A New and Enhanced Semidefinite Relaxation for a Class of Nonconvex Complex Quadratic Problems with Applications in Wireless Communications

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Complex QP and MIMO Detection

Conventional SDP

A New and Enhanced SDP

A Tightness Result

A Global Algorithm
Consider the following nonconvex Complex Quadratic Problem:

$$\begin{align*}
\min_{x \in \mathbb{C}^n} & \quad \frac{1}{2} x^\dagger Q x + R(c^\dagger x) \\
\text{s.t.} & \quad \ell_i \leq |x_i| \leq u_i, \; i = 1, \ldots, n, \\
& \quad \arg(x_i) \in A_i, \; i = 1, \ldots, n, 
\end{align*}$$

(CQP)

where $A_i \; (i = 1, \ldots, n)$ are $n$ discrete/continuous sets.
A Special Case: MIMO Detection

- Input-output relationship of the MIMO channel:

\[ r = Hx^* + v, \]

where each entry \( x_i^* \) of \( x^* \) belongs to a finite set of symbols, i.e.,

\[ x_i^* \in \{ e^{i\theta} \mid \theta = 2j\pi/M, \ j = 0, 1, \ldots, M - 1 \}, \ i = 1, 2, \ldots, n. \]
**A Special Case: MIMO Detection**

- Input-output relationship of the MIMO channel:

  \[ r = Hx^* + v, \]

  where each entry \( x_i^* \) of \( x^* \) belongs to a finite set of symbols, i.e.,

  \[ x_i^* \in \{ e^{i\theta} \mid \theta = 2j\pi/M, \, j = 0, 1, \ldots, M - 1 \}, \, i = 1, 2, \ldots, n. \]

- \( M = 4 \) and \( M = 8 \):

![a) 4-PSK Constellation](image1.png)

![b) 8-PSK Constellation](image2.png)
MIMOf detection problem formulation:

\[
\begin{align*}
\min_{x \in \mathbb{C}^n} & \quad \frac{1}{2} \| Hx - r \|^2_2 \\
\text{s.t.} & \quad |x_i| = 1, \quad \arg(x_i) \in A, \ i = 1, \ldots, n,
\end{align*}
\]

where

\[ A = \{2j\pi/M, \ j = 0, \ldots, M - 1\}. \]
MIMO detection problem formulation:

\[
\min_{x \in \mathbb{C}^n} \frac{1}{2} \|Hx - r\|^2_2
\]

s.t. \( |x_i| = 1, \ \text{arg} (x_i) \in A, \ i = 1, \ldots, n, \)

where

\[A = \{2j\pi/M, \ j = 0, \ldots, M - 1\}.\]

“Fifty Years of MIMO Detection: The Road to Large-Scale MIMOs” by Shaoshi Yang and Lajos Hanzo in 2015
Conventional SDP Relaxation

- Problem reformulation:

\[
\begin{align*}
\min_{\mathbf{X}, \mathbf{x}} & \quad \frac{1}{2} \mathbf{Q} \cdot \mathbf{X} + \mathbf{R} \left( \mathbf{c}^\dagger \mathbf{x} \right) \\
\text{s.t.} & \quad \ell_i^2 \leq X_{ii} \leq u_i^2, \ i = 1, \ldots, n, \\
& \quad \arg (x_i) \in \mathcal{A}_i, \ i = 1, \ldots, n, \\
& \quad \mathbf{X} = \mathbf{xx}^\dagger.
\end{align*}
\]

- Conventional SDP:

\[
\begin{align*}
\min_{\mathbf{X}, \mathbf{x}} & \quad \frac{1}{2} \mathbf{Q} \cdot \mathbf{X} + \mathbf{R} \left( \mathbf{c}^\dagger \mathbf{x} \right) \\
\text{s.t.} & \quad \ell_i^2 \leq X_{ii} \leq u_i^2, \ i = 1, \ldots, n, \\
& \quad \mathbf{X} \succeq \mathbf{xx}^\dagger.
\end{align*}
\]
What Happens in (CSDP)?

- Relax $\mathbf{X} = \mathbf{x}\mathbf{x}^\dagger$ to $\mathbf{X} \succeq \mathbf{x}\mathbf{x}^\dagger$, which is equivalent to

$$
\begin{pmatrix}
\mathbf{X} & \mathbf{x} \\
\mathbf{x}^\dagger & 1
\end{pmatrix} \succeq 0.
$$

- Drop $\arg (x_i) \in A_i$ for all $i = 1, 2, \ldots, n$. 

BUT this might be LOOSE and the relaxation gap might be LARGE!

Derive more valid constraints to reduce the relaxation gap and develop an enhanced SDP relaxation.
What Happens in (CSDP)?

- Relax $X = xx^\dagger$ to $X \succeq xx^\dagger$, which is equivalent to
  \[
  \begin{pmatrix}
  X & x \\
  x^\dagger & 1
  \end{pmatrix} \succeq 0.
  \]

- Drop $\arg (x_i) \in A_i$ for all $i = 1, 2, \ldots, n$.

- **BUT** this might be **LOOSE** and the relaxation gap might be **LARGE**!

- Derive more valid constraints to reduce the relaxation gap and develop an enhanced SDP relaxation.
A Useful Lemma

Our interested problem (P):

\[
\begin{align*}
\min_{\mathbf{X}, \mathbf{x}} & \quad \frac{1}{2} \mathbf{Q} \cdot \mathbf{X} + R (\mathbf{c}^\dagger \mathbf{x}) \\
\text{s.t.} & \quad \ell_i^2 \leq X_{ii} \leq u_i^2, \quad i = 1, \ldots, n, \\
& \quad \arg (x_i) \in \mathcal{A}_i, \quad i = 1, \ldots, n, \\
& \quad \mathbf{X} = \mathbf{xx}^\dagger.
\end{align*}
\]
A Useful Lemma

Our interested problem (P):

\[
\begin{align*}
\min_{\mathbf{X}, \mathbf{x}} \quad & \frac{1}{2} \mathbf{Q} \cdot \mathbf{X} + R (\mathbf{c}^\top \mathbf{x}) \\
\text{s.t.} \quad & \ell_i^2 \leq \mathbf{X}_{ii} \leq u_i^2, \quad i = 1, \ldots, n, \\
& \arg (\mathbf{x}_i) \in \mathcal{A}_i, \quad i = 1, \ldots, n, \\
& \mathbf{X} = \mathbf{x} \mathbf{x}^\top.
\end{align*}
\]

Lemma

The feasible set of (P) can be equivalently expressed as follows: \( \mathbf{X} \succeq \mathbf{x} \mathbf{x}^\top \) and

\[
\ell_i^2 \leq \mathbf{X}_{ii} \leq u_i^2, \quad \mathbf{x}_i = r_i e^{i\theta_i}, \quad \theta_i \in \mathcal{A}_i, \quad \mathbf{X}_{ii} = r_i^2, \quad i = 1, 2, \ldots, n.
\]

The constraints \( \mathbf{x}_i = r_i e^{i\theta_i}, \quad \theta_i \in \mathcal{A}_i \) and \( \mathbf{X}_{ii} = r_i^2 \) are still not convex, but they allow for SIMPLE convex relaxations.

Below we derive convex relaxations of these two nonconvex constraints, which lead to a new tighter SDP relaxation for problem (P).
Let us first consider the nonconvex set
\[
\{(x_i, r_i) | x_i = r_i e^{i\theta_i}, \ \theta_i \in A_i \}.
\]

Let \(G_{A_i}\) be its convex envelope. Then,
- if \(A_i = [\theta_i, \bar{\theta}_i]\) with \(\bar{\theta}_i - \theta_i < \pi\), then
  \[
  G_{A_i} = \left\{ (x_i, r_i) \ | \ |x_i| \leq r_i, a_i R(x_i) + b_i I(x_i) \geq r_i (a_i^2 + b_i^2) \right\},
  \]
  where \(a_i = \frac{\cos(\theta_i) + \cos(\bar{\theta}_i)}{2}\) and \(b_i = \frac{\sin(\theta_i) + \sin(\bar{\theta}_i)}{2}\);
- if \(A_i = [\theta_i, \bar{\theta}_i]\) with \(\bar{\theta}_i - \theta_i = \pi\), then
  \[
  G_{A_i} = \left\{ (x_i, r_i) \ | \ |x_i| \leq r_i, c_i R(x_i) + d_i I(x_i) \geq 0 \right\},
  \]
  where \(c_i = \cos\left(\frac{\theta_i + \bar{\theta}_i}{2}\right)\) and \(d_i = \sin\left(\frac{\theta_i + \bar{\theta}_i}{2}\right)\);
- if \(A_i\) is a discrete set, then \(G_{A_i}\) is a polyhedral set.
An Illustration of Convex Relaxation I
Let us consider the nonconvex set

\[
\{ (X_{ii}, r_i) \mid X_{ii} = r_i^2, \ r_i \in B_i \},
\]

where \( B_i = [\ell_i, u_i] \).

Let \( F_{B_i} \) be its convex envelope. Then,

\[
F_{B_i} = \{ (X_{ii}, r_i) \mid X_{ii} \geq r_i^2, \ X_{ii} - (\ell_i + u_i)r_i + \ell_i u_i \leq 0 \}.
\]
A New and Enhanced SDP Relaxation

- A new and enhanced SDP relaxation for problem (P):

\[
\begin{align*}
\min_{x, X, r} & \quad \frac{1}{2} Q \cdot X + R(c^\top x) \\
\text{s.t.} & \quad \ell_i \leq X_{ii} \leq u_i, \quad i = 1, \ldots, n, \\
& \quad (x_i, r_i) \in G_{A_i}, \quad i = 1, \ldots, n, \\
& \quad (X_{ii}, r_i) \in F_{B_i}, \quad i = 1, \ldots, n, \\
& \quad X \succeq xx^\top.
\end{align*}
\]

(ECSDP)

- Conventional SDP relaxation:

\[
\begin{align*}
\min_{x, X} & \quad \frac{1}{2} Q \cdot X + R(c^\top x) \\
\text{s.t.} & \quad \ell_i \leq X_{ii} \leq u_i, \quad i = 1, \ldots, n, \\
& \quad X \succeq xx^\top.
\end{align*}
\]

(CSDP)
Can we say somethings on the proposed (ECSDP)? Yes!
More on (ECSDP)

Can we say something on the proposed (ECSDP)? Yes!

- A theoretical and numerical comparison of (ECSDP) and (CSDP)
- A global branch-and-bound algorithm based on (ECSDP)
**Tightness of (ECSDP) for MIMO Detection**

- MIMO detection problem is

\[
\min_{x \in \mathbb{C}^n} \frac{1}{2} \| Hx - r \|^2_2 \\
\text{s.t.} \quad |x_i| = 1, \quad \arg(x_i) \in \mathcal{A}, \quad i = 1, \ldots, n,
\]

where

\[
\mathcal{A} = \{2j\pi/M \mid j = 0, \ldots, M - 1\}.
\]

---

**Theorem (Lu-L.-Zhang-Zhang, 2017)**

For any \( M \geq 2 \):
- Suppose

\[
\lambda_{\min}(H^\dagger H) \sin(\pi/M) > \|H^\dagger v\|_\infty
\]

holds true, then (ECSDP) is tight for MIMO detection problem;
MIMO detection problem is

\[
\min_{x \in \mathbb{C}^n} \frac{1}{2} \|Hx - r\|_2^2
\]

s.t. \[|x_i| = 1, \ \arg(x_i) \in \mathcal{A}, \ i = 1, \ldots, n,\]

where

\[\mathcal{A} = \{2j\pi/M \mid j = 0, \ldots, M - 1\}.\]

**Theorem (Lu-L.-Zhang-Zhang, 2017)**

For any \(M \geq 2\):

- Suppose

\[\lambda_{\min}(H^\dagger H) \sin(\pi/M) > \|H^\dagger v\|_\infty\]

holds true, then (ECSDP) is tight for MIMO detection problem;

- But (CSDP) is generically not tight for MIMO detection problem.
The condition $\lambda_{\min} (H^\dagger H) \sin (\pi/M) > \|H^\dagger v\|_\infty$ is easily checkable.

The condition essentially requires both the constellation number and the level of the noise to be below a certain threshold.
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An open question posed by So in 2010: Is the condition

$$\lambda_{\text{min}} (H^\dagger H) > \|H^\dagger v\|_\infty$$

sufficient for the conventional (complex) SDR being tight for the MIMO detection problem with $M \geq 3$?

Our answer: No in general! Need to modify the condition and also need to develop an enhanced SDP relaxation!
More Remarks on Tightness Results

- The condition $\lambda_{\text{min}}(H^\dagger H) \sin(\pi/M) > \|H^\dagger v\|_\infty$ is easily checkable.

- The condition essentially requires both the constellation number and the level of the noise to be below a certain threshold.

- An open question posed by So in 2010: Is the condition $\lambda_{\text{min}}(H^\dagger H) > \|H^\dagger v\|_\infty$ sufficient for the conventional (complex) SDR being tight for the MIMO detection problem with $M \geq 3$?

- Our answer: No in general! Need to modify the condition and also need to develop an enhanced SDP relaxation!

- Analysis results for $M = 2$: Jaldén, Ottersten (KTH); Luo, Kisialiou (UoM); So, Ma (CUHK); ...
Numerical comparison of (ECSDP) and (CSDP) on 100 randomly generated instances of problem (P) with different $M$ and different levels of noise

<table>
<thead>
<tr>
<th>$M$</th>
<th>$\sigma^2$</th>
<th>GapC</th>
<th>GapE</th>
<th>ClosedGap</th>
<th>TimeC</th>
<th>TimeE</th>
<th>ProbC</th>
<th>ProbE</th>
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<td>0.01</td>
<td>0.096</td>
<td>0.000</td>
<td>100.0%</td>
<td>0.05</td>
<td>0.08</td>
<td>0%</td>
<td>100%</td>
</tr>
<tr>
<td>6</td>
<td>0.01</td>
<td>0.106</td>
<td>0.000</td>
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<td>0.05</td>
<td>0.08</td>
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<td>100%</td>
</tr>
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<td>0.09</td>
<td>0%</td>
<td>99%</td>
</tr>
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<td>4</td>
<td>0.1</td>
<td>0.971</td>
<td>0.000</td>
<td>100.0%</td>
<td>0.05</td>
<td>0.08</td>
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<td>100%</td>
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<tr>
<td>6</td>
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<td>0.871</td>
<td>0.015</td>
<td>98.9%</td>
<td>0.05</td>
<td>0.08</td>
<td>0%</td>
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<td>0.07</td>
<td>0%</td>
<td>32%</td>
</tr>
<tr>
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<td>2.646</td>
<td>75.7%</td>
<td>0.05</td>
<td>0.07</td>
<td>0%</td>
<td>7%</td>
</tr>
<tr>
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<td>0.05</td>
<td>0.07</td>
<td>0%</td>
<td>0%</td>
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</table>
Quick Observations

- For the high SINR case (i.e., the low noise case), (ECSDP) is tight (for the MIMO detection problem).

- Our proposed (ECSDP) is generally much tighter than (CSDP).

- The CPU time of solving (ECSDP) is slightly larger than the one of solving (CSDP).
A Quick Review

- Our interested problem (P):

\[
\min_{\mathbf{X}, \mathbf{x}} \quad \frac{1}{2} \mathbf{Q} \cdot \mathbf{X} + \mathbf{R} (\mathbf{c}^\dagger \mathbf{x}) \\
\text{s.t.} \quad \ell_i^2 \leq X_{ii} \leq u_i^2, \quad i = 1, \ldots, n, \\
\arg (x_i) \in \mathcal{A}_i, \quad i = 1, \ldots, n, \\
\mathbf{X} = \mathbf{xx}^\dagger.
\]

- Our proposed SDP relaxation:

\[
\min_{\mathbf{x}, \mathbf{X}, \mathbf{r}} \quad \frac{1}{2} \mathbf{Q} \cdot \mathbf{X} + \mathbf{R} (\mathbf{c}^\dagger \mathbf{x}) \\
\text{s.t.} \quad \ell_i^2 \leq X_{ii} \leq u_i^2, \quad i = 1, \ldots, n, \\
(x_i, r_i) \in \mathcal{G}_{\mathcal{A}_i}, \quad i = 1, \ldots, n, \\
(X_{ii}, r_i) \in \mathcal{F}_{\mathcal{B}_i}, \quad i = 1, \ldots, n, \\
\mathbf{X} \succeq \mathbf{xx}^\dagger.
\]
A Global Algorithm for Problem (P)

**Theorem**

For set \( \mathcal{A}_i = [\theta_i, \bar{\theta}_i] \) with \( \bar{\theta}_i - \theta_i < \pi \), if \( (x_i, r_i) \in G_{\mathcal{A}_i} \), then

\[
    r_i \geq |x_i| \geq r_i \cos \left( \frac{\bar{\theta}_i - \theta_i}{2} \right);
\]

for set \( \mathcal{B}_i = [\ell_i, u_i] \), if \( (X_{ii}, r_i) \in F_{\mathcal{B}_i} \), then

\[
    0 \leq X_{ii} - r_i^2 \leq \frac{(u_i - \ell_i)^2}{4}.
\]

- (ECSDP) has a **guaranteed bounded relaxation gap**, albeit it is generally not tight.

- **Smaller** \( \mathcal{A}_i \) and \( \mathcal{B}_i \) will lead to **tighter** SDP.

- Combining (ECSDP) and the **branch-and-bound** framework yields a **GLOBAL** algorithm for solving problem (P).
<table>
<thead>
<tr>
<th>((m, n, M))</th>
<th>SNR</th>
<th># Iter</th>
<th>Time</th>
<th>Obj Val</th>
<th>Bound 1</th>
<th>Bound 2</th>
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<td>1.064</td>
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</table>
Quick Observations

- The proposed branch-and-bound algorithm based on (ECSDP) generally is efficient, i.e., the number of iterations (and the CPU time) generally is small.

- The relaxation gap of our proposed (ECSDP) is small (albeit it is generally not tight).
Concluding Remarks

- Consider a class of nonconvex complex quadratic problems
- Propose a new and enhanced SDP relaxation for the considered problem
- Show tightness of the proposed enhanced SDP for MIMO detection
- Develop a global algorithm for solving general problem (P)
Three Related Papers


Thank You!

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