Controllability of a multichannel system

Sergei A. Ivanov a,*, Jun Min Wang b

a Marine Geomagnetic Investigation Laboratory, SPbF IZMIRAN, Mendeleev line 1, 199034, St. Petersburg, Russia
b School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China

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Abstract

We consider the system consisting of $K$ coupled acoustic channels with the different sound velocities $c_j$. Channels are interacting at any point via the pressure and its time derivatives. Using the moment approach and the theory of exponential families with vector coefficients we establish two controllability results: the system is exactly controllable if

(i) the control $u_j$ in the $j$th channel acts longer than the double travel time of a wave from the start to the end of the $j$-th channel;
(ii) all controls $u_j$ act more than or equal to the maximal double travel time.

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1. Introduction and main results

We study the control system governed by the differential equation

$$y_{tt}(x,t) = Dy_{xx}(x,t) + Q(x)y(x,t) + R(x)y_t(x,t), \quad x \in (0,1),$$

(1)
where $D$ is a diagonal $K \times K$ matrix, $D = \text{diag}(c_j^2)^K$, $c_j > 0$, with different entries $c_j, c_i \neq c_j$, for $i \neq j$, $Q$ and $R$ are $C^1$-smooth matrix valued functions. The controls act in the channels at the boundary entering the Dirichlet boundary condition (BC) at one end. At the other end we have a homogeneous BC

$$y(0, t) = u(t), \quad y(1, t) = 0.$$  \hspace{2cm} (2)

All channel controls $u_j$ are quadratically integrable at any interval.

The unperturbed system with $R = Q = 0$ represents $K$ uncoupled channels governed by the string equation with constant coefficients. Such systems are controllable because each channel is exact controllable in time $T_i = 2/c_i$, this is the double travel time of a wave from the start to the end of the $j$-th channel.

We show that the system is exactly controllable in the state space $L^2(0, 1; \mathbb{C}^K) \oplus H^{-1}(0, 1; \mathbb{C}^K)$ for the following types of controls:

- **Type (i):** for all $j$ the control in the $j$th channel acts longer than $T_j$, or
- **Type (ii):** the whole control $u$ acts longer than or equal to $T_{\text{max}} = \text{max} T_j$ what is the double maximal travel time.

The first paper on controllability of a coupled system, namely, of a connected network of homogeneous strings with controls at the nodes, was written by Rolewicz [16]. In this statement the string network controllability problem was completely solved in [2]. Now many strong results in theory of control and stabilization of networks are obtained, see, for example, [20,21,1].

The main feature of the system (1), (2) is the presence of different wave modes propagate with different velocities and interact with each other. For the case of one unit velocity the controllability results are similar to an 1dim system. In [3] the control system governed by the equation with potential was studied

$$y_{tt}(x, t) = y_{xx}(x, t) + V(x)y(x, t), \quad x, t > 0 \quad y(x, t) \in \mathbb{C}^K,$$

$$y|_{t=0} = 0, \quad y(0, t) = u(t).$$

The shape–controllability in the filled domain was proved: for $u$ runs $L^2(0, T; \mathbb{C}^K)$ the first component $y(\cdot, T)$ of the state runs exactly $L^2(0, T; \mathbb{C}^K)$. If the system is considered on an spatial interval $(0, L)$ with the Dirichlet BC (2), we have controllability of Type (ii) in time $T = L$ as in the case of uncoupled channels.

In [19] an interesting case of the system of parallel coupled identical strings was considered. The authors prove the controllability and show that despite the damping, there exist sustained oscillations.

More interesting dynamic and more difficult control problems arise in the system possessing two-mode oscillations. In a series of papers, see, e.g., [4,5] the dynamic and the shape–controllability in the filled domain are studied. A classical example of a two-channel system with interacting modes is a Timoshenko beam in elasticity theory. In [9,17] the system with a distributed (in the right hand side of the equation) ‘1dim’ control of the form $u(t)\left(\frac{r_1(x)}{r_2(x)}\right)$ is studied. Here $r_1$ and $r_2$ are fixed functions. To obtain the controllability the authors use a non-physical condition that the velocities $c_1$ and $c_2$ are commensurable: $c_1/c_2$ is a rational number. The reason is that the authors consider the conservative BC and apply the moment approach what leads to a moment problem with respect to exponential family $\exp(ik_0t)$ where the real set of
frequencies \( \{k_n\} \) is not separable. Note that for a two-dimensional control \( (u_1(t), r_1(x), k_2(t), r_2(x)) \) we have a moment problem with respect to a 'vector' exponentials \( \eta_n \exp(ik_nt) \) and the vector coefficients \( \eta_n \) 'separate' the frequencies.

A two channel model can be used to describe the small vibrations of the system, composed by a steel beam and the reinforced concrete slab connected by some connectors, see [12]. The inverse problem connected with this model is studied in [13,6].

In [2] the controllability of a multichannel system governed by

\[
A(x)y_{tt}(x,t) = y_{xx}(x,t), \quad x \in (0, 1),
\]

\[
y(0,t) = u(t), \quad y(1,t) = 0, \tag{3}
\]

was studied under the assumptions that the matrix \( A \) can be smoothly diagonalized. Only the type (ii) controllability has been proved, using the same approach as in Sect. 3.2, namely the moment approach and factorization of the so called generating function.

The scheme of the work is the following. The equation (1) is written as one of the first order in time \( z_t = Az. \) For the perturbed system we find that the set \( \Phi \) eigenfunctions of \( A \) is quadratically close to the set of eigenfunctions of unperturbed operator what leads to Riesz basis property of \( \Phi \) and allow us to use the moment approach. This approach reduces the controllability problem to the moment one with respect to the vector exponential family \( \mathcal{E} = \{\eta_\lambda \exp(\lambda t)\}_{\lambda \in \sigma}, \) where the vector coefficients \( \eta_\lambda \) are expressed via the eigenfunctions corresponding to the eigenvalue \( \lambda. \)

The moment problem is always solvable in \( \mathbb{C}^2 \) if \( \mathcal{E} \) forms a Riesz basis in its span (in what follows, an \( \mathcal{L} \)-basis) A general theory of vector exponentials can be found in [2].

The unperturbed exponential family forms an \( \mathcal{L} \)-basis in

\[
\mathcal{L}_0 = L^2(0, T_1) \oplus L^2(0, T_2) \oplus \cdots \oplus L^2(0, T_K)
\]

(with K-dim defect). Because the perturbed family asymptotically close to the unperturbed exponential family, it forms an \( \mathcal{L} \)-basis in

\[
\mathcal{L}_\varepsilon = L^2(0, T_1 + \varepsilon) \oplus L^2(0, T_2 + \varepsilon) \oplus \cdots \oplus L^2(0, T_K + \varepsilon).
\]

This leads to the controllability of the system when controls are in this space (Type (i)).

In the last part of the paper we prove Type (ii) controllability which follows from the \( \mathcal{L} \)-basis property of \( \mathcal{E} \) in

\[
L^2(0, T_{\text{max}}; \mathbb{C}^K) = L^2(0, T_{\text{max}}) \oplus L^2(0, T_{\text{max}}) \oplus \cdots \oplus L^2(0, T_{\text{max}}).
\]

Many steps of the proof are similar to ones for the system (3) and we omit technical details. In the proof we construct an exponential type matrix function \( F \) such that \( F(\lambda)\eta_\lambda = 0 \) on the spectrum \( \sigma(A). \) This function is constructed from solutions of the evolution system what allow us to find the factorization of \( F. \) The factorization leads to the \( \mathcal{L} \)-basis property of \( \mathcal{E} \) in \( L^2(0, T_{\text{max}}; \mathbb{C}^K). \)

The last steps of the study of exponentials are based on the theory developed in [2].

2. Eigenfunctions and eigenvalues

Write the equation (1) (without control) as an equation of the first order in time. Let \( z = \begin{pmatrix} y \\ y_t \end{pmatrix}. \) Then
\[ \dot{z} = \begin{pmatrix} z_2 \\ Dz_1'' + Qz_1 + Rz_2 \end{pmatrix} = A z. \] (4)

Here \( A \) is the operator acting in

\[ H = \begin{pmatrix} H_0^1(0, 1; \mathbb{C}^K) \\ L^2(0, 1; \mathbb{C}^K) \end{pmatrix}, \]

with

\[ D(A) = \begin{pmatrix} H^2(0, 1; \mathbb{C}^K) \cap H_0^1(0, 1; \mathbb{C}^K) \\ H_0^1(0, 1; \mathbb{C}^K) \end{pmatrix}. \]

Introduce notations. By \( \{ \zeta_j \} \) we denote the standard basis in \( \mathbb{C}^K \). Set

\[ \Omega(x) = \text{diag}[e^{\int_0^x R_{jj}(s)ds}], \quad r_j = \int_0^1 R_{jj}(x)dx, \]

\[ f_{n,j}^\pm = \begin{pmatrix} \zeta_j n^{-1} \Omega(x) \sin(\pi nx) \\ \zeta_j \Omega(x) \sin(\pi nx) \end{pmatrix}. \] (5)

**Theorem 1.** (i) the eigenvalues of \( A \) have the asymptotic

\[ \lambda_{n,j}^\pm = \pm i \pi n c_j + r_j + O(\frac{1}{n}); \] (6)

(ii) the eigenfunctions of \( A \) can be written as

\[ \phi_{n,j}^\pm = f_{n,j}^\pm + O(\frac{1}{n}); \]

(iii) The family \( \Phi \) of eigenfunctions of \( A \) forms a Riesz basis in \( H \).

**Proof.** Let \( \lambda \) be an eigenvalue of \( A \) and \( (v w) \) be the corresponding eigenfunction. We see that \( w = v \) and then

\[ Dv'' + Qv + \lambda(R - \lambda)v = 0, \quad v(0) = v(1) = 0. \]

Consider the matrix equation

\[ D\Phi''(x, \lambda) + Q(x)\Phi(x, \lambda) + \lambda(R(x) - \lambda)\Phi(x, \lambda) = 0. \] (7)

By \( \Phi_{\pm}(x, \lambda) \) we denote the solutions satisfying the following BC

\[ \Phi_{\pm}(0, \lambda) = I, \quad \Phi_{\pm}'(0, \lambda) = \pm \lambda I, \]

and the differential system
\[ D\Phi''_\pm(x, \lambda) + Q_d(x, \lambda)\Phi_\pm(x, \lambda) = 0, \]  
(8)

where the matrix of the system is the diagonal part of the system matrices of (7)

\[ Q_d = [Q_{jj}(x)] + \lambda([R_{jj}(x)] - \lambda). \]

The system (8) is the set of scalar equations and the asymptotic of \( \Phi_\pm \) can be found in, e.g., [7]. In the strip \(|\Re \lambda| \leq \text{Const}\) we have the asymptotic

\[ \Phi_\pm(x, \lambda) = \text{diag} \left[ c_j \exp \left\{ \pm \frac{\lambda}{c_j} x \mp \frac{1}{2c_j} \int_0^x R_{jj}(s) \, ds \right\} \right] + O(1/\lambda). \]

(9)

Consider the solution \( \Phi_0 = (\Phi_+ - \Phi_-)/2 \) with the boundary conditions

\[ \Phi_0(0, \lambda) = 0, \quad \Phi'_0(0, \lambda) = \lambda I. \]

(10)

This solution has the asymptotic

\[ \Phi_0(x, \lambda) = \text{diag} \left[ c_j \sinh \left\{ \frac{\lambda}{c_j} x - \frac{1}{2c_j} \int_0^x R_{jj}(s) \, ds \right\} \right] + O(1/\lambda). \]

(11)

Let us return to equation (7) which can be written as

\[ \Phi'' + Q_d\Phi + Q_{nd}\Phi = 0. \]

(12)

Here \( Q_{nd} \) is the non-diagonal part of the matrix of the system

\[ Q_{nd} = (\{Q(x)\} + \lambda(R(x) - \lambda)) - Q_d. \]

We can write this differential equation with the BC (10) as an integral Volterra equation of the second kind

\[ \Phi(x, \lambda) = \Phi_0(x, \lambda) + \int_0^x \frac{1}{2\lambda} \{\Phi_+(x)\Phi_-(\xi) - \Phi_+(\xi)\Phi_-(x)\} Q_{nd}(\xi, \lambda)\Phi(\xi, \lambda) \, d\xi. \]

The diagonal entries of \( Q_{nd} \) are zeros and using the asymptotic (9) we find that the first iteration in the Neumann series for this Volterra equation is of order \( 1/\lambda \):

\[ \Phi(x, \lambda) = \Phi_0(x, \lambda) + O\left(\frac{1}{\lambda}\right). \]

(13)

Suppose for simplicity that all eigenvalues are geometrically simple and that \( \lambda = 0 \) is not an eigenvalue. The next statement is evident.
**Lemma 1.** \( \lambda \neq 0 \) is an eigenvalue of \( A \) if and only if for some \( \eta \in \mathbb{C}^K \)

\[
\Phi(1, \lambda) \eta = 0, \quad \eta \neq 0.
\]

(14)

In this case \( \Phi(x, \lambda) \eta \) is the corresponding eigenfunction.

From here and (11) we obtain the asymptotic of the eigenfunction \( \varphi_{n,j}^{\pm} \) and eigenvalues \( \lambda_{n,j}^{\pm} \) of \( A \).

Consider the orthonormal basis in \( H \)

\[
(\hat{\varphi}_{n,j}^{\pm})_{n=1, j=1}^{\infty, K}, \quad \hat{\varphi}_{n,j}^{\pm} = \begin{pmatrix} \zeta_j 1/\pi \sin \pi nx \\ \pm \xi_j \sin \pi nx \end{pmatrix}.
\]

The family \( (f_{nj}^{\pm})_{n=1, j=1}^{\infty, K} \) given in (5) is connected with this family by an isomorphism in \( H \)

\[
H \ni \begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} \Omega u \\ \Omega v \end{pmatrix} \in H.
\]

We conclude that the family \( (f_{nj}^{\pm})_{n=1, j=1}^{\infty, K} \) is a Riesz basis in \( H \).

Let us study the basis property of the family of the eigenfunctions \( \Upsilon \) of \( A \). Because the asymptotic is given for eigenvalues with large absolute values, we can not claim that the families are quadratically close, but we can claim this for subfamilies (for the tails):

A subfamily of \( \Upsilon \) corresponding to ‘large eigenvalues’ is quadratically close to a subfamily

\[
\{f_{nj}^{\pm}\}_{n \geq N}
\]

of the Riesz basis \( (\hat{\varphi}_{nj}^{\pm})_{n=1, j=1}^{\infty, K} \). From here we conclude that there is a subfamily \( \Upsilon^0 \) of \( \Upsilon \)

\[
\Upsilon^0 = \{\varphi_{n,j}^{\pm}\}_{n \geq M}
\]

which forms an \( L \)-basis in \( H \) and the codimension of its span is finite.

The whole family \( \Upsilon \) is an \( L \)-basis if and only if the angle between the span of \( \Upsilon^0 \) and the span of the rest first eigenfunctions is positive. The dimension of the latter span is finite and the angle is zero if and only if the spans have a common element, say, \( \varphi \):

\[
\varphi = \sum_{n < M} a_{n,j}^{\pm} \varphi_{n,j}^{\pm} = \sum_{n \geq M} a_{n,j}^{\pm} \varphi_{n,j}^{\pm}.
\]

Then, applying \( A^{-1} \) several times, we find that the coefficient corresponding to the eigenvalue with the minimal absolute value grows faster than the coefficient by the rest terms and has to be zero. We have proved that the family of eigenfunctions of \( A \) forms an \( L \)-basis with finite codimension of its span.

To show that the family of eigenfunctions is complete let us consider the unperturbed operator \( A_0 \)

\[
A_0z = \begin{pmatrix} 0 & I \\ Dd^2/dx^2 & 0 \end{pmatrix} z
\]
with the domain \(D(\mathcal{A})\). This is a skew adjoint operator with complete family of eigenfunctions and the difference \(\mathcal{A} - \mathcal{A}_0\) is a bounded operator. Now we can apply V.M. Keldysh’ result (see [8, pp. 170, Theorem 4.1]). Using this theorem we obtain the completeness. \(\square\)

The operator adjoint to \(\mathcal{A}\) differ from \(-\mathcal{A}\) by the sign of \(Q\). Later we will need the estimates of the eigenfunctions \(\theta_{n,j}^\pm(0)\). Its asymptotic is similar to one for the eigenfunctions for \(\mathcal{A}\). Denote by \(P\) the orthoprojector in \(C^{2K}\)

\[
P\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}.
\]

In the next section we will need in the following estimate.

\[
\frac{1}{C} n \leq \| \frac{d}{dx} P \theta_{n,j}^\pm(0) \|_{C^K} \leq Cn.
\]

### 3. Exponential bases and controllability

#### 3.1. Moment problem

Write the control system (1), (2) as an evolution system. As it is well known, the boundary Dirichlet control can be written as a term in the RHS

\[
\dot{z} = \mathcal{A}z + Bu
\]

with

\[
Bu(t) = \begin{pmatrix} 0 \\ \delta'(x) \end{pmatrix} u(t).
\]

This can be justified by using an integral identity instead (4), see, e.g., [2, Ch. IV, Sec. 2.1].

Apply the moment approach to the controllability problem of the system (1). Let the solution to the equation under zero initial conditions be written as the series with respect to \(\Phi\).

\[
z(x, t) = \sum_{n=1, j=1}^{\infty, K} c_{n,j}^\pm(t) \varphi_{n,j}^\pm(x).
\]

We may now consider this series formally, without the condition \(z(\cdot, t) \in H\).

Denote the system biorthogonal to \(\Upsilon\) by \(\Theta\). In what follows we will omit additional indices, if this can not confuse the reader. Using the Fourier representation of the control term

\[
Bu = \sum (Bu)_n(t) \varphi_n(x), \quad (Bu)_n = b_n = (Bu, \varphi_n)\_H,
\]

write (16) as the set of ordinary differential equation for the coefficient \(c_n\)

\[
\dot{c}_n = \lambda_n c_n + b_n, \quad c_n(0) = 0.
\]
The solution is
\[ c_n(T) = \int_0^T e^{\lambda_n(T-\tau)} (Bu)_n(\tau) d\tau. \]

Setting \( \tilde{u}(\tau) = u(T - \tau) \) we write this as the inner product
\[ c_n(T) = \left( \tilde{u}, \eta_n e^{\lambda_n \tau} \right)_{L^2(0,T;\mathbb{C}^K)}. \]  

(19)

Here \( \eta_n \) is the value of the derivative of the second part of \( \theta_n \) at zero
\[ \eta_n = B^\ast \theta_n = \frac{d}{dx} P \theta_n(0) = \frac{d}{dx} \left( (\theta_n)_{K+1}, (\theta_n)_{K+2}, (\theta_n)_{K+3}, \ldots, (\theta_n)_{2K} \right) \big|_{x=0} \]

(exactly this term appears in the integral identity, corresponding to the week solution).

Write this formal results as a lemma.

**Lemma 2.** The state \( z \)
\[ z = \sum_{n=1,j=1}^{\infty,K} z_{n,j}^\pm \varphi_{n,j}^\pm(x) \]

can be the final state of the system (17) at moment \( T \) if and only if the moment problem
\[ z_{n,j}^\pm = \left( \tilde{u}, \eta_n,j e^{\lambda_n \tau} \right)_{L^2(0,T;\mathbb{C}^K)}, \]  

has a solution \( u \) in \( L^2(0,T;\mathbb{C}^K) \).

3.2. Channel controls

Here we study the case where different controls \( u_j \) may act different time and show controllability of Type (i).

Consider the almost normalized by (15) vector exponential family
\[ \mathcal{E} = \left\{ \frac{1}{\pi n} \eta_n e^{i \lambda_n t} \right\}_{\lambda_n \in \sigma(A)} = \{ \epsilon_n \}. \]

(21)

The asymptotic of the spectrum is in (6). The main term of the asymptotic of the vectors \( \eta_n \) is
\[ \frac{1}{\pi n} \eta_{jn}^\pm = \zeta_j + O(1/n). \]  

(22)

We show that this asymptotic of the spectrum and of the eigenvectors in itself leads to controllability.
Theorem 2. For any $\varepsilon > 0$ the family $\mathcal{E}$ forms an $\mathcal{L}$-basis in the space

$$\mathcal{L}_\varepsilon = L^2(0, T_1 + \varepsilon) \oplus L^2(0, T_2 + \varepsilon) \oplus \cdots \oplus L^2(0, T_K + \varepsilon).$$

Proof. Set

$$\mathcal{E}^0 = \{\xi_j e^{i\pi \varepsilon n t}\}_{j=1,\ldots,K; n \in \mathbb{Z}}.$$  

Consider the “$j$-th” subfamily $\xi_j e^{i \pi \varepsilon n t}$ of $\mathcal{E}^0$. This family forms an orthogonal almost normalized basis in the subspace

$$\{\xi_j f(t), \ f \in L^2(0, T_j)\}$$

of $\mathcal{L}_0$. Thus, the family $\mathcal{E}^0$ forms an almost normalized orthogonal basis in $\mathcal{L}_0$.

Consider the isomorphism $\tilde{\Omega}$ in $\mathcal{L}_\varepsilon$

$$f \rightarrow \tilde{\Omega} f = \text{diag}[e^{ij\varepsilon}] f.$$  

The family $\tilde{\Omega}\mathcal{E}^0$ is then an $\mathcal{L}$-basis in $\mathcal{L}_\varepsilon$ and asymptotically close to $\mathcal{E}$ with the finite family $\{\xi_j\}$. The theorem now following from [2, Th. II.5.9]. Nevertheless we give some explanations which will be help in the next subsection.

Recall that a bounded and analytic in the upper half-plane $\mathbb{C}_+$ matrix valued function $G$ is an inner (in $\mathbb{C}_+$) function if it is unitary almost where on the real axis. If the determinant of $G$ is an entire function (has the form $a \exp(ib\varepsilon)$ with $|a| = 1$ and $b \geq 0$), it is called an entire singular (matrix valued) function ESF. See, for example [14,2]. By the Paley–Wiener theorem the inverse Fourier transform $F$ maps the space $L^2(0, T)$ onto the subspace $K_T = H^2_+ \ominus e^{izT}H^2_+$ of the Hardy space $H^2_+$ in the upper half-plane.

In our case the image of $\mathcal{L}_0$ under the inverse Fourier transform is

$$F \mathcal{L}_0 = \xi_1 K_{T_1} \oplus \xi_2 K_{T_2} \cdots \oplus \xi_K K_{T_K}.$$  

If we take the diagonal ESF

$$\Theta = [e^{iT_j z}]$$

we find

$$F \mathcal{L}_0 = K_\Theta = H^2_+(\mathbb{C}^K) \ominus \Theta H^2_+(\mathbb{C}^K).$$  

Now we can use [2, Th. II.5.9]. By this theorem the image $F \mathcal{E}$ forms an $\mathcal{L}$-basis in $K_{\exp(i\varepsilon)\Theta}$ for any $\varepsilon$. Then $\mathcal{E}$ forms an $\mathcal{L}$-basis in $\mathcal{L}_\varepsilon$.  

Theorem 3. (i) For any $\varepsilon > 0$ the system (1), (2) is controllable in the state space $\mathcal{A}H = L^2(0, 1; \mathbb{C}^K) \oplus H^{-1}(0, 1; \mathbb{C}^K)$ for $u \in \mathcal{L}_\varepsilon$ (the $j$-th channel control $u_j$ supported on $[0, T_j + \varepsilon]$).

(ii) For $u \in L^2(0, T; \mathbb{C}^K)$ the solution is in $C(0, T; \mathcal{A}H)$.  

Proof. (i) Take

\[ z = \sum z_n \varphi_n \in A_H. \]

Then this state is reachable if and only if we can solve the moment equality (20), which we write with respect to the family \( \mathcal{E} \), see (21)

\[ \frac{1}{\pi n} z_n = (\tilde{u}, e_n)_{\mathcal{L}_\varepsilon}. \tag{23} \]

The eigenvalues \( \lambda_n \) satisfy

\[ \frac{1}{C} n \leq |\lambda_n| \leq Cn, \quad C > 0. \]

For \( z \in A_H \) the left hand side of (23) forms an \( \ell^2 \)-sequence and because \( \mathcal{E} \) forms an \( \mathcal{L} \) basis in \( \mathcal{L}_\varepsilon \), this moment problem has a solution. From Lemma 2 we conclude that the state \( z \) is a reachable state.

(ii) Take a control from \( L^2(0, T; C^K) \). Extending if necessary channel controls \( u_j \) by zero we can consider \( u \) in \( \mathcal{L}_\varepsilon \) for some \( \varepsilon \). The formal solution is

\[ z(T) = \sum c_n(T) \varphi_n(x) = \sum \pi n (\tilde{u}, e_n)_{\mathcal{L}_\varepsilon} \varphi_n(x). \]

By (i) we know that the sequence of the inner products is in \( \ell^2 \). Therefore \( z(T) \in A_H \). The solution is in fact continuous in this space, see, e.g. [2].

Remark 1. We can say more (see the notation ‘B-controllability’ in [2]). Let \( T_\varepsilon = T_{\text{max}} + \varepsilon \). Then

(i) in the affine infinite dimensional set of controls \( u \in \mathcal{L}_\varepsilon \) transferring the system from the zero state to the given one \( (y(\cdot, T_\varepsilon), y_t(\cdot, T_\varepsilon)) \) there exists the control \( u_0 \) with the minimal norm.

(ii) norm \( \|u_0\|_{\mathcal{L}_\varepsilon} \) is equivalent to the norm of the final state: there exist the constant \( C > 0 \) such that

\[ \frac{1}{C} \left\| \begin{pmatrix} y(\cdot, T_\varepsilon) \\ y_t(\cdot, T_\varepsilon) \end{pmatrix} \right\|_{A_H} \leq \|u_0\|_{\mathcal{L}_\varepsilon} \leq C \left\| \begin{pmatrix} y(\cdot, T_\varepsilon) \\ y_t(\cdot, T_\varepsilon) \end{pmatrix} \right\|_{A_H}. \]

This inequalities means that the target space \( A_H \) is optimal for the controls from \( L^2 \).

Now we discuss the controllability of the system in the case when some channel controls are absent. The channels are coupled in weaker terms and we can not expect that the system will be exact controllable.

**Theorem 4.** Let one of the controls \( u_j \) be zero. Then the system is not exact controllable in \( A_H \) in any time.
Proof. Let \( u_1 \equiv 0 \). Denote by \( \tilde{\eta}_n \) the projections of \( \eta_n \) onto \((K - 1)\)-dim subspace orthogonal to \( \xi_1 \). Then, following Sec. 3.1, we obtain a moment problem with respect to the exponential family

\[
\{ \tilde{\eta}_{n,j}^\pm e^{\xi_n,j t} \}_{n,j \in \sigma}
\]

in \( L^2(0, T; \mathbb{C}^{K-1}) \).

If this family is not minimal in \( L^2(0, T; \mathbb{C}^{K-1}) \), then the system is not exactly controllable. Indeed, in this case not all eigen modes are reachable. Suppose that this family is minimal. Then the subfamily corresponding to \( j = 1 \) is minimal also. Let us consider the scaled subfamily

\[
E_{sc} = \left\{ \frac{1}{\pi n} \tilde{\eta}_{n,1}^\pm e^{\xi_n,1 t} \right\} = \{ \tilde{e}_n^\pm \}
\]

and denote the biorthogonal family by \( \tilde{\theta}_n^\pm \). From the definition of the biorthogonal elements we have

\[
\| \tilde{e}_n^\pm \| \cdot \| \tilde{\theta}_n^\pm \| \geq 1
\]

(the norms of the space \( L^2(0, T; \mathbb{C}^{K-1}) \)).

From the asymptotic (6) we see that \( \tilde{e}_n^\pm \to 0 \) what implies that the family \( \{ \tilde{\theta}_n^\pm \} \) is not uniformly bounded. By [2, Theorem 1.2.1 (b)] the moment problem with respect to \( E_{sc} \)

\[
(f, \tilde{e}_n^\pm) = d_n
\]

can not be solved for all elements \( \{ d_n \} \in \ell^2 \). Therefore the moment problem (19) can not be solved for all elements \( \{ \lambda_n d_n \} \{ d_n \} \in \ell^2 \). \( \square \)

Remark 2. If no vector \( \tilde{\eta}_{n,1}^\pm \) vanishes, the system is spectral controllable for large \( T \). Indeed, the spectrum of \( A \) is separated and is in a strip \( |\Re \lambda| \leq C \). In this case the scalar family \( \{ e^{\xi t} \} \) is minimal in \( L^2(0, T_1 + \cdots + T_K) \). Thus, \( T_1 + \cdots + T_K \) is a very rough upper estimate of the time of the spectral controllability.

3.3. Minimal time of controllability

Here we consider channel controls supported on the same interval \([0, T]\) (Type (ii)) and obtain the sharp time of controllability. To prove this we use the approach based on the Pavlov’ idea [15, 11], the Hardy space theory, and the theory of the differential equation with large parameter.

Theorem 5. The family \( E \) forms an \( L \)-basis in \( L^2(0, T_{\text{max}}; \mathbb{C}^K) \).

Proof. First, use an isomorphism

\[
f \to e^{-at} f
\]
with large enough \( a \). The isomorphism shifts the spectrum of \( \mathcal{E} \), i.e., the spectrum of \( \mathcal{A}^* \), in the left half-plane. To work with the Hardy space \( H^2_+ \) in the upper half-plane we replace \( \lambda_n \) by \( ik_n \) in the families of exponentials.

Keep the notation \( \mathcal{E} \) of the family and denote its elements as

\[
e_n = \frac{1}{\pi n} \eta_n e^{ik_nt}, \quad k_n = -i \tilde{\lambda}_n + i \alpha.
\]

The first step in the Pavlov’ approach is the following result.

**Lemma 3.** \( \mathcal{E} \) forms an \( \mathcal{L} \)-basis in \( L^2(0, \infty; \mathbb{C}^K) \).

**Proof of the lemma.** The family consists of \( K \) subfamilies.

\[
\mathcal{E}^{(j)} = \frac{1}{\pi n_{n,j}} \eta_{n,j} e^{ik_{n,j}t}.
\]

From (22) we see that the vector coefficients in subfamilies go to the vectors \( \xi_j \) of the standard basis. From (6) we see that \( k_n \) of any subfamily are separated and lie in a strip parallel to the real axis. Thus, the family is an union of Carleson series with orthogonal limit vectors and forms an \( \mathcal{L} \)-basis in \( L^2(0, \infty; \mathbb{C}^K) \) [14, Lecture 7.4]. The lemma is proved. \( \square \)

The second step in the Pavlov’ approach is to prove that the orthoprojector from the span of \( \mathcal{E} \) in \( L^2(0, \infty; \mathbb{C}^K) \) on its span in \( L^2(0, T_{\max}; \mathbb{C}^K) \) is an isomorphism. We need now the notations of matrix valued function in the Hardy spaces, exactly, strong functions, Blaschke–Potapov product (BPP), and outer functions. The definitions and the properties can be found in, e.g., [14], [2, Ch. I, Sec. 4].

**Definition 1.** An entire strong matrix valued function \( F \) is called the generating function (GF) of a family

\[
\mathcal{E}_\mu = \{\eta_\mu e^{-i\bar{\mu}t}\}_{\mu \in \sigma \subset \mathbb{C}_+},
\]

if it has the factorization

\[
F = F^+ e^\Pi = F^- e^{\Theta},
\]

here \( \Pi \) is a BPP, \( \sigma \) is the set of zeros of \( \det \Pi \), all zeros are simple, and \( \Pi(\mu) \eta_\mu = 0 \); \( F^\pm \) are inner functions in the upper and the lower half planes correspondingly, and \( \Theta \) is an ESF in the upper half-plane.

This definition is different from one in [2] by the order of factorizations. If we take the so-called associated functions \( \hat{F}(z) = F^*(-z) \), then (24) takes the form

\[
\hat{F} = \hat{\Pi} \hat{F}^+ = \hat{\Theta} \hat{F}^-.
\]

It is known [2] that if a family \( \mathcal{E}_\mu \) possesses a GF \( F \), then \( \mathcal{E}_\mu \) is minimal and complete in the Fourier image \( E = \mathcal{F}(H^2_+ \oplus \hat{\Theta} H^2_+) \).
Let us construct the GF of the family
\[ \tilde{\mathcal{E}} = \mathcal{E} \cup \mathcal{C}, \quad \mathcal{C} = \{ \zeta_j e^{-at} \}_1^K. \]
We add \( K \) exponentials to \( \mathcal{E} \) in order that to have a complete family. To construct the GF we start with the function \( G(z) = \Phi(1, z - ai) \), see (12), that has the required zeros, see (14).

**Proposition 1.** There exist an ESF \( \Theta_1 \) such that \( \Theta_1 G \) has the factorization
\[ \Theta_1 G = F_e^+ \Pi = \Theta F_e^- \]
and the matrix valued function \( \exp(izT_{\text{max}})\Theta^{-1} \) is bounded in the upper half-plane.

**Proof.** In fact this proposition has been proved in [2] for the system (3) and we show only the scheme of the proof of the estimate and omit the complicated transformation of the differential equations (7).

Write the equation (7) as the first order equation for \( W = (\lambda^{-1} \Phi) \)
\[ W' = \lambda \begin{pmatrix} 0 & I \\ B(x, \lambda) & 0 \end{pmatrix} W = \lambda B(x, \lambda) W \]
with
\[ B = D^{-1}(1 - R/\lambda^2 - Q/\lambda). \]
It is easy to see that the eigenvalues of \( B \) have the form \( \frac{1}{\varepsilon_j} (1 + O(1/\lambda)) \) and that the eigenvalues \( \mu_j^\pm \) of \( B \) are the eigenvalues of \( \pm \sqrt{B} \) and have the form
\[ \mu_j^\pm = \pm \frac{1}{\varepsilon_j} (1 + O(1/\lambda)). \]
The eigenfunctions are split into two groups and it is possible to find a change of variables \( W = UV \) such that the matrix of the system for \( V \) is block-diagonal
\[ V' = \begin{pmatrix} \lambda \sqrt{B} + D_+ & 0 \\ 0 & -\lambda \sqrt{B} + + D_- \end{pmatrix} V. \]
(25)
Here \( D_\pm(x, \lambda) \) are matrix valued functions bounded in the upper half-plane. The maximal eigenvalue of the hermitian part of the matrix is asymptotically \( |\Im \lambda|/c_{\text{min}} \) and from [10, Ch. 4, lemma 4.2] we obtain the estimate for \( V \)
\[ \|V(1, \lambda)\|_{C^2K} \leq C e^{|\Im \lambda|/c_{\text{min}}} \]
This gives
\[ \|G(\lambda)\|_{C^K} \leq C e^{\frac{1}{2} T_{\text{max}}|\Im \lambda|}. \]
Note that this estimate is sharp with respect to the coefficient \( \frac{1}{2} T_{\text{max}} \).
Because $\exp(izT_{\text{max}}/2)G(z)$ is bounded in $\mathbb{C}_+$, it may be factorized as

$$\exp(izT_{\text{max}}/2)G(z) = \Theta_+ F^+_e \Pi.$$

Similarly, in the lower half-plane the function $\exp(-izT_{\text{max}}/2)G(z)$ is bounded and

$$\exp(-izT_{\text{max}}/2)G(z) = \Theta_- F^-_e.$$

Now we are able to construct the GF as

$$F = \exp(izT_{\text{max}})\Theta_+^{-1} G(z) = F^+_e \Pi = \Theta F^-_e$$

with

$$\Theta = \exp(izT_{\text{max}})\Theta_+^{-1} \Theta_-.$$

The fact that the functions

$$\exp(izT_{\text{max}}/2)\Theta_+^{-1}, \exp(izT_{\text{max}}/2)\Theta_-$$

are bounded in $\mathbb{C}_+$ follows from the block diagonal system (25) similarly to the factorization in [2, Ch. I, Sec. 4].

Now we use the matrix analog of the Levin–Golovin theorem.

**Proposition 2.** [2] If the matrix norms of the GF and its inverse are bounded on the real axis

$$\|F(x)\| \leq \text{Const}, \quad \|F^{-1}(x)\| \leq \text{Const},$$

then the orthoprojector from the span of $\tilde{E}$ in $L^2(0, \infty; \mathbb{C}^K)$ onto $E = \mathcal{F}(H^2_+ \ominus \hat{\Theta} H^2_+)$ is an isomorphism.

We note that the necessary and sufficient condition when this orthoprojector is an isomorphism (the matrix analog of the Muckenhoupt condition) can be found in [18].

The estimates of the proposition follow from (13) and we have obtained that $\tilde{E}$ is an Riesz basis in $E$. From Proposition 1 we know that the matrix $\exp(izT_{\text{max}})\Theta^{-1}$ is bounded in the upper half-plane. The same is true for the associated ESF $\hat{\Theta}$. Then $e^{izT_{\text{max}}}H^2_+$ is a subset of $\Theta H^2_+$ and

$$E \subset \mathcal{F}(H^2_+ \ominus e^{izT_{\text{max}}}H^2_+) = L^2(0, T_{\text{max}}; \mathbb{C}^K).$$

Therefore the orthoprojector from the span of $\tilde{E}$ in $L^2(0, \infty; \mathbb{C}^K)$ on the span in $L^2(0, T_{\text{max}}; \mathbb{C}^K)$ is an isomorphism. Lemma 3 gives the $\mathcal{L}$-basis property of $\tilde{E}$. □

The next controllability result follows from the basis property similar to the proof of Theorem 3 based on Theorem 2.

**Theorem 6.** The system (1), (2) is controllable in the state space $\mathcal{A}H = L^2(0, 1; \mathbb{C}^K) \oplus H^{-1}(0, 1; \mathbb{C}^K)$ with the control space $L^2(0, T_{\text{max}}; \mathbb{C}^K)$. 

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