The Relevance of Harmonic Analysis in the Analysis of Quantum Systems

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30 June 2018

Fractional Signals and Systems, CMCAA 2018, Beijing
Collaborators

1. Starting new collaboration with Prof. Bing-Zhao Li (BIT); developing joint strategy to further develop new transforms and methods for Quantum and Nano Engineering

2. Background knowledge and perspectives on quantum metamaterials (QMMs) gained in conversations with Dr. Alexandre Zagoskin, Dr. Alexandre Balanov (Loughborough U), and Dr. Patrick Navez (U of S)

3. Some of the results on simulation of quantum dynamics and software have been developed with these collaborators: M. Everitt, J. Samson, S. Savel’ev, A. Zagoskin (LU), S. Heidel (Research Assistant Math and Stats, summer 2014), and J. Zuñiga-Anaya (ICT Research Computing, U of S)

4. Some of the software prototyping for D-transforms developed with Dr. I. Vlahu (PhD student, now graduated). Finally, the Broadband Redundancy based image processing algorithm has been developed jointly with Ali Melli (a PhD student in EE, U of S).
Overview

1. A fully quantum model of a QMM that is amenable to simulation; fractal and multi-scale structures

2. Nano-circuit inspired generalizations of the Fourier Transform

3. Qualitative analysis of quantum states and observables

4. The Broadband Redundancy: a noise cancellation phenomenon
A quantum model of a QMM that is amenable to simulation
The term **Quantum Engineering** refers to the art and science of designing and constructing functionally useful structures (devices) composed of qubits.


**Tradeoff:** Sometimes, a plethora of components (qubits) will be needed to engineer a useful device. At the same time the complexity of such a device (in terms of its dynamical properties) grows exponentially with the number of its components.

This typically prohibits a CAD approach to drafting. (One of the main reasons to hope for quantum computers is that they would enable simulation of quantum systems.)

Regardless of the availability of quantum computers, it is always desirable to have some computable models of a fairly complex qubit system in order to glean at least some insights.
A quick look at a semiclassical model of a QMM

Although, this is not what this talk is about!

Phase qubits (alt. qutrits) nonlinearly coupled to classical EM vector potential:

\[
\begin{align*}
\partial_t & \begin{bmatrix} \psi_{n,-1}(t) \\ \psi_{n,0}(t) \\ \psi_{n,1}(t) \end{bmatrix} = i \frac{\hbar}{\gamma} \begin{bmatrix} -E_c & \alpha_n(t)^2 & 0 \\ \alpha_n(t)^2 & 0 & \alpha_n(t)^2 \\ 0 & \alpha_n(t)^2 & -E_c \end{bmatrix} \begin{bmatrix} \psi_{n,-1}(t) \\ \psi_{n,0}(t) \\ \psi_{n,1}(t) \end{bmatrix} \\
\partial^2_t \alpha_m &= \gamma^2(\alpha_{m+1} - 2\alpha_m + \alpha_{m-1}) - \Re(\psi^*_m,0 \psi_{m,-1} + \psi^*_m,1 \psi_{m,0}) \alpha_m 
\end{align*}
\]

wherein \( n = 1 \ldots N; \quad m = -M + 1, \ldots, 0, 1, \ldots, N, N + 1, \ldots N + M. \)


Jaynes-Cummings, the epitome of a quantum model

\[ \mathcal{H} = I \otimes \mathcal{H}_F + \mathcal{H}_q \otimes I + \mathcal{H}_I : \quad \mathbb{H}_q \otimes \mathbb{H}_F \rightarrow \mathbb{H}_q \otimes \mathbb{H}_F. \]

where:

\[ \mathcal{H}_F = \hbar \omega (\hat{a}^{\dagger} \hat{a} + 1/2) \quad \text{photons in a cavity} \]

\[ \mathcal{H}_q = \frac{\hbar \Omega}{2} (|e\rangle\langle e| - |g\rangle\langle g|) = -\frac{\hbar \Omega}{2} \sigma_z \quad \text{qubit} \]

\[ \mathcal{H}_I = \hbar \lambda \sigma_x \otimes (\hat{a} + \hat{a}^{\dagger}) \quad \text{interaction (variant)} \]

Recall: \( \sigma_x = \sigma_+ + \sigma_- \) for \( \sigma_+ = |e\rangle\langle g| \) and \( \sigma_- = |g\rangle\langle e| \).
The best known features of JC variant $H_I = \hbar \lambda (\sigma_+ \otimes \hat{a} + \sigma_- \otimes \hat{a}^\dagger)$

- Block structure:
  \[
  \mathbb{H}_q \otimes \mathbb{H}_F = \text{span}\{\left|g\right\rangle\left|0\right\rangle\} \oplus \bigoplus_{n=1}^{\infty} \text{span}\{\left|g\right\rangle\left|n\right\rangle, \left|e\right\rangle\left|n-1\right\rangle\}
  \]

- Explicit eigenvalues:
  \[
  E_{\pm}(n) = \hbar(n \omega \pm \frac{1}{2} \Omega_n), \quad \Omega_n = \sqrt{(\omega - \Omega)^2 + 4\lambda^2 n}.
  \]

- When $\omega = \Omega$ (resonance) $\Omega_n = 2\lambda \sqrt{n}$. The square root has been confirmed experimentally at low energies.

A hint of self-similarity in a generalized JC

Replace $H_I$ by:

$$H_K = \hbar C_K \otimes (\hat{a} + \hat{a}^\dagger) : \left( \bigotimes_{k=1}^{K} \mathbb{H}_q \right) \otimes \mathbb{H}_F \rightarrow \left( \bigotimes_{k=1}^{K} \mathbb{H}_q \right) \otimes \mathbb{H}_F,$$

where $C_K = \sum_{k=1}^{K} \lambda_k \sigma_x^k$, $\sigma_x^k = I \otimes \ldots I \otimes \sigma_x \otimes I \ldots$
in the “limit” \( C[\Phi](x) = \hbar \sum_k \frac{1}{2^k} \Phi \left( x + \frac{(-1)^{\alpha_k(x)}}{2^k} \right) \quad x \in (0, 1). \)
Elements of the Haar basis

Let \( G(x) \equiv 1 \) for \( x \in (0, 1] \) and \( G(x) = 0 \) everywhere else on the real line, and

\[
G_{n,k}(x) = 2^{n/2} G(2^n x - k)
\]

\[
H(x) = [G_{1,0}(x) - G_{1,1}(x)]/\sqrt{2} \quad H_{n,k}(x) = 2^{n/2} H(2^n x - k)
\]
The basic properties of the Haar basis:

1. Denote $V_n = \text{span} \{ G_{n,k} : k = 0, 1, 2, \ldots 2^n - 1 \}$. Then, $V_0 \subseteq V_1 \subseteq V_2 \ldots$, and
   \[
   L_2[0, 1] = \bigcup_{n=0}^{\infty} V_n \quad \text{(multiresolution ladder)}.
   \]

2. Denote $W_n = V_{n+1}/V_n$:
   \[
   W_n = \text{span} \{ H_{n,k} : k = 0, 1, 2, \ldots 2^n - 1 \}
   \]
   \[
   L_2[0, 1] = V_0 \bigoplus_{n=0}^{\infty} W_n \quad \text{(direct sum decomposition)}. \quad (1)
   \]
   Use the orthogonal projections $\Pi_n : V_{n+1} \to W_n$.

3. \{ $G_{0,0}$ $\cup$ $\{ H_{n,k} : k = 0, 1, 2, \ldots 2^n - 1; n = 0, 1, 2, \ldots \}$ ON basis of $L_2[0, 1]$
In all references to this basis we will assume the canonical order in $W_n$ to be according to increasing $k$, so that, overall, the order of basis functions is fixed to be:

$$G_{0,0}, H_{1,0}, H_{1,1}, H_{2,0}, H_{2,1}, H_{2,2}, H_{2,3}, \ldots$$

We let $\mathcal{T}_\mathcal{H} : L_2[0, 1] \rightarrow \ell_2$ denote the Haar transform which assigns to a square integrable function, say, $f$ its ordered sequence of Haar coefficients:

$$c_0 = \int_0^1 f(x) \, dx, \text{ and } c_{n,k} = \int_0^1 f(x) H_{n,k}(x) \, dx.$$ 

Clearly, $\mathcal{T}_\mathcal{H}$ is a unitary transformation.
Operator $C$ defined in extends to a continuous operator $C : L_2[0, 1] \rightarrow L_2[0, 1]$. More precisely, $C$ has the following properties:

Let $D_n$ be the matrix of $\Pi_n C \Pi_n : W_n \rightarrow W_n$ in the canonical basis. Then,

$$D_0 = [0], \quad \text{while} \quad D_{n+1} = \frac{1}{2} \begin{bmatrix} D_n & I \\ I & D_n \end{bmatrix} \quad \text{for } n \geq 0.$$

Also, if $n > 0$, then $D_n$ is invertible, and its complete list of the eigenvalues is

$$\{ \pm (2k + 1)/2^n : k = 0, 1, \ldots 2^{n-1} - 1 \}.$$

In particular, $C$ preserves the direct sum decomposition, i.e. $CV_0 = V_0$ and $CW_n \subseteq W_n$ for all $n$. Specifically,

$$C = \mathcal{T}_\mathcal{H}^\dagger \left( I_1 \oplus \bigoplus_{n=0}^{\infty} D_n \right) \mathcal{T}_\mathcal{H}.$$
A fully quantum model of a QMM that is amenable to simulation; fractal and multi-scale structures

\[ \mathcal{T}_H C \mathcal{T}_H^\dagger = I_1 \bigoplus \bigoplus_{n=0}^{\infty} D_n \]

is a “matryoshka” of matrices:

![The Haar transformed C](image.png)

\[ \text{nz} = 8195 \]
Diagonalization of $C$:

Let $E_n$, $n \geq 1$, be the diagonal matrix whose diagonal entries are \(\{\pm (2k + 1)/2^n : k = 0, 1, \ldots 2^{n-1} - 1\}\) in the increasing order. Then,

\[
C = \mathcal{T}_\mathcal{H}^\dagger B^\dagger \left( l_1 \oplus \bigoplus_{n=0}^{\infty} E_n \right) B \mathcal{T}_\mathcal{H}.
\]

\[
u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad B := l_1 \oplus l_1 \oplus \bigoplus_{n=1}^{\infty} u^\otimes n
\]

($B$ is a unitary operator in $\ell_2$.)

$C$ is a bounded self-adjoint operator in $L_2[0, 1]$; its spectrum is $\sigma(C) = [-1/2, 1/2]$, and its norm $\|C\| = 1/2$. 

A fully quantum-mechanical model of a QMM that is amenable to analysis!

\[ \mathcal{H}_{QMM} = I \otimes \mathcal{H}_F + \lambda \hat{C} \otimes (\hat{a}^2 + (\hat{a}^2)^\dagger) + \Omega \, V \otimes I, \]

where:

\[ V = \hbar (x - 1/2) \quad \text{(potential)} \]

\[ \hat{C} := I_1 \oplus D_0 \oplus D_1^{-1} \oplus D_2^{-1} \ldots \quad \text{unbounded operator} \]

\[ \hat{C} C = C \hat{C} = I - \Pi_0. \]

- Use the characterization of \( C \) to efficiently simulate the dynamics. In particular, capture the characteristic of this system (involving an infinite number of qubits) via a restriction to \( V_N \otimes \mathcal{H}_F \).
- Deduce that the spectrum of \( \mathcal{H}_{QMM} \) has a continuous component (for some choices of \( \lambda, \omega, \Omega \)).
Cont. var. “Ising” \((L \sim \infty)\): \[
H_l = - \sum_{l=2}^{L} \sum_{k=1}^{l-1} \frac{J_{kl}}{2^l} \sigma^z_k \sigma^z_l
\]

**Figure:** The scaled limit of the Ising Hamiltonian is a potential, i.e. a scalar, real valued function on the interval \([0, 1]\). When all \(J_{kl} = 1\) it is precisely the *blancmange* function. For a generic choice of a randomly distributed \(J_{kl}\) the graph assumes different, typically discontinuous forms (the graph on the right is representative). However, all discontinuities are of a jump type and occur at diadic fractions, which ensures good compressibility of these functions in the Haar basis.
Nanocircuit-inspired generalizations of the Fourier Transform
A Brief History of Memristance: from mathematical concepts to nanoelectronics


3. Chua’s discovery stemmed from a purely mathematical insight. Many years later the concepts he proposed found experimental verification and became foundational in nanoelectronics.
RLCM circuit equation

\[ L \ddot{q} + R \dot{q} + M(q, \dot{q}) \dot{q} + \frac{q}{C} = v(t) \]

- Unlike the RLC circuit, this structure is nonlinear.
- In many cases the corresponding eigenvalue problem may be solved by engaging the Ansatz
  \[ q = \varphi(t) = 1 + \sum_{n=1}^{\infty} a_n e^{2\pi i n t} \]
  This leads to a recurrence relations for \( a_n \).
- The family of eigenfunctions \( \varphi_m(t), \ m = 1, 2, \ldots \) have the property
  \[ \varphi_m(t) = \varphi_1(mt) \]
  This is not an orthonormal system of functions! (cf. Sturm-Liouville.)
Waveforms $\varphi_m(t) = \varphi_1(mt)$ as a basis for $L_2[0, 1]$

The core idea (Arne Beurling, 1945):

Replace the trigonometric monomials $e_m = \exp(2\pi imt)$ by

$$\varphi_m(t) = \sum_{n=1}^{\infty} a_n e^{2\pi imnt} \quad (m > 0), \quad \varphi_{-m} = \bar{\varphi}_m, \quad \varphi_0(t) \equiv 1$$

Next, consider the linear map given by

$$D[e_m] = \varphi_m$$

This would be the relevant change of basis transform
Nano-circuit inspired generalizations of the Fourier Transform

**D in the Fourier basis:**

\[
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The inverse transform brings on board Number Theory

Recall the Dirichlet multiplication (convolution) of two sequences \((a_n), (b_n)\): The sequence \((c_n) = (a_n) \ast (b_n)\) is given by\(^1\)

\[
c_n = \sum_{d \mid n} a_d b_{n/d}.
\]

The inverse of \((a_n)\) exists iff \(a_1 \neq 0\); its terms are determined via recursion:

\[
b_1 = 1/a_1, \quad b_n = -\frac{1}{a_1} \sum_{d \mid n, d > 1} a_d b_{n/d} \quad n = 2, 3, \ldots,
\]

It turns out that if \((a_n)\) is the sequence from the matrix \(D\), then the matrix \(D^{-1}\) has the same structure as \(D\) and is defined via \((b_n)\). Notation: \(D = [(a_n)]\). Fact: \(D^{-1} = [(b_n)]\).

\(^1\)cf. product of Dirichlet series: \((\sum_{n \geq 1} a_n n^{-s})(\sum_{n \geq 1} b_n n^{-s}) = \sum_{n \geq 1} c_n n^{-s}\).
\[ D^{-1} = \]

\[
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\end{array}
\]
Examples of theorems

Theorem (Summability is nice)

If both \((a_n), (b_n)\) are absolutely summable then \((\varphi_n)\) form a good (i.e. unconditional) basis in \(L_2[0, 1]\) with the dual\(^a\)

\[ \chi_n = \sum_{d|n} b_d^* e_{\frac{n}{d}}, \text{ so that } \langle \chi_n | \varphi_m \rangle = \delta_{m,n} \]

\(^a\)i.e. one has a resolution of the identity \(I = \sum |\varphi_n\rangle \langle \chi_n|\)

Theorem (The frame bounds)

For an arbitrary \(f \in L_2[0, 1]\) we have

\[ A \|f\|^2 \leq \sum_n |\langle \chi_n | f \rangle|^2 \leq B \|f\|^2 \]

where \(A = (\sum |a_n|)^{-2}, B = (\sum |b_n|)^2\).
A very important special case tied to the periodized Riemann zeta function:

\[ a_n = n^{-\sigma - i\tau} \quad \text{so that} \quad b_n = \mu(n)n^{-\sigma - i\tau} \]

Luckily, \( \mu \) (the Möbius function) assumes only three values: \(-1, 0, 1\). So,

If \( \sigma > 1 \), then \( D_{[\sigma + i\tau]} \) is a homeomorphism in \( L^2[0, 1] \)

(Equivalently, the corresponding bases are good.)

Indeed, \( \sigma > 1 \) implies that both sequences \((a_n)\) and \((b_n)\) are absolutely summable.
Qualitative analysis of quantum states and observables
Wigner-Weyl transform explainer

Figure: The essence of the Wigner transform. Note: IF $\rho(x, y) = \rho(y, x)$ THEN $W[\hat{\rho}]$ is real-valued.
An alternative: the Q transform.

The Fourier coefficients $Z = [z_{k,l}]_{(k,l) \in \mathbb{Z}^2}$ of a real and integrable function (on $[0,1] \times [0,1]$) obey the symmetry: $z_{-k,-l} = z_{k,l}^*$, $(k, l) \in \mathbb{Z}^2$.

MAIN IDEA: One can reorganize these data in such way as to obtain a Hermitian matrix. It can be done via a linear reversible map.

A generalization is obtained by substituting any generalized Fourier transform ($D$-transform) for the Fourier transform.
Some properties and applications of the Q transform

- Q (the Fourier version) is an isometry between $L_2[0, 1]^\times 2$ and the space of Hilbert-Schmidt operators (in $\ell_2(\mathbb{Z})$). In every version Q takes real functions to self-adjoint operators (observables).

- Q transform makes it possible to define Sobolev spaces of observables. Moreover, one can show that
  \[ \|AB\|_\alpha \leq \|A\|_\alpha \|B\|_\alpha \quad \text{for } \alpha > 0. \]

- This property is used to establish regularity preserving properties of quantum flows.

- Also, in this way, Q transform gives a plethora of infinite-dimensional Lie algebras.

- Further applications in image processing.
Markovian master equation

\[ \partial_t A = i[\mathcal{H}, A] + \sum L_j^\dagger AL_j - \frac{1}{2} L_j^\dagger L_j A - \frac{1}{2} AL_j^\dagger L_j \]

\((e_n)_{n \in \mathbb{Z}} = \) the basis of \(\mathbb{H}\) that determines the Q-transform; \(\alpha > 0\). If

\[ \mathcal{H} = \sum_{n \in \mathbb{Z}} (an + b) |e_n \rangle \langle e_n | + C; \quad C, L_j \in H_\alpha^Q, \]

then

\[ A(0) \in H_\alpha^Q \Rightarrow A(t) \in H_\alpha^Q \]

Deduce compressibility of some quantum flows (e.g. in adiabatic QC)!
The Broadband Redundancy (BR), a noise cancellation phenomenon
The Broadband Redundancy: a noise cancellation phenomenon

We have already considered $\varphi(t) = F(-\sigma - i\tau, t) = \sum n^{-\sigma-i\tau} \exp 2\pi int$.

It is interesting to choose $\tau = \tau_n$, where (assuming the R. Hypothesis) $s_n = 1/2 + i\tau_n$ is the sequence of zeros of $\zeta(s) = F(s, 1)$ in the upper half plane.

A very long sequence of the initial $\tau_n$ have been computed with high accuracy (by Andrew Odlyzko); our numerical experiments rely upon these data.
Landau (1912):
For real number $\alpha \neq 0$ the sequence $(\alpha \tau_n)_{n=1}^{\infty}$ is uniformly distributed (mod 1).

Fujii (1990) quantifies Landau’s observation (under R.H.):
Whenever $\sigma > 0$ there is a positive constant $C$ such that
\[
\left| \frac{1}{N(T)} \sum_{\tau_n \leq T} k^{-\sigma - i\tau_n} \right| \leq C \frac{k^{1/2 - \sigma} \log k}{\log T} \quad \text{for all } k \geq 2 \text{ and } T \text{ large.}
\]
The foundational observation: BR

Theorem (Sowa 2017)
Assume \( \sigma > 0 \). Under the R.H.

\[
\lim_{T \to \infty} \frac{1}{N(T)} \sum_{n \leq T} F_{\sigma + i\tau_n}(t) = \exp(2\pi it),
\]

where the convergence type is as follows:

1. distributional whenever \( \sigma > 0 \),
2. in \( L_2 \)-norm whenever \( \sigma > 1 \),
3. uniform whenever \( \sigma > 3/2 \).
The Broadband Redundancy: a noise cancellation phenomenon

1. The BR in 1D: the Fourier transform is close to the average of some non-unitary transforms.

Theorem (The FT is the average of nonstandard FTs)

Let \( f \in H^\alpha(\mathbb{R}/\mathbb{Z}) \), i.e. \( \| f \|_\alpha^2 = \sum_{k \in \mathbb{Z}} (1 + k^2)\alpha |\hat{f}_k|^2 < \infty \). Then,

\[
\left\| \frac{1}{N(T)} \sum_{\tau_n \leq T} D_{[\sigma + i\tau_n]} \hat{f} - \hat{f} \right\|_{\ell_2} \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty, \quad (\alpha > 1/2, \sigma > \alpha + 1),
\]

and the rate of convergence depends on \( f \) only via the value of its \( \alpha \)-norm.
2. The BR in 2D: any observable is an average of “broad-band” observables.

Theorem

Let $\alpha > 1/2$ and $\sigma > \alpha + 1$. For $f \in H^{2\alpha}(\mathbb{T})$ we have

$$\left\| \frac{1}{N(T)} \sum_{\tau_n \leq T} D_{[\sigma + i\tau_n]}^{-1} \hat{f}(D_{[\sigma + i\tau_n]}^{-1}T - \hat{f}) \right\|_{\ell_2(\mathbb{Z} \times \mathbb{Z})} \to 0 \text{ as } T \to \infty,$$

and also

$$\left\| \frac{1}{N(T)} \sum_{\tau_n \leq T} Q[\sigma + i\tau_n] f - Qf \right\|_0 \to 0 \text{ as } T \to \infty.$$

In both cases the rate of convergence depends on $f$ only via its $2\alpha$-norm.
The Broadband Redundancy: a noise cancellation phenomenon

Image denoising with the Q-transform via a dissipative quantum process

Werner in a white noise haze

...denoised via Lindblad dissipation

It is natural to try and repeat this type of denoising via, say, 237 different channels and consider the average. However, this by itself is not sufficient to obtain a significant improvement.
The Broadband Redundancy: a noise cancellation phenomenon

Ali Melli’s idea: postprocessing (via TV minimization as in Compressed Sensing)

Signal Flow Graph for Redundant Channels
Proposed Denoising Algorithm
Result: Numerical experiments results

Figure 1: Visual and structural performance evaluation of the proposed algorithm
“Thoughts are but dreams till their effects be tried.”

— William Shakespeare

The shared Loughborough/Saskatchewan/BIT vision:

Q-MetaMaterials $\cup$ Q-SignalProcessing $\Rightarrow$ Improved Q-Imaging
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Thank you for your attention!