}$ inner vertices of degree 2 in $G$. Hence $\overline{G}$ has at least $[\frac{n}{2}]$ vertices of degree 2. Hence $\overline{G}$ satisfies the conditions of Theorem 12, which implies that $\overline{G}$ is hamiltonian, a contradiction. Hence $B_0$ has exactly three branches. Let $B_0 \in B_0$ with $|E(B_0)| = h_3(G)$. Taking two branches $B_1, B_2$ that are not $B_0$, there is a cycle $C_0$ in $G$ containing $B_1, B_2$ by the definition of an odd branch-bond. Now let $H$ be the subgraph obtained from $C_0$ by adding all vertices in $(\bigcup\limits_{i=3}^{\Delta(G)} V_i(G)) \setminus V(C_0)$ as isolated vertices in $H$. Then $H$ satisfies (I) and (II). We claim that $H \in E U_{h_3(G)-1}(G)$. It suffices to prove that $H$ satisfies (III) and (IV). Since $\Delta(G) \geq \frac{n-1}{4}$ and $\Delta(G) < \frac{n+4}{6}$, there are at most $n - 3(h_3(G) - 2) - \Delta(G) - 2 < 9$ vertices outside of $B_0 \cup N(u) \cup \{u\}$, where $u$ is a vertex of $G$ with maximum degree $\Delta(G)$. Hence $|E(B)| \leq 9 \leq h_3(G) - 2$ for every branch $B \in B(G) \setminus B_H(G)$, which implies that $H$ satisfies (III) and (IV). Hence $H \in E U_{h_3(G)-1}(G)$, which implies that $h(G) \leq h_3(G) - 1 \leq \frac{n-3}{6}$ by Theorem 8.

**Case 3.** $h_3(G) \geq \frac{n+4}{6}$.

By Theorem 12, $\overline{G}$ is hamiltonian, a contradiction. This completes the proof of Theorem 4. \(\Box\)

5. Sharpness

In this section, we discuss the sharpness of Theorems 2 and 4.

First we show that the result in Theorem 2 is sharp. To see this, we construct a graph $G_0$ from the graph $G_l$ ($l \geq 2$) depicted in Theorem 2 and a nontrivial complete graph $K_{\Delta-2}$ of order $\Delta - 2$. Let $w$ be a vertex of degree three in $G_l$ and $ww$ an edge of $G_l$. Now divide $ww$ and denote the new vertex by $x$, and obtain the graph $G_0$ by identifying $w$ with exactly one vertex of $K_{\Delta-2}$ and adding an edge $wx$, where $v$ is a vertex of $K_{\Delta-2}$ that is not the identified vertex. Note that $G_0$ is a 2-connected simple graph such that $\Delta(G_0) = \Delta$, $|V(G_0)| - \Delta = 3(l - 1) \geq 3$ and
Theorem 8

Theorem 6

Theorem 4

Theorem 13

Fig. 2

Theorem 2

Theorem 4

G

G

Fig. 1

Theorem 6


References

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