Input-to-state stability of an ODE-heat cascade system with disturbances

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Abstract: In this study, the authors consider the input-to-state stability of an ordinary differential equation (ODE)–heat cascade system with Dirichlet interconnection where the boundary control input is located at the right end of the heat equation and the disturbance is appeared as a non-homogeneous term in the ODE. Based on two backstepping transformations, they design a state feedback control law that guarantees the input-to-state stability of the closed-loop system. The well-posedness of the closed-loop system is presented by using the semi-group approach. Moreover, they design an output feedback control law by constructing an exponentially convergent observer. With the output feedback control, the input-to-state stability of the resulting closed-loop system is proven.

1 Introduction

We consider a control problem of an ordinary differential equation (ODE) cascaded by a heat equation with bi-directional interconnection, where the control actuator is located at the right end of the heat equation and the disturbance appears as a non-homogeneous term in the ODE.

Coupled ODE–partial differential equation (PDE) systems have been applied in engineering different aspects such as electromagnetic coupling, mechanical coupling, coupled chemical reactions etc. The model of the ODE–heat cascade system is widely used in the solid–gas interaction of the chemical reaction and heat diffusion with an insulated catalyst fixed at one point [1–4].

Recently, some powerful methods have been developed to deal with the ODE–PDE cascade systems subject to external disturbances. One is the active disturbance rejection control (ADRC) approach [5]. In [6], the stabilisation of the ODE–heat cascade system through Dirichlet interconnection with boundary control matched disturbance is considered. Using the backstepping method, the original system is transformed into a target system where the ODE part is stable and the control is only used to cope with the disturbance and to stabilise the PDE part. Through the ADRC approach, the disturbance is estimated and cancelled by the feedback loop. The sliding mode control (SMC) approach is another powerful method to reject disturbances [7]. In [8], the stabilisation of the heat PDE–ODE cascade system with Dirichlet and Neumann interconnection is considered, respectively. By the SMC approach integrated with the backstepping method, the closed-loop system in the sliding mode surface is shown to be exponentially stable. In [9], both the ADRC approach and SMC technique are applied to deal with the boundary stabilisation of a cascade of ODE–wave systems with control matched disturbance. Furthermore, an observer-based output feedback control law is designed to stabilise the ODE–wave system. In [10], the SMC approach is proposed for the Orr–Sommerfeld equation cascaded by both the Squire equation and ODE with boundary disturbances. The output regulation of coupled systems with disturbances is another important problem in control theory and engineering applications. In [11], the output regulation of the anti-stable coupled wave equations with disturbances is considered. A state-feedback regulator is designed to force the output of the coupled wave equations to track the reference signal. In [12], the output regulation of ODEs cascaded by a wave equation with disturbances is studied. Both the state-feedback and output-feedback regulators are proposed to solve the output regulation problem. In [13], a variable length crane system under the external disturbances and constraints is considered in the two-dimensional space. An active boundary control strategy is designed to reduce the longitudinal-transverse coupled vibrations of the flexible cable for the crane system.

In [14], a non-uniform gantry crane system with constrained tension described by a hybrid PDE–ODE system is studied. Cooperative control laws and adaption laws are proposed for handling parametric uncertainties and stabilising the system. In [15], flexible wings with flexibility and articulation of a robotic aircraft are considered. Boundary controls are designed to suppress the bending and twist displacements and the closed-loop system is proved to be uniformly bounded. A Lyapunov approach is also an effective tool of coping with disturbances [16–19].

In [20], two integral variants of the input-to-state stability (ISS) property for general nonlinear finite-dimensional systems are discussed. In [21], the ISS of the infinite-dimensional systems is studied and is proved that the existence of an ISS-Lyapunov function implies the ISS property of a general control system. In [22], the methods for constructing Lyapunov functions to establish the ISS and integral ISS (iISS) for some classes of nonlinear parabolic equations are discussed, and a stability criterion for interconnections of iISS parabolic systems is provided. Moreover, a small-gain theorem is generalised to study the ISS of the interconnected nonlinear infinite-dimensional systems. In [23, 24], the ISS of the general nonlinear infinite-dimensional systems are studied. In [25], the ISS of the delayed neural fields described by nonlinear integro-differential equations is considered. In [26], a nonlinear small-gain theorem for large-scale infinite-dimensional systems is developed, and the ISS of the large-scale infinite-dimensional systems is a special case of the theorem. Using the semi-group approach, the relation between ISS and integral ISS for linear infinite-dimensional systems with an unbounded control operator is studied in [27]. In [28, 29], the ISS properties of a diffusion equation subject to boundary disturbances with time-varying distributed coefficients are shown by using the Lyapunov approach, respectively. In [30], the ISS property of a wave equation with a boundary disturbance is considered by using a difference equation approach and Lyapunov approach. In [31], the ISS with respect to boundary disturbances for one-dimensional parabolic equations is considered in time-varying subsets of the state space.

In [32], the boundary stabilisation of a cascade of the Schrödinger...
equation–ODE system with both matched and unmatched disturbances is discussed.

Motivated by [2, chapter 15], it is known that for the ODE–PDE cascade systems, we can design state feedback laws to achieve exponential stabilisation by using the backstepping approach. Hence, when there is a disturbance located on the ODE, the ODE–PDE system is the ISS under the same feedback law.

In this study, we consider the ODE–heat cascade system with Dirichlet boundary interconnection

\[
\begin{align*}
\dot{X}(t) &= AX(t) + Bu(t,0) + Bd(t), \quad t > 0, \\
B_0X(0,t) + Bu_0(t) &= 0, \quad 0 < x < 1, t > 0, \\
u(t) &= CX(t), \quad t > 0, \\
u_0(0) &= U(t), \quad t > 0,
\end{align*}
\tag{1}
\]

where \(X(t) \in \mathbb{R}^{n \times 1}\) and \(u(x,t) \in \mathbb{R}\) are the states of ODE and heat PDE, respectively, \(U(t) \in \mathbb{R}\) is the control input to the entire system, \(A \in \mathbb{R}^{n \times n}\), \(B, B_0 \in \mathbb{R}^{n \times 1}, C \in \mathbb{R}^{1 \times n}\), \(c > 0\), \(d(t) \in \mathbb{R}\) is the external disturbance appeared in the ODE and is bounded: \(|d(t)| \leq \Delta\), where \(\Delta > 0\) is a constant. For simplicity, we propose the following assumption:

**Assumption (A1)** \((A,B)\) is stabilisable, \(A^{-1}\) exists and \(c\) satisfies

\[c \sin \sqrt{c} + CA^{-1}B \cos \sqrt{c} \neq 0.\]

The main contribution of the study focuses on the following aspects:

- We study the well-posedness of open-loop system (1) by using the semi-group method and Riesz basis approach.
- We design a state feedback control law for the cascade system (1) by using the backstepping approach. The state matrix of ‘\(X\)-part’ becomes a Hurwitz matrix in the target system, and the ‘heat-part’ is exponentially stable when \(d(t) \equiv 0\). Hence, when \(d(t) \neq 0\), the closed-loop system of (1) is shown to be input-to-state stable.
- The well-posedness of the target system, which is equivalent to the closed-loop system of (1), is proved by using the Riesz basis approach and semi-group method.
- Moreover, we consider the output feedback control by designing a state observer that is exponentially convergent. With the observer-based output feedback control, we obtain the ISS property of system (1). Therefore, the proposed control problem is different from those considered for infinite-dimensional interconnected systems without observers and output feedback controls (such as [31, 32]).

The paper is organised as follows. In Section 2, by applying the semi-group method and Riesz basis approach, the well-posedness of open-loop system (1) is established. Section 3 is devoted to designing the state feedback control to make the closed-loop system ISS stable, where two backstepping transformations and the Lyapunov method are used in the investigation. In Section 4, based on a state observer, the output feedback control is designed to make the closed-loop system ISS stable. The convergence of the observer system is also proved. The numerical simulations are presented in Section 5 for illustration of the effectiveness of our methods. Some concluding remarks are presented in Section 6.

### 2 Well-posedness of system (1)

In this section, we show the well-posedness of open-loop system (1) in the Hilbert space \(\mathcal{H} = \mathbb{R}^n \times L^2(0,1)\) with the inner product

\[\langle F_1, F_2 \rangle = \int_0^1 f_1(x)f_2(x) \, dx,\]

where \(F_i = [X_i, f_i] \in \mathcal{H}, i = 1, 2\). Define operator \(\mathcal{A}\) as follows:

\[\mathcal{A}[X, f] = [AX + Bf(0), f' + cf]\]

where \(X(0) = 1, 2\). Define operator \(\mathcal{A}\) as follows:

\[
\begin{align*}
\mathcal{A}[X, f] &= [AX + Bf(0), f' + cf] \quad \forall[X, f] \in D(\mathcal{A}), \\
D(\mathcal{A}) &= \left\{ [X, f] \in \mathbb{R}^n \times H^2(0,1) \left| f(0) = CX, f'(1) = 0 \right. \right\}.
\end{align*}
\tag{2}
\]

Then, system (1) can be written as an evolution equation in \(\mathcal{H}\)

\[
\frac{d}{dt}[X(t), u(t)] = \mathcal{A}[X(t)] + B_1U(t) + B_2d(t),
\tag{3}
\]

where

\[
B_1 = [0, \delta(x-1)]^T, \quad B_2 = [B_0, 0]^T.
\tag{4}
\]

It is obvious that \(B_1\) is an unbounded control operator and \(B_2\) is a bounded operator.

**Lemma 1:** Let \(\mathcal{A}\) be defined by (2). Then \(\mathcal{A}^{-1}\) exists and is compact on \(\mathcal{H}\). Hence, \(\sigma(\mathcal{A})\), the spectrum of \(\mathcal{A}\), consists of isolated eigenvalues only.

**Proof:** For any \([Y, g] \in \mathcal{H}\), we need to find \([X, f] \in D(\mathcal{A})\) such that \(\mathcal{A}[X, f] = [Y, g]\), which results in

\[
\begin{align*}
AX + Bf(0) &= Y, \\
f' + cf &= g, \\
f(0) &= CX, \quad f'(1) = 0.
\end{align*}
\]

A direct computation gives the unique solution

\[
\begin{align*}
X &= A^{-1}Y - Bf(0), \\
f(s) &= h(s) + \delta \cos \sqrt{c}(1 - s), \\
h(s) &= \frac{1}{\sqrt{c}} \int_0^s \sin \sqrt{c} \delta(1 - s)g(s) \, ds, \\
s &= \frac{CA^{-1}Y - Bh(0)}{\sqrt{c} \sin \sqrt{c} + CA^{-1}B \cos \sqrt{c}}, \\
f(0) &= h(0) + k \cos \sqrt{c},
\end{align*}
\]

where we have used the assumptions: \(A^{-1}\) exists and \(\sqrt{c} \sin \sqrt{c} + CA^{-1}B \cos \sqrt{c} \neq 0\).

Hence, we get the unique solution \([X, f] \in D(\mathcal{A})\) and \(\mathcal{A}^{-1}\) exists. The Sobolev embedding theorem [33, Theorem 4.12, p. 85] implies that \(\mathcal{A}^{-1}\) is compact on \(\mathcal{H}\). The proof is complete. \(\square\)

Now we consider the adjoint \(\mathcal{A}^*\), which has the following form:

\[
\begin{align*}
\mathcal{A}^*[Y, g] &= [A^*Y - C^*g(0), s' + cg] \quad \forall[Y, g] \in D(\mathcal{A}^*), \\
D(\mathcal{A}^*) &= \left\{ [Y, g] \in \mathbb{R}^n \times H^2(0,1) \left| Y(0) = -B^*Y, \quad Y(1) = 0 \right. \right\}.
\end{align*}
\tag{5}
\]

**Lemma 2:** Let \(\mathcal{A}^*\) be given by (5). Then the eigenvalues \(\sigma(\mathcal{A}^*) = \{\lambda_k^*, k \in \mathbb{N}\}\) and corresponding eigenvectors \(\{\Phi_k^*(x) = [X_k^*, f_k^*], k \in \mathbb{N}\}\) of \(\mathcal{A}^*\) have the following asymptotic expressions:

\[
\begin{align*}
\lambda_k^* &= c - k^2 \pi^2 + \Theta(k^{-1}), \\
X_k^* &= \Theta(k^{-1}), \quad \Theta(k^{-1}), \ldots, \Theta(k^{-1})^T, \\
f_k^*(x) &= \cos k \pi (1 - x) + \Theta(k^{-1}),
\end{align*}
\tag{6}
\]

for sufficiently large positive integers \(k\), and \(\Theta(k^{-1})\) denotes that there is a constant \(M_\Theta\) such that \(\Theta(k^{-1}) \leq M_\Theta k^{-1}\). Then there is a set of generalised eigenfunctions of \(\mathcal{A}^*\), which forms a Riesz basis for \(\mathcal{H}\).
Proof: By $A^*[X^*, f^*] = \lambda^*[X^*, f^*]$, we have

$$
\begin{align*}
\left\{ \begin{array}{l}
A^*X^* - C^*f^*(0) = \lambda^*X^*, \\
(f^*)'(x) = (\lambda^* - c)f^*(x), \\
(f^*)(0) = -B^*X^*, & (f^*)(1) = 0,
\end{array} \right.
\end{align*}
$$

(7)

and if only if

$$
f^*(x) = \frac{B^*X^*}{\sqrt{\lambda^* - c} \sinh \sqrt{\lambda^* - c}(1 - x)}
$$

(8)

with $X^*$ satisfying

$$
\left\{ \begin{array}{l}
\lambda^*X^* - A^* + \frac{C^*B^*}{\sqrt{\lambda^* - c} \coth \sqrt{\lambda^* - c}}X^* = 0.
\end{array} \right.
$$

(9)

Denote

$$
\Delta(\lambda^*) = \det \left( \lambda^*I - A^* + \frac{C^*B^*}{\sqrt{\lambda^* - c} \coth \sqrt{\lambda^* - c}} \right).
$$

(10)

Then $\sigma(\mathcal{A}^*) \subseteq C[\Delta(\lambda^*) = 0]$. We know

$$
\Delta(\lambda^*) = \coth \sqrt{\lambda^* - c} \eta_0 - (\lambda^* - c) + \xi_0(\lambda^* - c),
$$

where $\eta_0, (\lambda^* - c)$ and $\xi_0(\lambda^* - c)$ are polynomials of degree $n - 1$ and $n$, respectively. By $\Delta(\lambda^*) = 0$, it follows that

$$
\sinh \sqrt{\lambda^* - c} = \sinh \sqrt{\lambda^* - c} + o( (\lambda^* - c)^{-1}) = 0.
$$

(11)

From Rouche's theorem, the solutions of (11) can be expressed as

$$
\lambda^* = c - k^2x^2 + \Theta(k^-),
$$

$k \in \mathbb{N}$ and $k \to \infty$.

From (7) and (8), we have

$$
\begin{align*}
X^* = [\Theta(k^-), \Theta(k^-), \ldots, \Theta(k^-)]^T, \\
(f^*)'(x) = \cos k\pi(1 - x) + \Theta(k^-),
\end{align*}
$$

for sufficiently large positive integers $k$.

Let $\{\varepsilon_i, i = 1, 2, \ldots, n\}$ be a basis for $\mathbb{R}^n$ and $W_k(x) = [0, \cos k\pi(1 - x)] \in \mathcal{H}$. Then

$$
\left\{[\varepsilon_i, 0], W_k(x), i = 1, 2, \ldots, n, k \in \mathbb{N}\right\}
$$

form an orthogonal basis for $\mathcal{H}$. Since $\Phi_k^*(x) = [X_k^*, f_k^*(x)]^T$, $k \in \mathbb{N}$, then there exists some sufficiently large positive integer $N$, such that

$$
\sum_{k > N} \|W_k - \Phi_k^*\|_\mathcal{H} = \sum_{k > N} \Theta(k^-) < \infty.
$$

By Theorem 6.3 of [34], a modified classical Barri's theorem, there is a set of generalised eigenfunctions of $A^*$ which forms a Riesz basis for $\mathcal{H}$.

Let $B^*_k$ be the adjoint operator of $B_k$ given by

$$
B^*_k[X^*, f^*] = f^*(1), \quad \forall [X^*, f^*] \in \mathcal{H}.
$$

(12)

We have the well-posedness of system (3) as the following Proposition 1.

**Proposition 1:** $\mathcal{H}$ generates a $C_0$-semi-group $e^{\mathcal{H}t}$ on $\mathcal{H}$ and system (3) is well-posed in the sense that for initial datum $[X(0), u(\cdot, 0)] \in \mathcal{H}$ and $U, d(t) \in L^2_{loc}(0, \infty)$, system (3) admits a unique solution $[X(t), u(\cdot, t)] \in C(0, \infty; \mathcal{H})$.

$$
\begin{align*}
[X(t), u(\cdot, t)] &= e^{\mathcal{H}t}[X(0), u(\cdot, 0)] + \int_0^t e^{\mathcal{H}(t-r)}B_kU(r)\,dr \\
&\quad + \int_0^t e^{\mathcal{H}(t-r)}B_k\,dr
\end{align*}
$$

(13)

and for $T > 0$, there exists $C_T > 0$ such that

$$
\| [X(t), u(\cdot, t)] \|^2 \leq C_T \| [X(0), u(\cdot, 0)] \|^2 + \int_0^T \| U(r) \|^2 + \| d(r) \|_d \, dr.
$$

(14)

**Proof:** By Lemma 2, $\mathcal{H}$ generates a $C_0$-semi-group $e^{\mathcal{H}t}$ on $\mathcal{H}$, so does for $\mathcal{A}$. We write $A \simeq B$ instead of $c_iA \leq B \leq c_jA$ for brevity if not need to use explicitly the constants $c_i$, $c_j$. For any $Z_0 = [X_0^*, u_0^*] \in \mathcal{H}$, assume that

$$
Z_0 = \sum_{k \in \mathbb{N}} a_k \Phi_k^*, \quad \| Z_0 \|_\mathcal{H} \leq \sum_{k \in \mathbb{N}} |a_k|^2,
$$

where $(a_k)$ is a sequence in $\ell^2$. Then we have

$$
\| [X^*(t), u^*(x, t)] \|^2 = \| e^{\mathcal{H}t}Z_0 \|^2 \leq \sum_{k \in \mathbb{N}} |a_k| e^{\lambda_k t} f_k^*(1)^T f_k^*(1) \|.
$$

By (6), $\lambda_k = c - k^2x^2 + \Theta(k^-)$ and $f_k^*(1) = 1 + \Theta(k^-)$. Hence, for $T > 0$, by the Cauchy–Schwarz inequality, we have

$$
\int_0^T \| B_k^* e^{\lambda_k T} Z_0 \| \, dt \leq \int_0^T \sum_{k \in \mathbb{N}} |a_k| e^{-\lambda_k T} \sum_{k \in \mathbb{N}} e^{\lambda_k T} |f_k^*(1)|^2 \, dt \leq C_T \sum_{k \in \mathbb{N}} |a_k|^2 \leq D_T \| Z_0 \|_\mathcal{H},
$$

for some constant $C_T$ and $D_T$ that depend only on $T$. This shows that $B_k^* \mathcal{H}$ is admissible to $e^{\mathcal{H}t}$. Thus, $B_k$ is admissible to $e^{\mathcal{H}t}$. Due to the fact $B_k$ is bounded, system (3) admits a unique solution $[X, u] \in C(0, \infty; \mathcal{H})$ given by (13) and (14). The proof is complete.

**3 State feedback control for system (1)**

In this section, we show the ISS property of system (1) under a state feedback control designed by using the backstepping method.

**3.1 Backstepping transformation**

We introduce a backstepping transformation for $[X, u] \mapsto [X, w]$ (see [2, chapter 15, p. 256])

$$
\begin{align*}
X(t) &= X(t), \\
w(t, x) &= U(t, x) - \int_0^t q(x, y)u(y, t)\,dy - \gamma(x)X(t),
\end{align*}
$$

(15)

where
The inverse of the transformation (21) has the form
\[ q(x, y) = \int_0^{x-y} \gamma(\sigma)Bd\sigma, \] (16)
\[ \gamma(x) = [K, C, K(A - cI)] \begin{bmatrix} 0 & -BC \\ 0 & A - cI \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}, \] (17)
and \( I \) represents an \( n \times n \) identity matrix. This transformation brings system (1) into the following equivalent intermediate system [\( X, w \)]:
\[
\begin{aligned}
X(t) &= (A + BK)X(t) + Bw(0, t) + Bd(t), \\
w(x, t) &= w(x, t) + \int_0^t k(x, y)y(w, t) dy + \psi(x)X(t),
\end{aligned}
\] (18)
where \( K \) is chosen such that \( A + BK \) is Hurwitz and has real negative eigenvalues. The transformation (15) is invertible, and the inverse transformation \([X, w] \mapsto [X, u]\) has the form
\[
\begin{aligned}
X(t) &= X(t), \\
u(x, t) &= w(x, t) + \int_0^t k(x, y)y(w, t) dy + \psi(x)X(t),
\end{aligned}
\] (19)
where
\[
\begin{aligned}
k(x, y) &= \int_0^{x-y} \psi(\sigma)Bd\sigma, \\
\psi(x) &= [K, C][0 \ A - BK - cI] \begin{bmatrix} I \\ 0 \end{bmatrix}.
\end{aligned}
\] (20)
Next, we use transformation \([X, w] \mapsto [X, z]\) in the form (see [35, chapter 4, p. 30])
\[
\begin{aligned}
X(t) &= X(t), \\
z(x, t) &= w(x, t) - \int_0^x k(x, y)y(w, t) dy,
\end{aligned}
\] (21)
where
\[ k(x, y) = -cx_1 \frac{J_1(\sqrt{x^2 - y^2})}{\sqrt{x^2 - y^2}}, \quad 0 \leq y \leq x \leq 1, \] (22)
and \( I_1 \) is the modified Bessel function
\[ I_1(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+1}}{n!(n+1)!}. \]
Then, we obtain the target system
\[
\begin{aligned}
X(t) &= (A + BK)X(t) + Bz(0, t) + Bd(t), \\
z(x, t) &= z(x, t) + B_1(x)d(t), \\
z_0(t) &= 0, \\
z_1(t) &= -z_1(t), 
\end{aligned}
\] (23)
where
\[ B_1(x) = \left[ \int_0^x k(x, y)y(w, t) dy - \gamma(x) \right]B_1. \] (24)
The inverse of the transformation (21) has the form
\[
\begin{aligned}
X(t) &= X(t), \\
w(x, t) &= z(x, t) + \int_0^x p(x, y)z(y, t) dy,
\end{aligned}
\] (25)
where
\[ p(x, y) = -cx_1 \frac{J_1(\sqrt{x^2 - y^2})}{\sqrt{x^2 - y^2}}, \quad 0 \leq y \leq x \leq 1, \] (26)
and \( J_i \) is a Bessel function
\[ J_i(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x/2)^{2n+i}}{n!(n+1)!}. \]
From the boundary condition in (23), we have
\[
\begin{aligned}
z_0(1) &= w(1, t) - k(1, 1)w(1, t) - \int_0^1 k_1(y)w(y, t) dy \\
&= -z(1) = -w(1, t) + \int_0^1 k_1(y)w(y, t) dy,
\end{aligned}
\] (27)
which gives the control
\[
\begin{aligned}
U(t) &= \int_0^1 q_1(y)w(y, t) dy + \gamma(1)X(t) \\
&+ \int_0^1 k_1(y)w(y, t) dy - \int_0^1 q_1(y, y)u(y, t) d\tau - \gamma(1)X(t) \\
&+ \int_0^1 k_1(y)w(y, t) dy - \int_0^1 q_1(y, y)u(y, t) d\tau - \gamma(1)X(t)
\end{aligned}
\] (28)
and hence
\[
\begin{aligned}
l(x, t) &= w(1, t) - k(1, 1)w(1, t) - \int_0^1 k_1(y)w(y, t) dy \\
&= -z(1) = -w(1, t) + \int_0^1 k_1(y)w(y, t) dy,
\end{aligned}
\] (29)
which gives the control
\[
\begin{aligned}
U(t) &= \int_0^1 q_1(y)w(y, t) dy + \gamma(1)X(t) \\
&+ \int_0^1 k_1(y)w(y, t) dy - \int_0^1 q_1(y, y)u(y, t) d\tau - \gamma(1)X(t) \\
&+ \int_0^1 k_1(y)w(y, t) dy - \int_0^1 q_1(y, y)u(y, t) d\tau - \gamma(1)X(t)
\end{aligned}
\] (30)
and hence \( \sigma(\mathcal{A}) \) exists and is compact on \( \mathcal{H} \) and hence \( \sigma(\mathcal{A}) \), the spectrum of \( \mathcal{A} \) consists of isolated eigenvalues with finite algebraic multiplicity only.

**Lemma 3**: Let \( \mathcal{A} \) be given by (28). Then, \( \mathcal{A}^{-1} \) exists and is compact on \( \mathcal{H} \) and hence \( \sigma(\mathcal{A}) \), the spectrum of \( \mathcal{A} \) consists of isolated eigenvalues with finite algebraic multiplicity only.

**Proof**: For any given \([X, z] \in \mathcal{H}\), solve
\[
\mathcal{A}[X, z] = [(A + BK)X + Bz(0), z'] = [X, z].
\] (31)
We get
\[
\mathcal{A}[X, z] = [(A + BK)X + Bz(0), z'] = [X, z].
\] (32)
where

\[
\begin{align*}
\{A + BK\}X + Bz(0) = X_c, \\
z(0) = 0, \quad z(1) = -z(1),
\end{align*}
\]

with the solution

\[
\begin{align*}
x = (A + BK)^{-1}(X_c - Bz(0)), \\
z(x) = \int_0^x (\xi - 2)z_0(\xi) d\xi + \int_0^1 (x - \xi)z_0(\xi) d\xi.
\end{align*}
\]

Hence, we get the unique \([X, z] \in D(\sigma)\). \(\sigma^{-1}\) exists and is compact on \(\mathcal{H}\) by the Sobolev embedding theorem [33, Theorem 4.12, p. 85]. Therefore, \(\sigma(\sigma)\) consists of isolated eigenvalues of finitely algebraic multiplicity only. □

Now, we are in a position to consider the eigenvalue problem of \(\sigma\). By \(\sigma Y = \lambda Y\), where \(Y = [X, z]\), we have

\[
\begin{align*}
(A + BK)X + Bz(0) &= \lambda X, \\
z(0) &= 0, \\
z(1) &= -z(1).
\end{align*}
\]

Let \(\{\lambda^m, m \in \mathbb{N}\}\) and \(\{z^m(x), m \in \mathbb{N}\}\) satisfy the second and third equations in (31) with the following asymptotic expressions:

\[
\begin{align*}
\lambda^m &= -m^2 + \Theta(m^{-3}), \\
z^m(x) &= \cos m \pi x + \Theta(m^{-1}), \quad m > N. \tag{32}
\end{align*}
\]

**Lemma 4:** Let \(\sigma\) be given by (28), let \(\lambda^m, j = 1, 2, \ldots, n\) be the simple eigenvalue of \(A + BK\) with the corresponding eigenvector \(X^m\), and assume that

\[
\lambda^m \notin \{\lambda^j, m \in \mathbb{N}\}, \quad j = 1, 2, \ldots, n, \quad (33)
\]

where \(\lambda^m\) is defined in (32). Then, \(\sigma\) has the eigenvalues

\[
\{\lambda^m, j = 1, 2, \ldots, n\} \cup \{\lambda^m, m \in \mathbb{N}\}, \tag{34}
\]

and the corresponding eigenfunctions with respect to \(\lambda^m\) and \(\lambda^m\), respectively

\[
\begin{align*}
Z^m_j &= [X^m_j, 0], \\
Z^m_j(x) &= [X^m_j, z^m_j(x)] \tag{35},
\end{align*}
\]

where

\[
X^m_c = [\lambda^m I - (A + BK)]^{-1}B, \quad m \in \mathbb{N}, \tag{36}
\]

and \(z^m_j(x)\) is defined in (32). Moreover, \([Z^m_j, Z^m_j(x), j = 1, 2, \ldots, n, m \in \mathbb{N}\) forms a Riesz basis for \(\mathcal{H}\).

**Proof:** Let \(\lambda = \lambda^m\) and substitute \(X^m_j\) into the first equation of (31). Then, we have \(z(0) = 0\). Moreover

\[
\begin{align*}
z'(x) &= \lambda z(x), \\
z(0) &= \lambda z(0) = 0, \\
z(1) &= \lambda z(1) = -z(1),
\end{align*}
\]

only has a trivial solution \(z(x) = 0\). Thus, we get that \(\lambda^m, j = 1, 2, \ldots, n\) are eigenvalues of \(\sigma\) with corresponding eigenfunctions \([X^m_j, 0]\).

A simple calculation shows that the eigenvalue problem

\[
\begin{align*}
z'(x) &= \lambda z(x), \\
z(0) &= 0, \\
z(1) &= \lambda z(1),
\end{align*}
\]

has non-trivial solutions

\[
\lambda^m = \lambda^m_c(x), \quad m \in \mathbb{N},
\]

where \(\lambda^m_c\) and \(z^m_c(x)\) are given by (32). By the first equation of (31), we have

\[
X^m_c = [\lambda^m I - (A + BK)]^{-1}B.
\]

Therefore, \(\{\lambda^m, m \in \mathbb{N}\}\) are eigenvalues of \(\sigma\) with the corresponding eigenfunctions \([X^m_c, z^m_c(x)]\).

It is noted that \([X^m_j, j = 1, 2, \ldots, n]\) and \([z^m_j(x) = \cos m \pi x, m \in \mathbb{N}\) form two orthogonal bases for \(\mathbb{R}^n\) and \(L^2(0, 1)\), respectively. Define

\[
F_j = [X^m_j, 0], \quad F_m(x) = [0, z^m_j(x)].
\]

We have \([F_j, F_m(x), j = 1, 2, \ldots, n, m \in \mathbb{N}\) forming an orthogonal basis for \(\mathcal{H}\). It follows from (35) that

\[
\begin{align*}
\sum_{j=1}^n \|Z^m_j - F_j\|^2 + \sum_{m=0}^{\infty} \|Z^m_m(x) - F_m(x)\|^2 \\
&= \sum_{m=0}^n \|\lambda^m I - (A + BK)^{-1}B\|^2 \|Z^m_m\|^2 \\
&\quad + \sum_{m=N}^{\infty} \|z^m_m(x) - \cos m \pi x\|^2_{L^2(0, 1)} \\
&\quad + \sum_{m=N}^{\infty} \Theta(m^{-1}), \tag{37}
\end{align*}
\]

where \(\| \cdot \|_{\mathcal{H}}\) denotes the norm of a vector in \(\mathbb{R}^n\) and

\[
\|\lambda^m I - (A + BK)^{-1}B\|^2 \|Z^m_m\|^2 = B^T([\lambda^m I - (A + BK)]^{-1}B) = \|\lambda^m I - (A + BK)^{-1}B\|_{\mathcal{H}}^2.
\]

where

\[
Y = [I - \frac{1}{\lambda^m}(A + BK)]^{-1}(I - \frac{1}{\lambda^m}(A + BK))^T.
\]

Then for any \(m > N\), we have

\[
\left|I - \frac{1}{\lambda^m}(A + BK)\right|^{-1} = I + \Theta(m^{-3}).
\]

Thus

\[
\begin{align*}
\sum_{j=1}^n \|Z^m_j - F_j\|^2 + \sum_{m=0}^{\infty} \|Z^m_m(x) - F_m(x)\|^2 \\
&\leq \sum_{m=0}^{N} \|\lambda^m I - (A + BK)^{-1}B\|_{\mathcal{H}}^2 \\
&\quad + \sum_{m=N}^{\infty} \|B\|_{\mathcal{H}}^2 \left(I + \Theta(m^{-3})\right) \\
&\quad + \sum_{m=N}^{\infty} \|z^m_m(x) - \cos m \pi x\|^2_{L^2(0, 1)} \\
&\quad + \sum_{m=N}^{\infty} \Theta(m^{-3}) < \infty.
\end{align*}
\]

Then, by using Bari’s theorem, \([Z^m_j, Z^m_j(x), j = 1, 2, \ldots, n, m \in \mathbb{N}\) forms a Riesz basis for \(\mathcal{H}\). The proof is completed. □
Proposition 2: Let \( \mathcal{A} \) be given by (28). Then we have the following assertions:

i. \( \mathcal{A} \) generates a \( C_0 \)-semi-group \( e^{\mathcal{A}t} \) on \( \mathcal{H} \).

ii. The spectrum-determined growth condition \( \omega(\mathcal{A}) = s(\mathcal{A}) \) holds true for \( e^{\mathcal{A}t} \), where \( \omega(\mathcal{A}) = \lim_{t \to \infty} \frac{1}{t} \| e^{\mathcal{A}t} \| \) is the growth bound of \( e^{\mathcal{A}t} \), and \( s(\mathcal{A}) = \text{sup} \{ \text{Re} \lambda \mid \lambda \in \sigma(\mathcal{A}) \} \) is the spectral bound of \( \mathcal{A} \).

iii. The \( C_0 \)-semi-group \( e^{\mathcal{A}t} \) is exponentially stable in the sense

\[
\| e^{\mathcal{A}t} \| \leq M_t e^{-\mu t},
\]

where \( M_t > 0 \) and \( \mu > 0 \) are positive constants.

**Proof:** Since \( A + BK \) is Hurwitz, from (32), there exists a constant \( \mu > 0 \) such that

\[
\max \{ \text{Re} \lambda_j, \lambda_m^j, j = 1, \ldots, n, m \in \mathbb{N} \} = -\mu < 0. \quad (38)
\]

By Lemmas 3 and 4 and Theorem 2.3.5 of [37, p. 41], \( \mathcal{A} \) generates a \( C_0 \)-semi-group \( e^{\mathcal{A}t} \) in \( \mathcal{H} \), and the spectrum-determined growth condition \( \omega(\mathcal{A}) = s(\mathcal{A}) \) holds true for \( e^{\mathcal{A}t} \). Finally, by the eigenvalues of \( \mathcal{A} \) given by (32), (33), (34) and (38), there exists a positive constant \( M_t > 0 \) such that

\[
\| e^{\mathcal{A}t} \| \leq M_t e^{-\mu t}, \quad \forall t > 0.
\]

The proof is completed. \( \square \)

### 3.3 Input-to-state stability

Now we are in a position to consider the input-to-state stability for system (1).

**Theorem 1:** For every \( |d(t)| \leq \Delta \) and \( [X_0, u_0] \in \mathcal{H} \), with the feedback control \( U(t) \) given by (27), the closed-loop system of (1) has a unique solution \( [X, u] \in C(0, +\infty, \mathcal{H}) \) and satisfies the exponential ISS property, i.e. there are positive constants \( D_1, D_2, \) and \( D_3 \), independent of \( d(t) \) and \([X_0, u_0], \) such that

\[
\| X(t) \| + \| u(t) \| \leq D_1 e^{-\beta t} \| X_0 \| + \| u_0 \| + D_2 \Delta. \quad (39)
\]

where \( \cdot \| \cdot \| \) denote the norms in \( \mathbb{R}^n \) and \( L^1(0,1) \), respectively.

**Proof:** We consider the Lyapunov function

\[
V = X^T PX + \frac{a}{2} \| z \|^2 , \quad (40)
\]

where matrix \( P = P^T > 0 \) is the solution of the Lyapunov equation

\[
P(A + BK) + (A + BK)^T P = -Q \quad (41)
\]

for some \( Q = Q^T > 0 \), and the parameter \( a > 0 \) will be chosen later. It is easy to show, using (15), (19), (21) and (25), that there exist positive constants \( a, \beta_1, \beta_2, \) and \( \delta \), such that

\[
\| z \|^2 \leq \alpha_1 \| u \|^2 + \alpha_2 \| X \|^2 , \quad (42)
\]

Hence

\[
\delta \| X \|^2 + \| u \|^2 \leq V \leq \delta \| X \|^2 + \| u \|^2. \quad (43)
\]

Taking the time derivative of the Lyapunov function along the solution of the ODE-PDE system (23) and by means of Young’s inequality \( ab \leq (a^2/2) + (b^2/2) \), we get

\[
V' = -X^T Q X + 2X^T P B z_0 + 2X^T P B_d d
\]

\[
\leq -\frac{\delta}{2} \| X \|^2 + \frac{\delta}{2} \| z \|^2 + a \int_0^t (\| z \|_B(t)) \, dt
\]

\[
\leq -\frac{\delta}{2} \| X \|^2 + \frac{\delta}{2} \| z \|^2 + \frac{\delta}{2} \| z \|^2 + 2a \int_0^t (\| z \|_B(t)) \, dt.
\]

With Agmon’s inequality and Poincaré inequality, we have

\[
\| z \|^2 \leq 3 \| z \|^2 + 5 \| z \|^2, \quad \| z \|^2 \leq \frac{1}{4} \| z \|^2 - \frac{1}{4} \| z \|^2.
\]

Thus

\[
V' \leq -\frac{\delta}{2} \| X \|^2 - \frac{\delta}{2} \| X \|^2 - \frac{\delta}{2} \| z \|^2 + \frac{\delta}{2} \| z \|^2
\]

\[
\leq -\frac{\delta}{2} \| X \|^2 - \frac{\delta}{2} \| z \|^2 + \frac{\delta}{2} \| z \|^2 + \frac{\delta}{2} \| z \|^2 + 2a \int_0^t (\| z \|_B(t)) \, dt.
\]

Taking

\[
a \geq \frac{40}{\lambda_{\alpha}(Q)} \| \| z \|_B(t) \|^2,
\]

there exist \( C_1 > 0 \) and \( C_2 > 0 \), such that

\[
V(t) \leq -C_1 V(t) + C_2 \| d(t) \|^2. \quad (46)
\]

Then, we obtain

\[
V(t) \leq e^{-C_1 t} V(0) + \int_0^t e^{-C_1 (t-s)} C_2 \| d(s) \|^2 \, ds
\]

\[
\leq e^{-C_1 t} V(0) + C_2 \Delta^2 / C_1. \quad (47)
\]

where \( |d(t)| \leq \Delta \). Hence

\[
\| X(t) \|^2 + \| u(t) \|^2 \leq D_1 e^{-\beta_1 t} \| X_0 \|^2 + \| u_0 \|^2 + D_2 \Delta. \quad (48)
\]

where

\[
D_1 = \frac{\delta}{2}, \quad D_2 = C_1, \quad D_3 = \frac{C_2}{2C_1}.
\]

\( \square \)
Remark 1: When \(d(t) \equiv 0\), from Proposition 2, we know that system (23) is exponentially stable. Hence when \(d(t) \not\equiv 0\), system (23) is input-to-state stable from [21, Proposition 3] (or [27, Proposition 2.11]). In this study, we directly proved the ISS of system (23) (which is equivalent to the closed-loop system of (1)) by using the Lyapunov approach.

4 Output feedback control for system (1)

In this section, we design a state observer and an observer-based feedback control law to make the system (1) have the ISS property. To construct the observer, we need the additional assumption.

Assumption (A2) \((A,C)\) is observable.

We consider the case that \(u(0,t), CX(t)\) and \(B_1^T EX(t)\) are available for measurement: We design the following observer:

\[
\dot{\tilde{x}}(t) = A\tilde{x}(t) + LC\tilde{x}(t) - X(t) + \tilde{b}_u(t,0) \\
+ B_1\Delta \frac{B_1^T EX(t)}{B_1^T EX(t) - B_1^T EX(0)} \\
\dot{\tilde{u}}(t,0) = CX(t) + p_u u(t,0) - \tilde{u}(t,0), \\
\dot{\tilde{u}}(1,t) = U(t),
\]

where \(L\) is chosen such that \(A + LC\) is Hurwitz, \(\Delta\) is the bound of \(d(t)\), the matrix \(E = E^T > 0\) is the solution of the Lyapunov equation

\[E(A + LC) + (A + LC)^T E = -H\]

for some \(H = H^T > 0\), and the function \(p_u(x)\) and the constant \(p_u\) are observer gains to be determined.

Remark 2: In order to make the observer (49) exponentially convergent to system (1), measurements \(u(0,t), CX(t)\) and \(B_1^T EX(t)\) are necessary.

Since there is a nonlinear term in the first equation of (49), now we consider the existence of a generalised solution (see [38, Definition 2.10, p. 30]) of the ODE of (49). Define

\[
\tilde{u} := u - \tilde{u}, \quad \tilde{X} := X - \dot{X}.
\]

Set \(S(t) = B_1^T E\tilde{X}(t)\). It is noted that when \(S(t) \neq 0\) in some interval, the ODE of (49) admits a classical solution. Then \(\{B_1^T E\tilde{X}(t)\}/\{B_1^T E\tilde{X}(0)\}\) is discontinuous only on \(S(t) = 0\). Now we use the equivalent control method [38, p. 26] to find the sliding mode solution for the ODE of (49), i.e., we need to find a continuous function \(U^0\), called the equivalent control, so that

\[
\dot{\tilde{X}}(t) = A\tilde{X}(t) - LC\tilde{X}(t) + \tilde{b}_u(t,0) + B_1\Delta U^0(t).
\]

Since the equivalent control is needed only when \(S(t) = 0\) and hence \(S(t) = B_1^T E\tilde{X}(t)\), we find

\[
\begin{bmatrix}
U^0(t) = (B_1^T EB)^{-1}\Delta^{-1} B_1^T E(A + LC)\tilde{X}(t) \\
+ B_1^T EB\tilde{u}(0,0) + B_1^T EB d(t)
\end{bmatrix}, (53)
\]

\(U^0(t) \in [-1,1]\). Returning back to (49), the sliding mode is \(S(t) = B_1^T E\tilde{X}(t) = 0\) on which we have

\[
\dot{\tilde{X}}(t) = A\tilde{X}(t) - LC\tilde{X}(t) + \tilde{b}_u(t,0) + B_1\Delta U^0(t), \\
\frac{B_1^T E\tilde{X}(t)}{B_1^T E\tilde{X}(0)} = 0.
\]

Then by Theorem 2.4. of [38, p. 31], there exists a unique generalised solution to the ODE of (49).

Then, the observer error system is given by

\[
\begin{bmatrix}
\dot{\tilde{x}}(t) = (A + LC)\tilde{x}(t) + B\tilde{u}(0,t) \\
+ B_1 d(t) - B_1\Delta \frac{B_1^T E\tilde{x}(t)}{B_1^T E\tilde{x}(0)} \\
\end{bmatrix}, \quad \tilde{u}(x,t) = \tilde{u}_s(x,t) + c\tilde{u}(x,t) - p_u(t)\tilde{u}(0,t), \\
\tilde{u}(0,t) = -p_u\tilde{u}(0,t), \quad \tilde{u}(1,t) = 0.
\]

We introduce the following transformation (see [35, chapter 5, p. 54])

\[
\begin{bmatrix}
\tilde{X} = \tilde{X}, \\
\tilde{u}(x,t) = \tilde{w}(x,t) - \int_{r}^{t} \rho(x,y)\tilde{w}(y,t) dy,
\end{bmatrix},
\]

where

\[
\rho(x,y) = -c(1-y)\frac{I_1(\sqrt{c(2-x-y)(x-y)})}{\sqrt{c(2-x-y)(x-y)}}
\]

and \(I_1(x)\) is a modified Bessel function

\[
I_1(x) = \sum_{n = 0}^{\infty} \frac{(x/2)^{2n+1}}{n!(n+1)!}.
\]

The observer gains are

\[
p_u(x) = -2c\frac{I_1(\sqrt{c(2-x-y)}) + I_1(\sqrt{c(2-x+y)})}{\sqrt{c(2-x-y)}} \quad \text{and} \quad p_u = -\frac{3c}{2}.
\]

With the transformation (56), we transform the error system into the following equivalent ODE–heat system

\[
\begin{bmatrix}
\dot{\tilde{x}}(t) = (A + LC)\tilde{x}(t) + B\tilde{u}(0,t) + B_1 d(t) \\
- B_1\Delta \frac{B_1^T E\tilde{x}(t)}{B_1^T E\tilde{x}(0)} \\
\tilde{w}(x,t) = \tilde{w}_s(x,t), \\
\tilde{w}(0,t) = c\tilde{w}(0,t), \quad \tilde{w}(1,t) = 0.
\end{bmatrix}
\]

Proposition 3: For every \(|d(t)| \leq \Delta\) and \([\tilde{X}_w, \tilde{u}_w] \in \mathbb{R}^n \times L^2(0,1)\), the cascade ODE–heat system (55) is exponentially stable, i.e., there exist positive constants \(D_1\) and \(D_2\) independent of \(d(t)\) and \([\tilde{X}_w, \tilde{u}_w]\), such that

\[
\tilde{X}(t) + \|\tilde{u}(t)\| \leq D_1 e^{-D_2 t} \left[\|\tilde{X}_w\| + \|\tilde{u}_w\|\right].
\]

where \(1\) and \(\| \cdot \|\) denote the norms in \(\mathbb{R}^n\) and \(L^2(0,1)\), respectively.

Proof: We consider the Lyapunov function

\[
\tilde{V}(t) = \tilde{X}^T E\tilde{X} + \frac{c}{2} \|\tilde{w}\|^2.
\]
where the parameter $\alpha > 0$ will be chosen later. Similar to the proof of Theorem 1, there exist positive constants $\bar{\eta}$ and $\underline{\eta}$ such that
\[
\bar{\eta}(\| \hat{X} \| + \| \hat{\mu} \|) \leq \bar{\eta}(\| \hat{X} \| + \| \hat{\mu} \|).
\] (62)

Taking the time derivative of the Lyapunov function along the solution of the ODE–PDE system (59) and by means of Young's inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we get
\[
\begin{align*}
\dot{V} & = -\hat{X}^T H \hat{X} + 2 \hat{X}^T E B \hat{w}(0) + 2 \hat{X}^T E B \hat{d} \\
& = -2 \hat{X}^T E B \Delta \frac{B_1^T E \hat{X}(t)}{B_1^T E \hat{X}(t)} + \alpha \| \hat{w}(0) \| - \alpha \| \hat{\mu} \|
\end{align*}
\] (63)
\[
\leq - \frac{\lambda_{\text{max}}(H)}{2} \| \hat{X} \|^2 - \left( \alpha c - \frac{2|EB|^2}{\lambda_{\text{max}}(H)} \right) \| \hat{w}(0) \|^2 - 2(\Delta - \| \hat{d}(t) \|) \| \hat{X} \|^2
\] (64)
\[
\leq - \frac{\lambda_{\text{max}}(H)}{2} \| \hat{X} \|^2 - \left( \alpha c - \frac{2|EB|^2}{\lambda_{\text{max}}(H)} \right) \| \hat{w}(0) \|^2 - \frac{\alpha}{4} \| \hat{\mu} \|^2.
\]

With Poincaré inequality, we have
\[
- \| \hat{\mu} \|^2 \leq \frac{1}{2} \| \hat{w}(0) \|^2 - \frac{1}{4} \| \hat{\mu} \|^2.
\]

From $|\hat{d}(t)| \leq \Delta$, we have
\[
\dot{V} \leq - \frac{\lambda_{\text{max}}(H)}{2} \| \hat{X} \|^2 - \frac{2|EB|^2}{\lambda_{\text{max}}(H)} \| \hat{w}(0) \|^2
\]
\[
- \frac{\alpha}{4} \| \hat{\mu} \|^2.
\]
(65)

Taking $\alpha \geq \| (2|EB|^2)/(c - (1/2)\lambda_{\text{max}}(H)) \|$, there exists a constant $C_1 > 0$ such that
\[
\dot{V}(t) \leq -C_1 \| \hat{V} \|.
\]

From (62), we have
\[
\| \hat{X}(t) \|^2 + \| \hat{\mu}(t) \|^2 \leq D_1 e^{-\alpha t} \| \hat{X}_0 \|^2 + \| \hat{\mu}_0 \|^2.
\] (66)

where
\[
D_1 = \frac{\bar{\eta}}{2}, \; D_2 = C_1.
\]

This completes the proof. □

Replacing $X(t)$ and $u(y, t)$ with $\hat{X}(t)$ and $\hat{\mu}(y, t)$ in the feedback control (27), respectively, we obtain an output feedback control law as follows:
\[
U(t) = \int_0^t q_1(1, y) \hat{u}(y, t) \, dy + \gamma(1) \hat{X}(t)
\]
\[
+ \int_0^t k_1(1, y) \hat{u}(y, t) - \int_0^t q(y, t) \hat{u}(y, t) \, dt - \gamma(1) \hat{X}(t) \, dy
\]
\[
+ \int_0^t k_1(1, y) \hat{u}(y, t) - \int_0^t q(y, t) \hat{u}(y, t) \, dt - \gamma(1) \hat{X}(t) \, dy
\]
\[
\quad - \frac{1 - c}{2} \| \hat{u}(1, t) - \int_0^t q(1, y) \hat{u}(y, t) \, dy - \gamma(1) \hat{X}(t) \|.
\] (67)

Then, \( X, \hat{X}, \hat{u} \in \mathbb{R}^n \times L^2(0, 1) \), we define the Lyapunov function as follows:
\[
\begin{align*}
\dot{V}(t) & = \dot{\hat{X}}(t) \left[ + \| \dot{\hat{u}}(t) \|^2 + \| \hat{X}(t) \|^2 + \| \hat{\mu}(t) \|^2 \right] \\
& \leq C e^{-C_1 t} \| \hat{X}(t) \|^2 + \| \hat{\mu}(t) \|^2 + \| \hat{X}_0 \|^2 + \| \hat{\mu}_0 \|^2 + C_1 \Delta.
\end{align*}
\] (68)

where $\| \cdot \|$ and $\| \cdot \|$ denote the norms in $\mathbb{R}^n$ and $L^2(0, 1)$, respectively.

\textbf{Proof:} The transformation
\[
\begin{align*}
\dot{\hat{X}}(t) & = \dot{\hat{X}}(t), \\
\dot{\hat{u}}(x, t) & = \dot{\hat{u}}(x, t) - \int_0^t q(x, y) \hat{u}(y, t) \, dy - \gamma(x) \hat{X}(t),
\end{align*}
\] (69)

converts system (49) into
\[
\begin{align*}
\dot{\hat{X}}(t) & = (A + BK) \hat{X}(t) + B \hat{w}(0, t) - LC \hat{X}(t) \\
& + \frac{B_1^T E \hat{X}(t)}{B_1^T E \hat{X}(t)} \| \hat{w}(0) \| - \gamma(x) \hat{X}(t)
\end{align*}
\] (70)
\[
\begin{align*}
\dot{\hat{w}}(x, t) & = \hat{w}(x, t) + \gamma(x) \hat{X}(t)
\end{align*}
\] (71)

\begin{align*}
\dot{\hat{w}}(1, t) & = U(t) - \int_0^t q(x, y) \hat{w}(y, t) \, dy - \gamma(1) \hat{X}(t).
\end{align*}

where $q(x, y)$ and $\gamma(x)$ are given by (16) and
\[
\begin{align*}
B_4(x) & = q(x, 0) + \gamma(x) L, \\
B_4(x) & = p_0 q(x, 0) + p_1 - \int_0^1 q(x, y) p(y) \, dy.
\end{align*}
\]

Next, using the transformation
\[
\begin{align*}
\hat{X}(t) & = \hat{X}(t), \\
\hat{z}(x, t) & = \hat{w}(x, t) - \int_0^1 k(x, y) \hat{u}(y, t) \, dy
\end{align*}
\] (72)

we formulate system (70) into
\[
\begin{align*}
\dot{\hat{X}}(t) & = (A + BK) \hat{X}(t) + B \hat{w}(0, t) - LC \hat{X}(t) \\
& + \frac{B_1^T E \hat{X}(t)}{B_1^T E \hat{X}(t)} \| \hat{w}(0) \| - \gamma(x) \hat{X}(t)
\end{align*}
\] (73)
\[
\begin{align*}
\dot{\hat{z}}(x, t) & = \hat{z}(x, t) + B_4(x) \hat{X}(t) + B_4(x) \hat{w}(0, t)
\end{align*}
\] (74)
\[
\begin{align*}
\hat{z}(1, t) & = - \hat{z}(1, t).
\end{align*}
\]

where
\[
\begin{align*}
B_4(x) & = \gamma(x) - \int_0^1 k(x, y) B_4(y) \, dy, \\
B_4(x) & = p_0 q(x, 0) - \int_0^1 k(x, y) B_4(y) \, dy.
\end{align*}
\] (74)

We define the Lyapunov function as follows:
\[
\dot{V}(t) = \dot{X}^T F \dot{X} + \frac{b}{2} \| \dot{z} \|^2 ,
\]

where \( \| \cdot \| \) represents the \( L^2 \) norm, matrix \( F = F^T > 0 \) is the solution of the Lyapunov equation

\[
F(A + BK) + (A + BK)^T F = - G
\]

for some \( G = G^T > 0 \), and the parameter \( b > 0 \) will be chosen later. Taking the time derivative of the Lyapunov function along the solution of system (73), we obtain

\[
\dot{V} = - \dot{X}^T G X + 2 \dot{X}^T FB \dot{z}(0) - 2 \dot{X}^T (LC)^T \dot{X} + 2 \dot{X}^T FB \frac{\partial}{\partial x} \int_0^t \dot{z}(x) \dot{z}(x) \, dx + b \int_0^t \dot{z}(x) \dot{z}(x) \, dx \\
\leq - \frac{\lambda_{\text{min}}(G)}{4} \| \dot{X} \|^2 + \frac{4|FB|^2}{\lambda_{\text{min}}(G)} \| \dot{z}(0) \|^2 + \frac{4|FLC|^2}{\lambda_{\text{min}}(G)} \| \dot{X} \|^2 \\
+ 4 \Delta \| FB \| \| \dot{z}(0) \|^2 + b \int_0^t \dot{z}(x) \dot{z}(x) \, dx \\
+ b \int_0^t \dot{z}(x) B(x) C \dot{X} \, dx + b \nu(0) \int_0^t \dot{z}(x) B(x) \, dx \\
+ b \int_0^t \dot{z}(x) \int_0^t k(x,y) \gamma(y) B(x) \Delta \, dy \, dx \\
- b \int_0^t \dot{z}(x) B(x) \Delta \, dy \\
\leq - \frac{\lambda_{\text{min}}(G)}{4} \| \dot{X} \|^2 + \frac{4|FLC|^2}{\lambda_{\text{min}}(G)} \| \dot{X} \|^2 + \frac{4|FB|^2}{\lambda_{\text{min}}(G)} \| \dot{z}(0) \|^2 \\
- b \| \dot{z} \|^2 + \frac{b}{5} \| \dot{z} \|^2 + 6b \int_0^t B(x) C \dot{X} \, dx \\
+ 6b \int_0^t \dot{z}(x) \, dx + b \nu(0) \int_0^t \dot{z}(x) \, dx \\
+ 6b \int_0^t \int_0^t k(x,y) \gamma(y) B(x) \Delta \, dy \, \dot{y} \, \dot{z} \, dx
\]

With Agmon's inequality and Poincaré inequality, we have

\[
\| \dot{z}(0) \|^2 \leq 3 \| \dot{z}(1) \|^2 + 5 \| \dot{z} \|^2
\]

and

\[
- \| \dot{z} \|^2 \leq \frac{1}{2} \| \dot{z}(1) \|^2 - \frac{1}{4} \| \dot{z} \|^2.
\]

Thus

\[
\dot{V} \leq - \frac{\lambda_{\text{min}}(G)}{4} \| \dot{X} \|^2 + \frac{4|FLC|^2}{\lambda_{\text{min}}(G)} \| \dot{X} \|^2 + \frac{4|FB|^2}{\lambda_{\text{min}}(G)} \| \dot{z}(0) \|^2 \\
- \frac{b}{16} \| \dot{z} \|^2 + \frac{b}{5} \| \dot{z} \|^2 + 6b \int_0^t B(x) C \dot{X} \, dx + 6b \nu(0) \int_0^t \dot{z}(x) \, dx \\
+ 6b \int_0^t \int_0^t k(x,y) \gamma(y) B(x) \Delta \, dy \, \dot{y} \, \dot{z} \, dx
\]

From the boundary condition \( \dot{z}(0) = C \dot{X} + p_0 \nu(0) \) in system (73), we have

\[
\dot{V} \leq - \frac{\lambda_{\text{min}}(G)}{4} \| \dot{X} \|^2 + \frac{4|FLC|^2}{\lambda_{\text{min}}(G)} \| \dot{X} \|^2 + \frac{4|FB|^2}{\lambda_{\text{min}}(G)} \| \dot{z}(0) \|^2 \\
- \frac{b}{16} \| \dot{z} \|^2 + \frac{b}{5} \| \dot{z} \|^2 + 6b \int_0^t B(x) C \dot{X} \, dx + 6b \nu(0) \int_0^t \dot{z}(x) \, dx \\
+ 6b \int_0^t \int_0^t k(x,y) \gamma(y) B(x) \Delta \, dy \, \dot{y} \, \dot{z} \, dx
\]

Taking \( b \geq 80|FB|^2/\lambda_{\text{min}}(G) \), there exist constants \( D_1 > 0 \), \( D_2 > 0 \) and \( D_3 > 0 \), such that

\[
\dot{V} \leq - D_1 \dot{V} + D_2 \dot{V} + D_3 \Delta^2,
\]

where \( \dot{V} \) is given by (61).

Now, we define a new Lyapunov function

\[
E(t) = \dot{V}(t) + \epsilon \dot{V}(t),
\]

where the parameter \( \epsilon > 0 \) will be determined later. Taking the time derivative of (81), we get

\[
\dot{E} \leq - \dot{V} + \epsilon \dot{V} \leq - D_1 \dot{V} - \epsilon \dot{E}(t) + D_2 \Delta^2.
\]

Taking \( \epsilon = \frac{D_2}{D_1} \), it can be obtained that

\[
\dot{E}(t) \leq - \dot{f}(t) + D_2 \Delta^2,
\]

where

\[
f = \min \{ D_1, D_2 \}
\]

Then, we have

\[
E(t) \leq e^{-ft} E(0) + \int_0^t e^{-f(t-s)} D_2 \Delta^2 \, ds \leq e^{-ft} E(0) + \frac{D_2}{f} \Delta^2.
\]

The proof is completed. \( \square \)

5 Numerical simulations

In this section, we present the numerical simulations for the ISS property of the ODE–heat system under both the state and output feedbacks. A finite method is applied to compute the solutions of the ODE and the heat equation. We take the steps for space and time as 0.02 and 0.0001, respectively, and choose \( A = 10 \), \( B = -5 \), \( K = 3 \), \( L = -4 \), \( C = 3 \), \( c = 10 \), \( B_1 = 1 \), and the disturbance \( d(t) = 0.15 \sin(2t) \).

Fig. 1 shows the states of open-loop system (1). The initial data are chosen as \( x(0) = 0 \) and \( u(x,0) = x \). It is seen that without control \( x(t) \) and \( u(x,t) \) are not stabilised in Fig. 1. When \( d(t) = 0 \), with the feedback (27) both the ODE and heat equation are stabilised to zero as shown in Fig. 2.

Fig. 3 displays the states of system (1) with the disturbance and state feedback (27). It is seen that \( X(t) \) and \( u(x,t) \) are ultimately bounded and the bounds are dependent on the bound of the disturbance \( \Delta = 0.15 \), which means system (1) is input-to-state stable.
Fig. 4 demonstrates the states of system (1) with the output feedback (67). It is seen that $X(t)$ and $u(x, t)$ are also ultimately bounded. Figs. 5 and 6 present the states of the observer (49) with the output feedback (67) and the error system (55), respectively.

The initial conditions are chosen as $X^0 = 0$, $u^0(x, 0) = x^4$, $\tilde{X}(0) = 0$, and $\tilde{u}(x, 0) = x^5$. It is seen that $X(t)$ and $u(x, t)$ are ultimately bounded, and the states of the error system are obviously convergent to zero. Combining Figs. 4–6, it is shown that under the observer-based output feedback control (67), system (1) is input-to-state stable.

6 Conclusion

In this study, we studied the input-to-state stability of an ODE–heat cascade system with Dirichlet interconnection and disturbances located on the ODE. First, we designed a state feedback control designed by using the backstepping method. Without disturbance, the closed-loop system is actually exponentially stable. Hence, when there are disturbances, the closed-loop system is ISS stable.
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8 References


Fig. 5 States of the closed-loop observer system
(a) Estimated ODE state $x(t)$, (b) Estimated PDE $u(x,t)$

Fig. 6 States of the error system (55)
(a) Error $\hat{x}(t)$, (b) Error $u(x,t)$

Then, we constructed a state observer, which was proved to be exponentially convergent to system (1). Finally, an output feedback control law was proposed and the ISS of the closed-loop system was also proved.

The ISS of the coupled ODE–PDE and PDE–PDE systems will be investigated in our future research. Systems with disturbances located at the boundaries of the PDE, in which the input operators are unbounded in the state space, will also be considered.

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