LARGE DEGREE VERTICES IN LONGEST CYCLES OF GRAPHS, I

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Abstract

In this paper, we consider the least integer $d$ such that every longest cycle of a $k$-connected graph of order $n$ (and of independent number $\alpha$) contains all vertices of degree at least $d$.

Keywords: longest cycle, large degree vertices, order, connectivity, independent number.

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1. Introduction

1.1. Basic notation and terminology

All graphs considered here are simple and finite. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [2]. Let $G$ be a graph. For a vertex $v \in V(G)$ and a subgraph $H$ of $G$, we use $N_H(v)$ and $d_H(v)$ to denote the set and the number of neighbors of $v$ in $H$, respectively. We call $N_H(v)$ the neighborhood of $v$ in $H$ and $d_H(v)$ the degree of $v$ in $H$. We use $d_H(u, v)$ to denote the distance between two vertices $u, v \in V(H)$ in $H$. For two subgraphs $H$ and $L$ of a graph $G$, we set $N_L(H) = \bigcup_{v \in V(H)} N_L(v)$. When no confusion occurs, we will denote $N_G(v)$ and $d_G(v)$ by $N(v)$ and $d(v)$, respectively. We set $N[x] = N(x) \cup \{x\}$.

Throughout this paper, we denote the order, the connectivity and the independent number of a graph $G$, by $n(G)$, $\kappa(G)$ and $\alpha(G)$, respectively.

1.2. Motivation and main results of this paper

By the definition every Hamilton cycle of a graph passes through every vertex of the graph. Thus, in non-Hamiltonian graphs, a (longest) cycle through some special vertices should be also interesting for the same topic. There are many results on the problem whether a graph has a (longest) cycle through some special vertices, for example, any given vertex set [8]; large degree vertices, see [1, 7, 9]. Unlike most research of the existence of some (longest) cycle passing through special vertices in the literature, we put our attention to the problem to determine the least integer $d$ such that every longest cycle of a graph passes all vertices of degree at least $d$, using some additional conditions of order, of connectivity or of independence number.

The following known result gave a partly answer for the above problem.

**Theorem 1** (Li and Zhang [6]). Let $G$ be a 2-connected graph of order $n \geq 8$. Then every longest cycle of $G$ contains all vertices of degree at least $n - 4$.

We firstly extend Theorem 1 to $k$-connected graphs for any $k \geq 2$ and shall give a complete answer for the above problem by using the order of a graph and its connectivity.

**Theorem 2.** Let $G$ be a graph of connectivity $\kappa(G) \geq k \geq 2$ and of order $n \geq 6k - 4$. Then every longest cycle of $G$ contains all vertices of degree at least $n - 3k + 2$.

The bound on the degree in Theorem 2 is sharp. We construct a graph as follows. Let $R = 2K_2 \cup (k - 2)P_3$, $S = kK_1$ and $T = (n - 4k + 1)K_1$ are vertex-disjoint. Let $R'$ be the subset of $V(R)$ each vertex of which is either
a vertex of a $K_2$ or a center of a $P_3$ in $R$, and let $s'$ be a fixed vertex of $S$ and $x$ a vertex not in $R \cup S \cup T$. Let $L(k, n)$ be the graph with $V(L(k, n)) = \{x\} \cup V(R) \cup V(S) \cup V(T)$, and $E(L(k, n)) = E(R) \cup \{r's', rs, s'x, sx, st, xt : r' \in R', r \in V(R), s \in V(S) \backslash \{s'\}, t \in V(T)\}$. One can check that $L(k, n)$ is $k$-connected and the degree of $x$ is $n - 3k + 1$, but there is a longest cycle (in the subgraph induced by $V(R) \cup V(S)$) excluding $x$.

![Graph L(4, 21)](image)

The bound $n \geq 6k - 4$ is also sharp. This can be seen from the complete bipartite graph $K_{3k-3,3k-2}$ of order $6k - 5$. However, the longest cycles of $K_{3k-3,3k-2}$ exclude some vertices of degree $3k - 3 = n - 3k + 2$.

Now we define $\varphi(k, n)$ to be the least integer such that every longest cycle of a $k$-connected graph $G$ of order $n$ contains all vertices of degree at least $\varphi(k, n)$ in $G$.

To avoid the discussions of the petty cases, we put our considerations on 2-connected graphs, i.e., we always assume that $k \geq 2$. Note that if $n \leq k$, then there are no $k$-connected graphs of order $n$. Hence $\varphi(k, n)$ will be meaningless. Is $\varphi(k, n)$ well-defined for all pairs $(k, n)$ with $n \geq k + 1$? No. Under the condition that it holds “every $k$-connected graph on $n$ vertices is Hamiltonian” (e.g., $n = k + 1$), $\varphi(k, n)$ does not exist (or we may say $\varphi(k, n) = -\infty$). So we should take the pair $(k, n)$ such that there exist some $k$-connected graphs of order $n$ which are not Hamiltonian. This implies that $n \geq 2k + 1$ from the well-known Dirac’s theorem [4]. On the other hand, there indeed exist such graphs when $n \geq 2k + 1$ (for example, complete bipartite graphs $K_{k,n-k}$). So $\varphi(k, n)$ is well-defined if and only if $n \geq 2k + 1$.

From Theorem 2 and the construction of $L(k, n)$, we have

$$\varphi(k, n) = n - 3k + 2, \text{ for } n \geq 6k - 4.$$
How about the cases when $2k + 1 \leq n \leq 6k - 5$? First we construct a graph as follows: if $n$ is odd, then let $L(k, n) = K_{(n-1)/2, (n+1)/2}$; if $n$ is even, then let $L(k, n) = K_{n/2 - 1, n/2 + 1}$. Note that every longest cycle of $L(k, n)$ excludes some vertices of degree $\left\lceil n/2 \right\rceil - 1$. This shows that $\varphi(k, n) \geq \left\lceil n/2 \right\rceil$. On the other hand, we have the following result (one may compare it with the results in [1] and [9] where they replaced “every cycle” with “there exists some cycle” under the condition that “$G$ is 2-connected”).

**Theorem 3.** Let $G$ be a $k$-connected graph on $n \leq 6k - 5$ vertices. Then every longest cycle of $G$ contains all vertices of degree at least $\left\lceil n/2 \right\rceil$.

Instead of Theorems 2 and 3, we shall prove the following theorem in Section 3.

**Theorem 4.** Let $G$ be a graph of connectivity $\kappa(G) \geq k \geq 2$ and of order $n \geq 2k + 1$. Then every longest cycle of $G$ contains all vertices of degree at least $d = \max \left\{ \left\lceil n/2 \right\rceil, n - 3k + 2 \right\}$.

Now we have a complete formula

$$
\varphi(k, n) = \max \left\{ \left\lceil n/2 \right\rceil, n - 3k + 2 \right\}, \text{ for all } n \geq 2k + 1.
$$

In the following we consider the same problem by using an additional condition of independent number. We use $\varphi(k, \alpha, n)$ to denote the least integer such that for every $k$-connected graph $G$ of order $n$ and of independent number $\alpha$, every longest cycle of $G$ contains all vertices of degree at least $\varphi(k, \alpha, n)$. As the analysis above, we should take the triple $(k, \alpha, n)$ such that there exists a $k$-connected graph of order $n$ and independent number $\alpha$ that is not Hamiltonian. This requires $\alpha \geq k + 1$ from Chvátal-Erdős’s theorem [3]; and $\alpha \leq n - k$, since every $k$ connected graph of order $n$ has independent number at most $n - k$ (note that an independent set excludes the $k$ neighbors of some vertex). On the other hand, for triple $(k, \alpha, n)$ with $k + 1 \leq \alpha \leq n - k$, the graph $kK_1 \cup ((\alpha - 1)K_1 \cup K_{n - k - \alpha + 1})$ is a $k$-connected graph of order $n$ and independent number $\alpha$ that is not Hamiltonian. Thus $\varphi(k, \alpha, n)$ is well-defined if and only if $k + 1 \leq \alpha \leq n - k$.

By the definition of $\varphi(k, n)$, we can see that

$$
\varphi(k, n) = \max \{ \varphi(k, \alpha, n) : k + 1 \leq \alpha \leq n - k \}, \text{ for all } n \geq 2k + 1.
$$

Using a result in [10], we can prove the following result.

**Theorem 5.** Let $G$ be a $k$-connected graph of order $n$ and of independent number $\alpha$. Then every longest cycle of $G$ contains all vertices of degree more than

$$
d_0 = \frac{(\alpha - k)n - k\alpha + k^2 + \alpha^2 - 2\alpha}{\alpha}.
$$
Taking $\alpha = k + 1$ in the above theorem, we can obtain the following correspondence.

**Theorem 6.** Let $G$ be a graph of connectivity $\kappa(G) \geq k \geq 2$, of order $n \geq 2k + 1$ and of independent number $k + 1$. Then every longest cycle of $G$ contains all vertices of degree at least

$$d = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 1.$$

The bound on $d$ in Theorem 6 is sharp. We construct a graph $L(k, k + 1, n)$ by joining each vertex of $R = kK_1$ to all vertices of $S = rK_{q+1} \cup (k + 1 - r)K_q$, where

$$n - k = q(k + 1) + r, \quad 0 \leq r \leq k.$$

Note that $L(k, k + 1, n)$ has a longest cycle excluding some vertices of degree

$$q + k - 1 = \left\lfloor \frac{n-k}{k+1} \right\rfloor + k - 1 = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 2.$$

By Theorem 6, the above equality implies that

$$\varphi(k, k + 1, n) = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 1, \text{ for all } n \geq 2k + 1.$$

Thus, in the following we will assume that $\alpha \geq k + 2$. For the case $k = 2$, we have the following result.

**Theorem 7.** Let $G$ be a 2-connected graph of order $n \geq 8$ and independent number $\alpha \geq 4$. Then every longest cycle of $G$ contains all vertices of degree at least

$$d = \left\lfloor \frac{n-5}{\alpha} \right\rfloor (\alpha - 2) + \max \left\{ 3, n - 4 - \left\lfloor \frac{n-5}{\alpha} \right\rfloor \alpha \right\}$$

i.e.,

$$d = \begin{cases} 
q(\alpha - 2) + 3, & 0 \leq r \leq 2, \\
q(\alpha - 2) + r + 1, & 3 \leq r < \alpha,
\end{cases}$$

where

$$n - 5 = q\alpha + r, \quad 0 \leq r < \alpha.$$

The bound on $d$ in Theorem 7 is sharp when $q \geq 1$ (i.e., when $n \geq \alpha + 5$). We construct extremal graphs as follows. If $0 \leq r \leq 2$, then let $R = rK_{q+2} \cup (2 - r)K_{q+1}$ and $T = (\alpha - 2)K_q$; if $3 \leq r < \alpha$, then let $R = 2K_{q+2}$ and $T = (r - 2)K_{q+1} \cup (\alpha - r)K_q$. Let $s', s, x$ be three vertices not in $R \cup T$. Let $L(2, \alpha, n)$
be a graph with the vertex set $V(L(2, \alpha, n)) = \{s', s, x\} \cup V(R) \cup V(T)$ and the edge set

$$E(L(2, \alpha, n)) = E(R) \cup E(T) \cup \{s'r, sr, s'x, sx, st, xt : r \in V(R), t \in V(T)\}.$$ 

One can check that $L(2, \alpha, n)$ is a 2-connected graph of order $n$ and of independent number $\alpha$, and $x$ has degree $d - 1$. But there is a longest cycle of $G$ excluding $x$. By Theorem 7, this implies that for $n \geq \alpha + 5$,

$$\varphi(2, \alpha, n) = \begin{cases} q(\alpha - 2) + 3, & 0 \leq r \leq 2, \\ q(\alpha - 2) + r + 1, & 3 \leq r < \alpha, \end{cases}$$

where

$$n - 5 = q\alpha + r, 0 \leq r < \alpha.$$

For the case $q = 0$, the above construction does not give the exact value of $\varphi(2, \alpha, n)$, since the independent number of the constructed graph is less than $\alpha$. What is its exact values for this case?

Note that $n \leq \alpha + 4$ in this case. Also note that in our assumption $n \geq \alpha + 2$.

We have three cases: $n = \alpha + 2$, $n = \alpha + 3$ and $n = \alpha + 4$.

**Theorem 8.** Let $G$ be a 2-connected graph of independent number $\alpha \geq 4$ and of order $n$ such that $\alpha + 2 \leq n \leq \alpha + 4$. Then every longest cycle of $G$ contains all vertices of degree at least

$$d = \begin{cases} n - \alpha + 1, & n - \alpha = 2, 3, \\ \alpha, & n - \alpha = 4. \end{cases}$$

Now we will show the sharpness of the bound in Theorem 8. For the case $n = \alpha + 2$, consider the graph $L(2, \alpha, \alpha + 2) = K_{2,\alpha}$. Note that every longest cycle of $L(2, \alpha, \alpha + 2)$ excludes some vertices of degree 2.

For the case $n = \alpha + 3$, consider the graph $L(2, \alpha, \alpha + 3) = K_{3,\alpha}$. Note that every longest cycle of $L(2, \alpha, \alpha + 3)$ excludes some vertices of degree 3.

Now we consider the case $n = \alpha + 4$. We construct the graph $L(2, \alpha, \alpha + 4)$ by combining a cycle $C_6$ and a star $K_{1,\alpha-3}$ in such a way: choosing two vertices $u, v$ in $C_6$ with distance 3, joining the center $x$ of the star to $u$ and $v$, and joining all the end-vertices of the star to $v$. Note that one longest cycle of $L(2, \alpha, \alpha + 4)$ excludes $x$ of degree $\alpha - 1$.

Therefore, we give formulas for $\alpha \geq 4$,

$$\varphi(2, \alpha, \alpha + 2) = 3,$$

$$\varphi(2, \alpha, \alpha + 3) = 4,$$

$$\varphi(2, \alpha, \alpha + 4) = \alpha.$$

The bound of $d_0$ in Theorem 5 seems not sharp for $\alpha \geq k + 2$ (at least it is not sharp when $k = 2$). We propose the following conjecture.
Conjecture 9. Let $G$ be a $k$-connected graph, $k \geq 3$, of independent number $\alpha \geq k + 2$ and of order $n \geq \max\{2\alpha + 1, \alpha + 3k + 1\}$. Then every longest cycle of $G$ contains all vertices of degree at least

$$d = \begin{cases} q(\alpha - k) + k + 1, & 0 \leq r \leq k, \\ q(\alpha - k) + k + 2, & k + 1 \leq r \leq 2k + 1, \\ q(\alpha - k) + r - k + 1, & 2k + 2 \leq r < \alpha + k, \end{cases}$$

where

$$n - 2k - 1 = q(\alpha + k) + r, \quad 0 \leq r < \alpha + k.$$

We remark that if the conjecture is true, then the bound on $d$ is sharp. We construct a graph as follows. If $0 \leq r \leq k$, then let $R = rK_{2q+2} \cup (k-r)K_{2q+1}$ and $T = (\alpha-k)K_q$; if $k+1 \leq r \leq 2k+1$, then let $R = (r-k-1)K_{2q+3} \cup (2k+1-r)K_{2q+2}$ and $T = K_{q+1} \cup (\alpha-k-1)K_q$; if $2k+2 \leq r < \alpha + k$, then let $R = kK_{2q+3}$ and $T = (r-2k)K_{q+1} \cup (\alpha+k-r)K_q$, and let $S = kK_1$. Let $x$ be a vertex not in $R \cup S \cup T$. Let $L(k, \alpha, n)$ be a graph with $V(L(k, \alpha, n)) = \{x\} \cup V(R) \cup V(S) \cup V(T)$ and

$$E(L(k, \alpha, n)) = E(R) \cup E(T) \cup \{sr, sx, st, xt : r \in V(R), s \in V(S), t \in V(T)\}.$$  

One can check that $L(k, \alpha, n)$ is a 2-connected graph of order $n$ and of independent number $\alpha$, and $x$ has degree $d - 1$. But there is a longest cycle of $G$ excluding $x$.

2. Preliminaries

Let $G$ be a graph and $x, y \in V(G)$. An $x$-path is a path with $x$ as one of its end vertices; an $(x, y)$-path is one connecting $x$ and $y$. If $Y$ is a subset of $V(G)$, then an $(x, Y)$-path is one connecting $x$ and a vertex in $Y$ with all internal vertices in
Let $P$ and $Q$ be paths instead of $(x, V(H))$-path and $V(H)$-path, respectively. It is convenient to denote a path $P$ with end-vertices $x, y$ by $P(x, y)$.

For a cycle $C$ with a given orientation and a vertex $x$ on $C$, we use $x^+$ to denote the successor, and $x^-$ the predecessor of $x$ on $C$. In the following, we always assume that $C$ has an orientation, $\overrightarrow{C}$. For two vertices $x, y$ on $C$, $\overrightarrow{C}[x, y]$ or $\overrightarrow{C}[y, x]$ denotes the path from $x$ to $y$ along $\overrightarrow{C}$. Similarly, if $x, y$ are two vertices in a path $P$, $P[x, y]$ denotes the subpath of $P$ between $x$ and $y$. For an arbitrary path $P$ or cycle $C$, we use $l(P)$ or $l(C)$ to denote the length (the number of edges) of it.

We first give some lemmas on longest cycles of graphs.

**Lemma 10.** Let $C$ be a longest cycle of a graph $G$, and $P = P(u, v)$ be a $C$-path. Then $l(\overrightarrow{C}[u, v]) \geq l(P)$.

**Proof.** Otherwise, $\overrightarrow{C}[v, u]uPv$ is a cycle longer than $C$, a contradiction. \hfill \blacksquare

Lemma 10 can be extended to the following.

**Lemma 11.** Let $C$ be a longest cycle of a graph $G$, $H$ be a component of $G - C$ and $P = P(u, v)$ be a $C$-path of length at least 2 with all internal vertices in $H$. Then

$$l\left(\overrightarrow{C}[u, v]\right) \geq l(P) + 2|N_C(H) \cap V(\overrightarrow{C}[u^+, v^-])|.$$ 

**Proof.** We use induction on $|N_C(H) \cap V(\overrightarrow{C}[u^+, v^-])|$. If $N_C(H) \cap V(\overrightarrow{C}[u^+, v^-]) = \emptyset$, then we are done by Lemma 10. Now we suppose that $N_C(H) \cap V(\overrightarrow{C}[u^+, v^-]) \neq \emptyset$.

Let $x$ be a vertex in $N_C(H) \cap V(\overrightarrow{C}[u^+, v^-])$. Let $P' = P'(x, x')$ be an $(x, P - \{u, v\})$-path with all internal vertices in $H - P$. Then $P_1 = P[u, x']x'P'$ and $P_2 = P'x'P'[x', v]$ are two $C$-paths with end-vertices $u, x$ and $x, v$, respectively, and with all internal vertices in $H$. Clearly, the length of $P_1$ and $P_2$ are at least 2. By induction hypothesis,

$$l\left(\overrightarrow{C}[u, x]\right) \geq l(P_1) + 2|N_C(H) \cap V(\overrightarrow{C}[u^+, x^-])|,$$

$$l\left(\overrightarrow{C}[x, v]\right) \geq l(P_2) + 2|N_C(H) \cap V(\overrightarrow{C}[x^+, v^-])|.$$ 

Note that $l(P_1) + l(P_2) = l(P) + 2l(P') \geq l(P) + 2$, and

$$|N_C(H) \cap V(\overrightarrow{C}[u^+, x^-])| + |N_C(H) \cap V(\overrightarrow{C}[x^+, v^-])| = |N_C(H) \cap V(\overrightarrow{C}[u^+, v^-])| - 1.$$
We have

\[ l \left( \overline{C}[u,v] \right) = l \left( \overline{C}[u,x] \right) + l \left( \overline{C}[x,v] \right) \geq l(P) + 2 \left| N_C(H) \cap V \left( \overline{C}[u^+,v^-] \right) \right| . \]

The assertion is proved.

**Lemma 12.** Let \( G \) be a graph, \( C \) be a longest cycle of \( G \) and \( H \) be a component of \( G - C \).

1. If \( u \in N_C(H) \). then \( u^+, u^- \notin N_C(H) \).
2. If \( u, v \in N_C(H) \), then \( u^+v^+, u^-v^- \notin E(G) \).

**Proof.** The assertion (1) can be deduced from Lemma 10. Now we prove the assertion (2).
Suppose that \( u, v \in N_C(H) \). Then let \( P \) be a \((u,v)\)-path of length at least 2 with all internal vertices in \( H \). If \( u^+v^+ \in E(G) \), then

\[ C' = \overline{C}[u^+,v]vPu\overline{C}[u,v^+]v^+u \]

is a cycle longer than \( C \), a contradiction. Thus we conclude that \( u^+v^+ \notin E(G) \), and similarly, \( u^-v^- \notin E(G) \).

Let \( G \) be a graph and \( yz \in E(G) \), we define the contraction of \( G \) at \( yz \), denoted by \( G \cdot yz \), as the graph with \( V(G \cdot yz) = V(G) \setminus \{y\} \), and \( E(G \cdot yz) = E(G - y) \cup \{xz : xy \in E(G) \text{ and } x \neq z\} \).

**Lemma 13.** Let \( G \) be a graph and \( yz \in E(G) \). If there is a cycle \( C \) in \( G \cdot yz \), then there is a cycle \( C' \) in \( G \) with length at least \( l(C) \) such that \( V(C') \subseteq V(C) \cup \{y\} \).

**Proof.** If \( C \) does not contain \( z \), then \( C \) is also a cycle of \( G \) and we are done. So we assume that \( z \in V(C) \). By the definition of contraction, \( zz^+ \in E(G) \) or \( yz^+ \in E(G) \), and \( zz^- \in E(G) \) or \( yz^- \in E(G) \). Let

\[ C' = \begin{cases} 
C, & \text{if } zz^+ \in E(G) \text{ and } zz^- \in E(G), \\
\overline{C}[z,z^-]z^-yz, & \text{if } zz^+ \in E(G) \text{ and } zz^- \notin E(G), \\
yz^+\overline{C}[z^+,z], & \text{if } zz^+ \notin E(G) \text{ and } zz^- \in E(G), \\
yz^+\overline{C}[z^+,z^-]z^-y, & \text{if } zz^+ \notin E(G) \text{ and } zz^- \notin E(G).
\end{cases} \]

Then \( C' \) is a required cycle.

We will use the following theorems from [3, 10].

**Theorem 14** (Chvátal and Erdős [3]). If \( G \) is a graph of order \( n \geq 3 \) with \( \alpha(G) \leq \kappa(G) \), then \( G \) is Hamiltonian.
**Theorem 15** (O, West and Wu [10]). If $G$ is a graph of order $n$ with $\alpha(G) \geq \kappa(G)$, then $G$ has a cycle of length at least $$\frac{\kappa(G)(n + \alpha(G) - \kappa(G))}{\alpha(G)}.$$ 

**Theorem 16** (O, West and Wu [10]). If $G$ is separable, then $V(G)$ admits a partition $(V_1, V_2)$ such that $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2]).$

Now we prove some more lemmas.

**Lemma 17.** Let $G$ be a nonseparable graph. Then for any two distinct vertices $u, v$ of $G$, $G$ contains a $(u, v)$-path of order at least $[n(G)/\alpha(G)]$.

**Proof.** If $G$ is complete, then the result is trivially true. Now we assume that $G$ is not complete, i.e., $\alpha(G) \geq 2$. So $G$ is 2-connected. If $\alpha(G) \leq \kappa(G)$, then by Theorem 14, $G$ has a Hamilton cycle $C$, and either $\overrightarrow{C}[u, v]$ or $\overleftarrow{C}[u, v]$ is a required path. Now we assume that $\alpha(G) > \kappa(G)$.

Let $C$ be a longest cycle of $G$. By Theorem 15, $l(C) \geq 2n(G)/\alpha(G)$. Since $G$ is 2-connected, we may choose a $(u, C)$-path $P_1$ and a $(v, C)$-path $P_2$ such that they are vertex-disjoint. Let $u'$ and $v'$ be the end-vertices of $P_1$ and $P_2$, respectively, on $C$ (possibly $u = u'$ or $v = v'$, or both). Then $P_1u'\overrightarrow{C}[u', v']v'P_2$ or $P_1u'\overleftarrow{C}[u', v']v'P_2$ is a required path. \(\blacksquare\)

**Lemma 18.** Let $G$ be a 2-connected graph. Let $C$ be a subgraph of $G$ with at least two vertices, and $H$ be an induced subgraph of $G - C$. Then $G$ contains a $C$-path $P$ such that $$|V(P) \cap V(H)| \geq \left\lceil \frac{n(H)}{\alpha(H)} \right\rceil.$$ 

**Proof.** We use induction on $n(H)$. If $H$ has only one vertex, say $x$, then $n(H) = \alpha(H) = 1$. Since $G$ is 2-connected, there is a $C$-path passing through $x$, which is a required path. Now we assume that $H$ has at least two vertices.

Suppose first that $H$ is nonseparable. Let $P_1(u, u')$ and $P_2(v, v')$ be two vertex-disjoint paths between $H$ and $C$ with all internal vertices in $G - (H \cup C)$, where $u, v \in V(H)$ and $u', v' \in V(C)$. By Lemma 17, $H$ contains a $(u, v)$-path $P'$ of order at least $[n(H)/\alpha(H)]$. Thus $P = P_1uP'vP_2$ is a required path.

Now we suppose that $H$ is separable. By Theorem 16, there is a partition $(V_1, V_2)$ of $V(H)$ such that $\alpha(H) = \alpha(G[V_1]) + \alpha(G[V_2])$. Let $H_1 = G[V_1]$ and $H_2 = G[V_2]$. Note that $n(H) = n(H_1) + n(H_2)$. It is not hard to see that

$$\frac{n(H)}{\alpha(H)} \leq \max \left\{ \frac{n(H_1)}{\alpha(H_1)}, \frac{n(H_2)}{\alpha(H_2)} \right\} \Rightarrow \frac{n(H_1)}{\alpha(H_1)} = \alpha(H_1).$$
By induction hypothesis, there is a $C$-path $P$ such that

$$|V(P) \cap V(H)| \geq \frac{n(H_1)}{\alpha(H_1)} \geq \frac{n(H)}{\alpha(H)},$$

and $P$ is a required path. ■

Let $L_1$ be the graph obtained from $C_6$ by adding a new vertex $x$, and by
adding three edges from $x$ to three pairwise nonadjacent vertices of the $C_6$, and
let $L_2 = K_3 \lor 4K_1$. Set

$$\mathcal{L} = \{G : L_1 \subseteq G \subseteq L_2\}.$$

We prove the following lemma to complete this section.

**Lemma 19.** Let $G$ be a 2-connected graph with $n(G) \leq 7$ and $x$ be a vertex of $G$
with $d(x) \geq 3$. If there is a longest cycle of $G$ excluding $x$, then $G \in \mathcal{L}$.

**Proof.** Let $C$ be a longest cycle of $G$ excluding $x$ and let $H$ be the component
of $G - C$ containing $x$.

We first suppose that $H$ has at least two vertices. By the 2-connectedness of $G$,
there are two independent edges from $H$ to $C$. Let $yy'$ and $zz'$ be such two
edges, where $y, z \in V(H)$ and $y', z' \in V(C)$. Let $P$ be a $(y, z)$-path of $H$. Then
$P' = y'yPzz'$ is a path with length at least 3. By Lemma 10, $l(C[y', z']) \geq 3$
and $l(C[y', z']) \geq 3$. Thus $l(C) \geq 6$. Note that $n(H) \geq 2$, we have $n(G) \geq 8$,
a contradiction.

Thus we conclude that $H$ has the only one vertex $x$. By Lemma 12, $x$ cannot
be adjacent to two successive vertices on $C$. Since $d(x) \geq 3$, there will be at least
three vertices on $C$ adjacent to $x$, and at least three vertices on $C$ nonadjacent
to $x$. Thus $l(C) \geq 6$. Since $n(G) \leq 7$ and $x \notin V(C)$, we have $l(C) \leq 6$. Thus $C$
has exactly 6 vertices and $x$ is adjacent to three pairwise nonadjacent vertices of $C$. This implies that $L_1 \subseteq G$.

Let $C = y_1z_1y_2z_2y_3z_3y_1$ such that $N(x) = \{y_1, y_2, y_3\}$. By Lemma 12, \{x, z_1, z_2, z_3\} is an independent set. This implies that $G \subseteq L_2$. 

3. Proofs of main results

In this section, we shall present the proof of main results.

Proof of Theorem 4. Let $C$, with an orientation $\vec{C}$, be a longest cycle of $G$. We assume on the contrary that there is a vertex $x$ in $V(G - C)$ with $d(x) \geq d = \max\{\lceil n/2 \rceil, n - 3k + 2\}$.

An $(x, C)$-fan is a collection of $(x, C)$-paths such that they have the only vertex $x$ in common. Since $G$ is $k$-connected, there is an $(x, C)$-fan with $s \geq k$ paths $P_i = P_i(x, z_i)$, $1 \leq i \leq s$, where $z_i \in V(C)$. We choose the $(x, C)$-fan such that $s$ is as large as possible. We suppose that $z_1, z_2, \ldots, z_s$ appear in this order along $\vec{C}$. Thus

\begin{equation}
\ell(C) = \sum_{i=1}^{s} \ell(\vec{C}[z_i, z_{i+1}]),
\end{equation}

where the subscripts are taken modulo $s$.

By Menger’s theorem, there is a vertex $y_i \in V(P_i - x)$ such that $S = \{y_i : 1 \leq i \leq s\}$ is a vertex-cut of $G$ separating $x$ and $C - S$. We choose $y_i$ in such a way that $d_{P_i}(x, y_i)$ is as small as possible (note that $y_i$ is possibly equal to $z_i$). Clearly

\begin{equation}
N_C(x) \subseteq S.
\end{equation}

Let $H$ be the component of $G - S$ containing $x$. Then

Claim 20. For every vertex $y_i \in S$, either $N_H(y_i) = \{x\}$ or $|N_H(y_i)| \geq 2$.

Proof. Suppose on the contrary that $|N_H(y_i)| = 1$ and $y_i' \neq x$ is the vertex in $N_H(y_i)$. Then $y_i'$ is the neighbor of $y_i$ on $P_i[x, y_i]$. Let $S' = (S\setminus\{y_i\}) \cup \{y_i'\}$. Then $S'$ is a vertex-cut of $G$ separating $x$ and $C - S'$ such that $d_{P_i}(x, y_i') < d_{P_i}(x, y_i)$, contradicting the choice of $S$. 

If $H$ has only one vertex $x$, then $d(x) = |S| = s$. By Lemma 12, $\ell(\vec{C}[z_i, z_{i+1}]) \geq 2$ for all $i \in \{1, 2, \ldots, s\}$. By (1), $\ell(C) \geq 2s = 2d(x)$ and

\[ n \geq \ell(C) + 1 \geq 2d(x) + 1 \geq n + 1, \]

a contradiction.
If $H$ has exactly two vertices, then let $x'$ be the vertex in $V(H)\backslash\{x\}$. By Claim 20, every vertex $y_i$ in $S$ is adjacent to $x$. Hence $N(x) = S \cup \{x'\}$ and $s = d(x) - 1$. Note that $d(x') = d_S(x') + 1$ and $d(x') \geq k$, since $G$ is $k$-connected. We have $d_S(x') \geq k - 1$. By Lemma 12, $l(\overline{C}[z_i, z_{i+1}]) \geq 2$ for all $i$. Moreover, if $x'y_i \in E(G)$, then $P = P_1[z_i, y_i]y_ixy_{i+1}P_{i+1}[y_{i+1}, z_{i+1}]$ is a $C$-path of length at least 3, by Lemma 10, $l(\overline{C}[z_i, z_{i+1}]) \geq 3$. This implies that $l(C) = \sum_{i=1}^s l(\overline{C}[z_i, z_{i+1}]) \geq 3s(k) + 2(s - d_S(x')) \geq 2s + d_S(x') \geq 2d(x) + k - 3 \geq n + k - 3$, and $n \geq l(C) + 2 \geq n + k - 1 \geq n + 1$, a contradiction.

Now it remains to consider the case when $H$ has at least three vertices.

By $b(x)$ we denote the number of vertices in $V(G) \backslash N[x]$. Then $b(x) = n - 1 - d(x) \leq 3k - 3$. Hence, by (2),

$$l(C) \leq s + b(x) \leq s + 3k - 3.$$  

**Claim 21.** Every vertex in $V(H)\backslash\{x\}$ is not a cut-vertex of $H$.

**Proof.** Suppose, otherwise, that $x' \neq x$ is a cut-vertex of $H$. Let $H_1$ and $H_2$ be two components of $H - x'$ such that $x \in V(H_1)$.

We claim that for every vertex $y_i \in S$, $N_{H_1}(y_i) \neq \emptyset$. Otherwise, every $(x, y_i)$-path with all internal vertices in $H$ will pass through $x'$, so is $P_1[x, y_i]$. Let $S' = (S \backslash \{y_i\}) \cup \{x'\}$. Then $S'$ is a vertex-cut of $G$ separating $x$ and $C - S'$ such that $d_{P_1}(x, x') < d_{P_1}(x, y_i)$, a contradiction. Thus as we claimed, $N_{H_1}(y_i) \neq \emptyset$.

For every $y_i \in S$, let $w_i$ be a vertex in $N_{H_1}(y_i)$. Now we claim that $l(\overline{C}[z_i, z_{i+1}]) \geq 4$ for those $i$ such that $N_{H_2}(y_i) \neq \emptyset$. Suppose $N_{H_2}(y_i) \neq \emptyset$. Let $w'_i$ be a neighbor of $y_i$ in $H_2$. Then $H$ has a $(w'_i, w_{i+1})$-path $P$ of length at least 2. Thus $P' = P_1[z_i, y_i]y_iw'_iw_{i+1}y_{i+1}P_{i+1}[y_{i+1}, z_{i+1}]$ is a path of length at least 4 with all internal vertices in $G - C$. By Lemma 10, $l(\overline{C}[z_i, z_{i+1}]) \geq 4$.

Note that $|N_S(H_2)| \geq k - 1$, since $G$ is $k$-connected and $N_S(H_2) \cup \{x'\}$ is a vertex-cut. Therefore,

$$l(C) \geq 4(k - 1) + 2(s - k + 1) = 2s + 2k - 2 \geq s + 3k - 2,$$

contradicting (3). \hfill \Box

**Claim 22.** $H$ is a star with center $x$.

**Proof.** Suppose, otherwise, $H$ has an $x$-path $xx''x''$ (say) of length 2. Then there is an $(x'', S)$-fan with $k$ internally disjoint paths $Q_i = Q_i(x'', y_{j_i})$, $1 \leq j_i < j_2 < \cdots < j_k \leq s$, such that they have the only vertex $x''$ in common. We set $S' = \{y_{j_i} : 1 \leq i \leq k\}$.

Note that at most one path of $Q_i$ passes through $x$. We will prove that $l(\overline{C}[z_{j_i}, z_{j_i+1}]) \geq 4$ for those $j_i$ such that $y_{j_i} \in S'$ and $Q_i$ does not pass through $x$. 
Suppose that \( y_{j_1} \in S' \) and \( Q_i \) does not pass through \( x \). Let \( w_{j_i} \) be the neighbor of \( y_{j_i} \) on \( Q_i \). Then \( w_{j_i} \neq x \). If \( \ell(Q_i) \geq 2 \), then let \( v_{j_i} \) be a neighbor of \( w_{j_i} \) on the path \( Q_i[x'', w_{j_i}] \); if \( \ell(Q_i) = 1 \), then \( (w_{j_i}, x'') \) and we let \( v_{j_i} = x'' \). Then \( v_{j_i} \neq x \). By Claim 20, \( y_{j_i+1} \) has a neighbor \( w'_{j_i+1} \) in \( H \) other that \( w_{j_i} \). We claim that \( H \) has a \((w_{j_i}, w'_{j_i+1})\)-path of length at least 2. Otherwise \( w_j, w'_{j_i+1} \in E(G) \) and \( w_j, w'_{j_i+1} \) is a cut-edge of \( H \). By Claim 21, every vertex of \( V(H) \setminus \{x\} \) is not a cut-vertex of \( H \). This implies that \( w'_{j_i+1} = x \) and \( w_j \) has only one neighbor \( x \) in \( H \), contradicting the fact that \( v_{j_i} \in N_H(w_j) \) and \( v_{j_i} \neq x \). Thus as we claimed, \( H \) has a \((w_{j_i}, w'_{j_i+1})\)-path \( P \) of length at least 2. Thus \( P' = P_{j_i}[z_{j_i}, y_{j_i}, w_{j_i}, w'_{j_i+1}, y_{j_i+1}, w'_{j_i+1}] \) is a path of length at least 4 with all internal vertices in \( G - C \). By Lemma 10, \( l(C[z_{j_i}, z_{j_i+1}]) \geq 4 \).

Thus we conclude that there are at least \( k - 1 \) segments \( C[z_{j_i}, z_{j_i+1}] \) of length at least 4. Hence

\[
l(C) \geq 4(k - 1) + 2(s - k + 1) = 2s + 2k - 2 \geq s + 3k - 2,
\]

a contradiction. □

By Claim 22, \( H = K_{1,n(H)-1} \). Let

\[
S_0 = \{y_i \in S : N_H(y_i) = \{x\}\}, \quad S_2 = S \setminus (S_0 \cup S_1),
S_1 = \{y_i \in S : |N_H(y_i) \setminus \{x\}| = 1\}, \quad s_i = |S_i|, \quad i \in \{0, 1, 2\}.
\]

Thus \( s = s_0 + s_1 + s_2 \).

Let \( y_{j_i}, 1 \leq j_1 < j_2 < \cdots < j_{s_1+s_2} \leq s \), be the vertices in \( S_1 \cup S_2 \). Since \( G \) is \( k \)-connected,

\[
s_1 + s_2 \geq |N_S(x')| \geq k - 1
\]

for any \( x' \in V(H) \setminus \{x\} \), and

\[
s_1 + (n(H) - 1)s_2 \geq |E(H - x, S)| \geq (k - 1)(n(H) - 1).
\]

If \( s_1 + s_2 = 1 \), then without loss of generality we assume that \( x'y_1 \in E(G) \), where \( x' \in V(H) \setminus \{x\} \) and \( y_1 \in S_1 \cup S_2 \). Note that \( \{x, y_1\} \) is a vertex-cut of \( G \), implying that \( k = 2 \). Since \( z_1 P_1[z_1, y_1, y_1'x'y_2, y_2] \) is a path of length at least 3, by Lemma 10, \( l(C[z_1, z_2]) \geq 3 \) and by symmetry, \( l(C[z_1, z_2]) \geq 3 \). Thus

\[
l(C) \geq 3 + 3 + 2(s - 2) = 2s + 2 > s + 3k - 3,
\]

a contradiction. Now we conclude that \( s_1 + s_2 \geq 2 \).
Claim 23. For every vertex $y_j \in S_1 \cup S_2$,

$$l(\overrightarrow{C}[z_{j_i}, z_{j_{i+1}}]) \geq \begin{cases} 
3 + 2|N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])|; & y_j \in S_1, \\
4 + 2|N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])|; & y_j \in S_2,
\end{cases}$$

where the subscripts are taken modulo $s_1 + s_2$.

Proof. For any $y_j \in S_1 \cup S_2$, we let $w_{j_i}$ be a vertex in $N_H(y_j) \setminus \{x\}$. If $y_j \in S_1$, then by Claim 20, $w_{j_i} x \in E(G)$. Thus

$$P = P_{j_i}[z_{j_i}, y_j] y_j x w_{j_{i+1}} y_{j_{i+1}} P_{j_{i+1}}[y_{j_{i+1}}, z_{j_{i+1}}]$$

is a $C$-path of length at least 3. If $y_j \in S_2$, then let $w'_{j_i}$ be a vertex in $N_H(y_j) \setminus \{x, w_{j_{i+1}}\}$. Thus $P = P_{j_i}[z_{j_i}, y_j] y_j w'_{j_i} x w_{j_{i+1}} y_{j_{i+1}} P_{j_{i+1}}[y_{j_{i+1}}, z_{j_{i+1}}]$ is a $C$-path of length at least 4. Note that $N_C(H) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-]) = N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])$. By Lemma 11, we have the assertion.

By (4) and (5), we have

$$l(C) = \sum_{i=1}^{s_1+s_2} l(\overrightarrow{C}[z_{j_i}, z_{j_{i+1}}]) \geq 2s_0 + 3s_1 + 4s_2 = 2s + s_1 + 2s_2.$$ 

By (4) and (5), we have

$$l(C) \geq 2s + s_1 + 2s_2 = 2s + \frac{n(H) - 3}{n(H) - 2}(s_1 + s_2) + \frac{1}{n(H) - 2}(s_1 + (n(H) - 1)s_2)$$

$$\geq 2s + \frac{n(H) - 3}{n(H) - 2}(k - 1) + \frac{n(H) - 1}{n(H) - 2}(k - 1) = 2s + 2k - 2 \geq s + 3k - 2,$$

a contradiction.

The proof is complete. 

Proof of Theorem 5. If $\alpha \leq \kappa(G)$, then $G$ is Hamiltonian by Theorem 14 and we are done. Now suppose that $\alpha > \kappa(G)$. Let $C$ be a longest cycle of $G$ with an orientation, $\overrightarrow{C}$. Assume for contradiction that there exists a vertex $x$ of degree more than $d_0$ such that $x \not\in V(C)$. Let $H$ be the component of $G - C$ containing $x$. Then $|N_C(H)| \geq k$, since $G$ is $k$-connected. Let $N_C(H) = \{z_1, z_2, \ldots, z_s\}$, where $s = |N_C(H)|$. Hence

$$d(x) \leq |V(H - x)| + |N_C(H)| \leq n + s - l(C) - 1.$$
By Lemma 12, \( \{x, z_1^+, z_2^+, \ldots, z_s^+\} \) is an independent set of \( G \). Thus, we obtain that \( s + 1 \leq \alpha \). Therefore, by (6) and by the hypothesis of \( d(x) > d_0 \) and by Theorem 15,

\[
d_0 < d(x) \leq n + s - l(C) - 1 \leq n + \alpha - 1 - \frac{\kappa(G)(n + \alpha - \kappa(G))}{\alpha} - 1
\]

\[
= n + \alpha - 2 - \frac{\kappa(G)(n + \alpha - \kappa(G))}{\alpha} = d_0,
\]

a contradiction. This completes the proof of Theorem 5.

In order to use the induction method, we prove the following stronger theorem instead of Theorem 7.

**Theorem 24.** Suppose \( \alpha \geq 4 \) and \( n \geq 3 \) are two integers and \( d \) is defined as in Theorem 7. Let \( G \) be a 2-connected graph with \( n(G) \leq n \) and \( \alpha(G) \leq \alpha \). Then every longest cycle of \( G \) contains all the vertices of degree at least \( d \), unless \( G \in \mathcal{L} \).

**Proof.** We use induction on \( n(G) \). If \( G \) has only three or four vertices, then \( G \) is Hamiltonian and the result is trivially true. Now we assume that \( G \) has at least five vertices and assume that the assertion holds for all graphs with order less than \( n(G) \). This implies that \( n \geq 5 \) and \( q \geq 0 \).

Suppose that \( q = 0 \). Then \( r = n - 5 \). If \( n \leq 7 \), then \( r \leq 2 \) and \( d = 3 \). By Lemma 19, \( G \in \mathcal{L} \) or every longest cycle contains all vertices of degree at least \( d \). If \( n \geq 8 \), then \( r \geq 3 \) and \( d = r + 1 = n - 4 \). By Theorem 1, every longest cycle contains all vertices of degree at least \( d \). Thus we are done. So in the following, we assume that \( q \geq 1 \) (i.e., \( n \geq \alpha + 5 \)).

Let \( C \) be a longest cycle of \( G \). We suppose on the contrary that there is a vertex \( x \) in \( V(G - C) \) with \( d(x) \geq d \). Let \( H \) be the component of \( G - C \) containing \( x \).

Let \( b = n - 1 - d \). Then

\[
b = \begin{cases} 
2q + r + 1, & 0 \leq r \leq 2, \\
2q + 3, & 3 \leq r < \alpha.
\end{cases}
\]

By \( b(x) \) we denote the number of vertices in \( V(G) \setminus N[x] \). Then

\[(7)\]

\[
b(x) \leq b \leq 2q + 3.
\]

Suppose first that \( H \) has only one vertex \( x \). By Lemma 12, \( x \) is nonadjacent to every vertex of \( N_C^+(x) \). Thus \( b \geq b(x) \geq d(x) \geq d \). By comparing the formulas of \( b \) and \( d \), we can see that \( r = 2 \) and \( \alpha = 4 \). Since \( q \geq 1 \), we have \( d \geq \alpha + 1 \geq 5 \). But in this case \( N_C^+(x) \) is an independent set with \( d(x) \geq 5 \) vertices, a contradiction. This implies that \( H \) has at least two vertices.
Note that
\[ d - \alpha = \begin{cases} 
(q - 1)(\alpha - 2) + 1, & 0 \leq r < 2, \\
(q - 1)(\alpha - 2) + r - 1, & 3 \leq r < \alpha.
\end{cases} \]
We have \( d - \alpha \geq (q - 1)(\alpha - 2) + 1 \), and
\[ \left\lceil \frac{d - \alpha}{\alpha - 2} \right\rceil \geq \left\lceil \frac{(q - 1)(\alpha - 2) + 1}{\alpha - 2} \right\rceil = \frac{q}{\alpha - 2}. \]

Suppose that there is some component of \( G - C \) other than \( H \). Let \( G' \) be the graph obtained from \( G \) by removing all other components of \( G \), i.e., \( G' = G[V(C) \cup V(H)] \). Then \( G' \) is 2-connected, \( n(G') < n(G) \), \( \alpha(G') \leq \alpha(G) \), and \( d_{G'}(x) = d(x) \). By induction hypothesis, every longest cycle of \( G' \) contains \( x \). This implies that there is a cycle in \( G' \), and then in \( G \), longer than \( C \), a contradiction. Hence we conclude that there is only one component \( H \) of \( G - C \), i.e., \( G - C = H \).

**Claim 25.** \( N(x) = (V(H) \cup N_C(H)) \setminus \{x\} \).

**Proof.** Suppose that there is a vertex \( y \) in \( H \) such that \( xy \notin E(G) \). We choose a vertex \( z \in N(y) \) in such a way that if \( G - y \) is 2-connected, then let \( z \) be an arbitrary neighbor of \( y \); if \( G - y \) is separable, then let \( z \) be a neighbor of \( y \) which is an inner-vertex of some end-block of \( G - y \). In any case, \( \{y, z\} \) is not a cut-vertex and thus \( G' = G \cdot yz \) is 2-connected. Note that \( n(G'') < n(G) \), \( \alpha(G') \leq \alpha(G) \), and \( d_{G'}(x) = d(x) \). By induction hypothesis, every longest cycle of \( G' \) contains \( x \). This implies that there is a cycle in \( G' \) longer than \( C \). But if \( G' \) contains such a cycle, then so is \( G \) by Lemma 13, a contradiction. This implies that \( x \) is adjacent to all the vertices in \( V(H) \setminus \{x\} \).

Note that every vertex in \( V(H) \setminus \{x\} \) is not a cut-vertex of \( H \). Suppose that there is a vertex \( z \) in \( N_C(H) \) such that \( xz \notin E(G) \). It is not difficult to see that there is a neighbor \( y \) of \( y \) in \( H \) such that \( \{y, z\} \) is not a vertex-cut of \( G \). Thus \( G' = G \cdot yz \) is 2-connected. Note that \( n(G'') < n(G) \), \( \alpha(G') \leq \alpha(G) \), and \( d_{G'}(x) = d(x) \). By induction hypothesis, every longest cycle of \( G' \) contains \( x \). This implies that there is a cycle in \( G' \), and then in \( G \), longer than \( C \), a contradiction. Now we conclude that \( x \) is adjacent to all the vertices in \( (V(H) \cup N_C(H)) \setminus \{x\} \).

By Claim 25, \( \alpha(H) = \alpha(H - x) \) and \( d_H(x) = n(H - x) \). By Lemma 18, there is a \( C \)-path \( P = P(u, v) \) such that
\[ |V(P) \cap V(H - x)| \geq \frac{n(H - x)}{\alpha(H - x)} = \frac{d_H(x)}{\alpha(H)}. \]

By Claim 25, we can choose \( P \) such that it satisfies the above inequality and \( x \in V(P) \). Thus
\[ |V(P)| \geq |V(P) \cap V(H - x)| + |\{u, v, x\}| \geq \frac{d_H(x)}{\alpha(H)} + 3. \]
By Claim 25, \( d_H(x) = d(x) - |N_C(H)| \geq d - |N_C(H)| \). Note that the union of \( N^+_C(H) \) and an independent set of \( H \) form an independent set of \( G \). This implies that \( \alpha(H) \leq \alpha(G) - |N_C(H)| \leq \alpha - |N_C(H)| \). Together with the above inequality, we have

\[
|V(P)| \geq \left\lfloor \frac{d - |N_C(H)|}{\alpha - |N_C(H)|} \right\rfloor + 3 = \left\lfloor \frac{d - \alpha}{\alpha - |N_C(H)|} \right\rfloor + 4 \geq \left\lfloor \frac{d - \alpha}{\alpha - 2} \right\rfloor + 4.
\]

By (8), \( l(P) = |V(P)| - 1 \geq q + 3 \). By Lemma 11,

\[
l(C) = l(C[u, v]) + l(C[v, u]) \geq 2l(P) + 2(|N_C(H)| - 2) \geq 2q + 2|N_C(H)| + 2.
\]

Thus

\[
b(x) = |V(C)\setminus N_C(H)| \geq 2q + 2|N_C(H)| + 2 - |N_C(H)| \geq 2q + 4,
\]

contradicting (7).

The proof is complete. \( \blacksquare \)

**Proof of Theorem 8.** The case \( n = \alpha + 2 \) is trivial. The only 2-connected graphs with independent number \( \alpha \) and order \( \alpha + 2 \) are \( K_{2, \alpha} \) and \( K_{1,1,\alpha} \). Note that every longest cycle of them contains all (the two) vertices with degree at least 3. For the case \( n = \alpha + 4 \), the bound on \( d \) in Theorems 7 and 8 are equal. So the result can be deduced by Theorem 7 immediately.

Now we consider the case \( n = \alpha + 3 \). Let \( G \) be a 2-connected graph with independent number \( \alpha \) and order \( \alpha + 3 \), let \( C \) be an arbitrary longest cycle of \( G \), and let \( x \) be a vertex of \( G \) of degree at least 4. If \( C \) contains \( x \), then we have nothing to prove. So we assume that \( x \in V(G - C) \). If \( x \) is an isolated vertex of \( G - C \), then \( d_C(x) = d(x) \geq 4 \). By Lemma 12, \( l(C) \geq 8 \). Thus

\[
\alpha(G) \leq \alpha(G[V(C)]) + \alpha(G - C) \leq \alpha(C) + |V(G - C)| = \left\lfloor \frac{l(C)}{2} \right\rfloor + n - l(C)
\]

\[
= n - \left\lfloor \frac{l(C)}{2} \right\rfloor \leq n - 4,
\]

a contradiction. Thus we conclude that \( x \) has a neighbor \( x' \) in \( G - C \). Since \( G \) is 2-connected, \( G \) has a \( C \)-path \( P \) passing through the edge \( xx' \). Note that \( l(P) \geq 3 \), and by Lemma 10, \( l(C) \geq 6 \). Thus

\[
\alpha(G) \leq \alpha(G[V(C)]) + \alpha(G - C) \leq \left\lfloor \frac{l(C)}{2} \right\rfloor + n - l(C) - 1
\]

\[
= n - \left\lfloor \frac{l(C)}{2} \right\rfloor - 1 \leq n - 4,
\]

a contradiction. \( \blacksquare \)
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