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Pointwise stabilisation of a string with time delay in the observation

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ABSTRACT
Pointwise stabilisation is proposed in this paper for a string equation where the observation signal is subject to a time delay. Different from the boundary control, the feedback stabiliser is acting at the middle joint of the string. Well-posedness of the open-loop system and solvability of the observer are shown first. An observer system is then designed to estimate the state at the time interval when the observation is available, and a predictor system is designed to predict the state at the time interval when the observation is not available. Pointwise output feedback controller is introduced to make the closed-loop system asymptotically stable for the non-smooth initial values and exponentially stable for the smooth initial values, respectively. Simulation results demonstrate that the output feedback based on the observer and predictor effectively stabilises the pointwise control system with time delay.

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String equation; time delay; pointwise control; estimated state feedback; stability

1. Introduction

In industrial systems, it is common that a number of actuators are combined with some joints. One of the most significant examples is pointwise control system mentioned by such references as Ammari, Tucsnak, and Henrot (2000), Ammari, Liu, and Tucsnak (2002), Ammari and Tucsnak (2000), Chen, Delfour, Krall, and Payre (1987), Chen et al. (1989) and Tucsnak (1998). A string equation with a pointwise interior actuator is considered by Tucsnak (1998) where the uniform decay rate is gained by the spectrum analysis method. Later stabilisation of the same pointwise control string system with a Neumann right-hand boundary condition is characterised completely by Ammari et al. (2000) dependent on the positions of the actuator while the fastest decay rate is obtained with the actuator at the middle of string actuator. Stabilisation and control of N serially connected Euler–Bernoulli beams are considered by Chen et al. (1987), while design and behaviour of dissipative joints for coupled beams are analysed in Chen et al. (1989). Both uniform and non-uniform energy decays of an Euler–Bernoulli beam subject to a pointwise feedback force are shown in Ammari and Tucsnak (2000). Obviously, it is difficult to stabilise pointwise control systems whose stability properties depend on the location of the joints in an unrobust way since it may need more actuators (see Ammari & Tucsnak, 2000). Furthermore, the energy decay rates for both Euler–Bernoulli and Rayleigh beams with pointwise shear force and bending moment are studied by Ammari et al. (2002). And it is indicated that the energy decays exponentially independently of the position of the actuator for the Euler–Bernoulli beam, while the energy exponential decay depends on the position of the actuator for the Rayleigh beam. Later on, by exerting the distribute control on the two connected Rayleigh beams, the exponential stability is shown to be robust with respect to the location of the joint by the Riesz basis approach (Guo, Wang, & Zhou, 2008), while well-posedness and exponential stabilisation of the hinged Reyleigh beam is obtained by the well-posed theory (Weiss & Curtain, 2008).

On the other hand, in a physical control system, there frequently exists a time delay between the controller and observation signal, such as the process of sampled-data control (see Karafyllis & Krstic, 2013; Zhong, 2006). Existence of time delay may deform or damage stability of the originally stable system (see Gumowski & Mira, 1968). It is indicated in Datko (1997) and Datko, Lagnese, and Polis (1986) that even a small amount of time delay may make the string equation system unstable. For distributed parameter control systems, time delay in observation or control may lead to difficult mathematical challenge (see Fleming, 1988). Stabilization may be obtained by transforming the linear system with delayed control action into systems without

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delays (Artstein, 1982). Later on, for the robust control of time-delay systems, controller parameterization and design to controller implementation have been analyzed (Zhong, 2006). A class of switched linear systems with time delay have been stabilized by proposing switching law (Zhao et al., 2012). By the methods of Lyapunov-Krasovskii funtional approach and linear matrix inequalities, asymptotic stability of the switched systems with time delays or the neutral networks have been considered (Wang et al. 2011, Wang et al. 2012, or Zhang et al. 2015). More effectively, such designs as smith predictor are commonly available for finite-dimensional systems in order to compensate arbitrarily long input delays. Such compensation have recently been introduced for distributed parameter systems in Krstic (2008). Compensations of arbitrary long delay for the input are established in the unstable reaction–diffusion and wave equation, respectively (Krstic, 2009, 2011). Indeed stability of the reaction–diffusion equation is shown to be robust with respect to long input delay after designing these kinds of compensation (see Krstic, 2009). For a string equation with the output feedback loop which is subject to a time delay, the necessary and sufficient conditions have been given to guarantee the exponential stability of the closed-loop system if the time delay is equal to the even multiples of the wave propagation time (Wang, Guo, & Krstic, 2011). The other kinds of feedback where a certain delay is included as a part of the control law have been proven to make the vibrating string system exponentially stable by Gugat (2010a, 2010b), and Gugat and Tucsnak (2011). For the distributed parameter systems where the boundary observation is suffered from an arbitrary time delay, based on the design of the observer and predictor, for instance, an output feedback law has been introduced to stabilise a one-dimensional wave equation (Guo, Xu, & Hammouri, 2012). This method of the observer–predictor-based scheme has also effectively been applied to stabilise an Euler–Bernoulli beam with output delay (see Guo & Yang, 2009). To our knowledge, stabilisation of pointwise control systems with time delay, especially that with delayed observation, has been rarely involved in even recent research.

This paper considers the system as following that the actuator is constrained in the middle point of the string:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \ -1 < x < 1, t > 0, \\ w(-1,t) = w_x(1,t) = 0, \ t \geq 0, \\ w(0^-,t) = w(0^+,t), \ t \geq 0, \\ w_x(0^+,t) - w_x(0^-,t) = u(t), \ t \geq 0, \\ y(t) = w_t(0, t-\tau), \ t \geq \tau, \\ w(x,0) = w_0(x), \ w_t(x,0) \\ \quad = w_1(x), \ -1 \leq x \leq 1, \end{cases} \tag{1}$$

where x is the position, t is the time, w represents the state, $u(t)$ is control, $(w_0, w_1)^T$ is the initial value, $\tau(>0)$ is a given time delay, and $y(t)$ is observation which is suffered from the time delay τ. In this system, the left boundary end of the string is fixed while the right boundary end is free, and the observation signal is subject to the time delay.

The system above is considered in Hilbert space as follows:

$$\begin{aligned} \mathcal{H} = \big\{ & (f,g)^T \in H^1((-1,0)\cup(0,1)) \\ & \times L^2((-1,0)\cup(0,1)) \mid f(-1) = f'(1) = 0 \big\} \end{aligned} \tag{2}$$

with the inner-product-induced norm:

$$\begin{aligned} E_0(t) &= \frac{1}{2} \parallel (w(\cdot,t), w_t(\cdot,t))^T \parallel_{\mathcal{H}}^2 \\ &= \frac{1}{2}\int_{-1}^{1} [w_x^2(x,t) + w_t^2(x,t)]dx. \end{aligned} \tag{3}$$

Based on the Salamon–Weiss well-posed infinite-dimensional system theory, we prove that the open-loop system is well-posed. Then we design the observer system at the time interval when the observation signal is available, and design the predictor system at the time interval when the observation signal is not available. Naturally, we construct a control law with the estimated state by the state of the observer and predictor. We prove that the closed-loop system is asymptotically stable for non-smooth initial values, and exponentially stable for smooth initial values. Numerical simulations are illustrated to show stabilised effectiveness of the output feedback controller. By the approach of the observer- and predictor-based scheme, we provide a unified methodology to solve stabilisation of the pointwise control distributed parameter system where observation signal is suffered from a time delay.

We proceed as follows. The next section shows the well-posedness of the original open-loop system. In Section 3, we design the observer and predictor systems. The stability properties of the closed-loop system under the estimated state feedback control are then illustrated in Section 4. Section 5 gives simulation results and Section 6 concludes the paper.

2. Well-posedness of open-loop system

We introduce a new variable $z(x, t) = w_t(0, t - \tau x)$. Then the system (1) becomes

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \ -1 < x < 1, t > 0, \\ w(-1,t) = w_x(1,t) = 0, \ t \geq 0, \\ w(0^-,t) = w(0^+,t), \ t \geq 0, \\ w_x(0^+,t) - w_x(0^-,t) = u(t), \ t \geq 0, \\ \tau z_t(x,t) + z_x(x,t) = 0, \ 0 < x < 1, t > 0, \\ z(0,t) = w_t(0,t), \ t \geq 0, \\ w(x,0) = w_0(x), \ w_t(x,0) \\ \quad = w_1(x), \ -1 \leq x \leq 1, \\ z(x,0) = z_0(x), \ \ 0 \leq x \leq 1, \\ y(t) = z(1,t), \ t \geq 0, \end{cases} \tag{4}$$

where z_0 is the initial value of the new variable z.

With the state variable $(w(\cdot,t), w_t(\cdot,t), z(\cdot,t))^T$, we consider the system (4) in the energy state space $\mathbb{H} = \mathcal{H} \times L^2(0,1)$. The norm of $(w(\cdot,t), w_t(\cdot,t), z(\cdot,t))^T$ in $\mathbb{H}$ is defined by the energy as follows:

$$\begin{aligned} E_1(t) &= \frac{1}{2} \| (w(\cdot,t), w_t(\cdot,t), z(\cdot,t))^T \|_{\mathbb{H}}^2 \\ &= \frac{1}{2}\int_{-1}^{1} [w_x^2(x,t) + w_t^2(x,t)]dx + \frac{1}{2}\int_0^1 z^2(x,t)dx. \end{aligned} \tag{5}$$

The input space and the output space are the same $U = Y = \mathbb{C}$.

Lemma 2.1: *The system* (4) *is well-posed: for each $u \in L^2_{loc}(0,\infty)$ and initial datum $(w_0, w_1, z_0)^T \in \mathbb{H}$, there exists a unique solution $(w(\cdot,t), w_t(\cdot,t), z(\cdot,t))^T \in C(0,\infty;\mathbb{H})$ of* (4) *and for each $T > 0$ there exists a constant $C_T > 0$ such that*

$$\begin{aligned} &\| (w(\cdot,T), w_t(\cdot,T), z(\cdot,T))^T \|_{\mathbb{H}}^2 + \int_0^T | y(t) |^2 dt \\ &\leq C_T\left[\| (w_0, w_1, z_0)^T \|_{\mathbb{H}}^2 + \int_0^T | u(t) |^2 dt\right]. \end{aligned} \tag{6}$$

Proof: It is known that the following system

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \ -1 < x < 1, t > 0, \\ w(-1,t) = w_x(1,t) = 0, \ t \geq 0, \\ w(0^-,t) = w(0^+,t), \ t \geq 0, \\ w_x(0^+,t) - w_x(0^-,t) = u(t), \ t \geq 0, \\ y_w(t) = w_t(0,t), \ t \geq 0, \end{cases} \tag{7}$$

can be written as

$$\sum(\mathcal{A},\mathcal{B},\mathcal{C}) : \begin{cases} \frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathcal{A}\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathcal{B}u(t) \\ y_w(t) = \mathcal{C}\begin{pmatrix} w \\ w_t \end{pmatrix} = w_t(0,t) \end{cases} \tag{8}$$

where $\mathcal{A} : D(\mathcal{A})(\subset \mathcal{H}) \longrightarrow \mathcal{H}$ is defined as follows:

$$\begin{cases} \mathcal{A}(f,g)^T = (g, f'')^T, \ \forall (f,g)^T \in D(\mathcal{A}), \\ D(\mathcal{A}) = \{(f,g)^T \in \mathcal{H} \cap (H^2((-1,0)\cup(0,1)) \\ \quad \times H^1((-1,0)\cup(0,1))) \mid \\ \quad f(0^-) = f(0^+), f'(0^-) = f'(0^+)\}, \end{cases} \tag{9}$$

and

$$\mathcal{B} = \begin{pmatrix} 0 \\ -\delta(x) \end{pmatrix}, \mathcal{C} = (0, \langle \cdot, \delta(x)\rangle). \tag{10}$$

Here $\delta(\cdot)$ denotes the Dirac function. Obviously, both $\mathcal{B}$ and $\mathcal{C}$ are unbounded operators.

By the well-posed linear infinite-dimensional system theory (see Curtain, 1997; Guo & Zhang, 2005; Tucsnak & Weiss, 2009), it is equivalent to showing that $\mathcal{C}$ is admissible for $e^{\mathcal{A}t}$, $\mathcal{B}^*$ is admissible for $e^{\mathcal{A}^*t}$ and the transfer function is bounded on some right-half complex plane.

A direct computation shows that

$$\mathcal{C}\mathcal{A}^{-1}(\varphi,\psi)^T = \varphi(0), \ \ \forall (\varphi,\psi)^T \in \mathcal{H}, \tag{11}$$

which tells us that $\mathcal{C}\mathcal{A}^{-1}$ is bounded.

Define

$$\begin{aligned} \rho(t) = &\int_{-1}^{0} (x+1)w_x(x,t)w_t(x,t)dx \\ &+ \int_0^1 (x-1)w_x(x,t)w_t(x,t)dx. \end{aligned} \tag{12}$$

Then $E_0(t) = E_0(0)$ and $|\rho(t)| \leq E_0(t)$ for $\forall t \geq 0$. Noticing that

$$\dot\rho(t) = w_t^2(0,t) + w_x^2(0,t) - E_0(t), \tag{13}$$

we have that

$$\int_0^T w_t^2(0,t)dt \leq (T+2)E_0(0), \tag{14}$$

which together with (11) show that $\mathcal{C}$ is admissible for $e^{\mathcal{A}t}$.

Actually, a straightforward computation gives that

$$\begin{cases} \mathcal{A}^*(\varphi,\psi)^T = (-\psi, \varphi'')^T, \ \ \forall (\varphi,\psi)^T \in D(\mathcal{A}^*), \\ D(\mathcal{A}^*) = \{(\varphi,\psi)^T \in \mathcal{H} \cap (H^2((-1,0)\cup(0,1)) \\ \quad \times H^1((-1,0)\cup(0,1))) \mid \\ \quad \psi(0^-) = \psi(0^+), \varphi'(0^-) = \varphi'(0^+)\}, \end{cases} \tag{15}$$

and $\mathcal{B}^* = -(0, \langle \cdot, \delta(x)\rangle)$. Similar computation as that above shows that $\mathcal{B}^*$ is admissible for $e^{\mathcal{A}^*t}$.

Finally, the transfer function for the system (7) is found to be

$$H(s) = -\frac{1}{2}\cdot\frac{e^{2s}-e^{-2s}}{e^{2s}+e^{-2s}},\ s>0,$$

which is obviously bounded on some right-half complex plane.

Therefore, system (7) is well-posed in the sense of D. Salamon (see Curtain, 1997): for any $u \in L^2_{\text{loc}}(0,\infty)$ and $(w_0, w_1)^T \in \mathcal{H}$, there exists a unique solution $(w(\cdot,t), w_t(\cdot,t))^T \in C(0,\infty;\mathcal{H})$ of (8); and for any $T > 0$, there exists a constant $D_T > 0$ such that

$$\begin{aligned} &\| (w(\cdot,T), w_t(\cdot,T))^T \|_{\mathcal{H}}^2 + \int_0^T |\, y_w(t)\,|^2\, dt \\ &\le D_T\left[\| (w_0, w_1)^T \|_{\mathcal{H}}^2 + \int_0^T |\, u(t)\,|^2\, dt\right]. \end{aligned} \tag{16}$$

On the other hand, the analytic expression of the 'z' part equation

$$\begin{cases} \tau z_t(x,t) + z_x(x,t) = 0, \\ z(0,t) = w_t(0,t), \quad z(x,0) = z_0(x), \end{cases} \tag{17}$$

can be obtained:

$$z(x,t) = \begin{cases} z_0\left(x - \dfrac{t}{\tau}\right), & x \ge \dfrac{t}{\tau}, \\ w_t(0, t - x\tau), & x < \dfrac{t}{\tau}, \end{cases} \tag{18}$$

by integrating along the characteristic line.

Therefore, we have, for any $T > 0$,

$$\int_0^1 |\, z(x,T)\,|^2\, dx = \begin{cases} \displaystyle\int_0^{1-T/\tau} z_0^2(x)dx \\ \quad + \frac{1}{\tau}\int_0^T w_t^2(0,t)dt, & T \le \tau, \\ \displaystyle\frac{1}{\tau}\int_{T-\tau}^T w_t^2(0,t)dt, & T > \tau, \end{cases}$$

and

$$\int_0^T |\, y(t)\,|^2\, dt = \begin{cases} \tau \displaystyle\int_{1-T/\tau}^1 z_0^2(x)dx, & T \le \tau, \\ \tau \displaystyle\int_0^1 z_0^2(x)dx \\ \quad + \int_0^{T-\tau} w_t^2(0,t)dt, & T > \tau. \end{cases}$$

These together with (16) gives (6). ■

3. Observer and predictor design

In this section, we proceed two steps to estimate the state of (1) via the observer and predictor.

Step 1: Construct an observer to estimate the state $\{(w(x,s), w_s(x,s))^T, s \in [0, t-\tau], t > \tau\}$ from the known observation $\{y(s+\tau), s \in [0, t-\tau], t > \tau\}$.

Since the observation $\{y(s+\tau), s \in [0, t-\tau], t > \tau\}$ is known and $\{(w(x,s), w_s(x,s))^T, s \in [0, t-\tau], t > \tau\}$ satisfies

$$\begin{cases} w_{ss}(x,s) = w_{xx}(x,s), \ -1 < x < 1, s > 0, \\ w(-1,s) = w_x(1,s) = 0, \ s \ge 0, \\ w(0^-,s) = w(0^+,s), \ s \ge 0, \\ w_x(0^+,s) - w_x(0^-,s) = u(s), \ s \ge 0, \\ y(s+\tau) = w_s(0,s), \ s \ge 0, \\ w(x,0) = w_0(x), \ w_s(x,0) = w_1(x), \ -1 \le x \le 1. \end{cases} \tag{19}$$

We can construct naturally a Luenberger observer for system (19) as following, for $k_1 > 0$:

$$\begin{cases} \widehat{w}_{ss}(x,s) = \widehat{w}_{xx}(x,s), \ -1 < x < 1, 0 < s \\ \quad \le t-\tau, t > \tau, \\ \widehat{w}(-1,s) = \widehat{w}_x(1,s) = 0, \ 0 \le s \le t-\tau, t \ge \tau, \\ \widehat{w}(0^-,s) = \widehat{w}(0^+,s), \ 0 \le s \le t-\tau, t \ge \tau, \\ \widehat{w}_x(0^+,s) - \widehat{w}_x(0^-,s) = u(s) \\ \quad + k_1[\widehat{w}_s(0,s) - y(s+\tau)], \ 0 \le s \le t-\tau, t \ge \tau, \\ \widehat{w}(x,0) = \widehat{w}_0(x), \ \widehat{w}_s(x,0) = \widehat{w}_1(x), \ -1 \le x \le 1, \end{cases} \tag{20}$$

where $(\widehat{w}_0, \widehat{w}_1)^T$ is the (arbitrarily assigned) initial state of observer.

In order for (20) to be an observer for (19), we have to show its convergence. To do this, let

$$\varepsilon(x,s) = \widehat{w}(x,s) - w(x,s), \ 0 \le s \le t-\tau, t > \tau. \tag{21}$$

Then by (19) and (20), $\epsilon(x,s)$ satisfies

$$\begin{cases} \varepsilon_{ss}(x,s) = \varepsilon_{xx}(x,s), \ -1 < x < 1, 0 < s \le t-\tau, t > \tau, \\ \varepsilon(-1,s) = \varepsilon_x(1,s) = 0, \ 0 \le s \le t-\tau, t \ge \tau, \\ \varepsilon(0^-,s) = \varepsilon(0^+,s), \ 0 \le s \le t-\tau, t \ge \tau, \\ \varepsilon_x(0^+,s) - \varepsilon_x(0^-,s) = k_1\varepsilon_s(0,s), \ 0 \\ \quad \le s \le t-\tau, t \ge \tau, \\ \varepsilon(x,0) = \widehat{w}_0(x) - w_0(x), \ \varepsilon_s(x,0) \\ \quad = \widehat{w}_1(x) - w_1(x), \ -1 \le x \le 1. \end{cases} \tag{22}$$

The system (22) can be written as follows:

$$\frac{d}{ds}\begin{pmatrix} \varepsilon(\cdot,s) \\ \varepsilon_s(\cdot,s) \end{pmatrix} = \mathbb{B}\begin{pmatrix} \varepsilon(\cdot,s) \\ \varepsilon_s(\cdot,s) \end{pmatrix} \tag{23}$$

where the operator $\mathbb{B} : D(\mathbb{B})(\subset \mathcal{H}) \to \mathcal{H}$ is defined as follows:

$$\begin{cases} \mathbb{B}\left(f, g\right)^T = \left(g, f''\right)^T, \ \forall (f, g)^T \in D(\mathbb{B}), \\ D(\mathbb{B}) = \{(f, g)^T \in \mathcal{H} \cap (H^2((-1, 0) \cup (0, 1)) \\ \quad \times H^1((-1, 0) \cup (0, 1))) \mid \\ f(0^-) = f(0^+), \, f'(0^+) - f'(0^-) = k_1 g(0)\}. \end{cases} \tag{24}$$

It is known that $\mathbb{B}$ generates an exponentially stable C_0-semigroup on $\mathcal{H}$ by Ammari et al. (2000) and Liu (1988), that is to say, for any $(w_0, w_1)^T \in \mathcal{H}$ and $(\widehat{w}_0, \widehat{w}_1)^T \in \mathcal{H}$, there exists a unique solution $(\epsilon(\,\cdot, s), \epsilon_s(\,\cdot, s))^T$ of (22) which satisfies

$$\| \, (\varepsilon(\cdot, s), \varepsilon_s(\cdot, s))^T \, \|_{\mathcal{H}} \le M e^{-\omega s} \, \| \, (\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T \, \|_{\mathcal{H}}, \forall s \in [0, t - \tau], t > \tau, \tag{25}$$

for some positive constants M and ω.

Step 2: Predict $\{(w(x, s), w_s(x, s))^T, s \in (t - \tau, t], t > \tau\}$ by $\{(\widehat{w}(x, s), \widehat{w}_s(x, s))^T, s \in [0, t - \tau], t > \tau\}$.

This can be achieved by solving (1) with estimated initial value $\{(\widehat{w}(x, t - \tau), \widehat{w}_s(x, t - \tau))^T, t > \tau\}$ obtained from (19):

$$\begin{cases} \widehat{w}^t_{ss}(x, s) = \widehat{w}^t_{xx}(x, s), \ -1 < x < 1, t - \tau < s \le t, t > \tau, \\ \widehat{w}^t(-1, s) = \widehat{w}^t_x(1, s) = 0, \ t - \tau \le s \le t, t \ge \tau, \\ \widehat{w}^t(0^-, s) = \widehat{w}^t(0^+, s), \ t - \tau \le s \le t, t \ge \tau, \\ \widehat{w}^t_x(0^+, s) - \widehat{w}^t_x(0^-, s) = u(s), \ t - \tau \le s \le t, t \ge \tau, \\ \widehat{w}^t(x, t - \tau) = \widehat{w}(x, t - \tau), \ \widehat{w}^t_s(x, t - \tau) \\ \quad = \widehat{w}_s(x, t - \tau), \ -1 \le x \le 1, t \ge \tau. \end{cases} \tag{26}$$

We finally get the estimated state variable by

$$\widetilde{w}(x, t) = \widehat{w}^t(x, t), \quad \widetilde{w}_t(x, t) = \widehat{w}^t_s(x, t), \ t > \tau. \tag{27}$$

Theorem 3.1: *For any $t > \tau$, we have*

$$\left\| w(\cdot, t) - \widetilde{w}(\cdot, t), w_t(\cdot, t) - \widetilde{w}_t(\cdot, t))^T \right\|_{\mathcal{H}} \le M e^{-\omega(t - \tau)} \left\| (\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T \right\|_{\mathcal{H}}, \tag{28}$$

where $(\widehat{w}_0, \widehat{w}_1)^T$ is the initial state of observer (20), $(w_0, w_1)^T$ is the initial state of original system (1), and M, ω are constants in (25).

Proof: Let

$$\varepsilon^t(x, s) = \widehat{w}^t(x, s) - w(x, s), \quad -1 \le x \le 1, t - \tau \le s \le t, t > \tau. \tag{29}$$

Then $\epsilon^t(x, s)$ satisfies

$$\begin{cases} \varepsilon^t_{ss}(x, s) = \varepsilon^t_{xx}(x, s), \ -1 < x < 1, t - \tau < s \le t, t > \tau, \\ \varepsilon^t(-1, s) = \varepsilon^t_x(1, s) = 0, \ t - \tau \le s \le t, t \ge \tau, \\ \varepsilon^t(x, t - \tau) = \varepsilon(x, t - \tau), \ \varepsilon^t_s(x, t - \tau) \\ \quad = \varepsilon_s(x, t - \tau), \ -1 \le x \le 1, t \ge \tau, \end{cases} \tag{30}$$

which is a conservative system, that is,

$$\| \, (\varepsilon^t(\cdot, t), \varepsilon^t_s(\cdot, t))^T \, \|_{\mathcal{H}} = \| \, (\varepsilon(\cdot, t - \tau), \varepsilon_s(\cdot, t - \tau))^T \, \|_{\mathcal{H}} \, .$$

This together with (25) and (27) gives (28). ■

4. Stability for closed-loop system

Since the feedback $u(t) = k_2 w_t(0, t) (k_2 > 0)$ stabilises exponentially the system (1), and we have the estimation $\widetilde{w}_t(0, t)$ of $w_t(0, t)$, it is natural to design the estimated state feedback control law as follows:

$$u^*(t) = \begin{cases} 0, \quad t \in [0, \tau], \\ k_2 \widehat{w}^t_s(0, t), \quad t > \tau, k_2 > 0, \end{cases} \tag{31}$$

under which, the closed-loop system becomes a system of partial differential equations (32)–(34):

$$\begin{cases} w_{tt}(x, t) = w_{xx}(x, t), \ -1 < x < 1, t > 0, \\ w(-1, t) = w_x(1, t) = 0, \ t \ge 0, \\ w(0^-, t) = w(0^+, t), \ t \ge 0, \\ w_x(0^+, t) - w_x(0^-, t) = u^*(t), \ t \ge 0, \\ w(x, 0) = w_0(x), \ w_t(x, 0) = w_1(x), \ -1 \le x \le 1, \end{cases} \tag{32}$$

$$\begin{cases} \widehat{w}_{ss}(x, s) = \widehat{w}_{xx}(x, s), \\ \quad -1 < x < 1, 0 < s \le t - \tau, t > \tau, \\ \widehat{w}(-1, s) = \widehat{w}_x(1, s) = 0, \ 0 \le s \le t - \tau, t \ge \tau, \\ \widehat{w}(0^-, s) = \widehat{w}(0^+, s), \ 0 \le s \le t - \tau, t \ge \tau, \\ \widehat{w}_x(0^+, s) - \widehat{w}_x(0^-, s) = u^*(s) + k_1[\widehat{w}_s(0, s) \\ \quad -y(s + \tau)], 0 \le s \le t - \tau, t \ge \tau, \\ \widehat{w}(x, 0) = \widehat{w}_0(x), \ \widehat{w}_s(x, 0) = \widehat{w}_1(x), -1 \le x \le 1, \end{cases} \tag{33}$$

and

$$\begin{cases} \widehat{w}^t_{ss}(x, s) = \widehat{w}^t_{xx}(x, s), \\ \quad -1 < x < 1, t - \tau < s \le t, t > \tau, \\ \widehat{w}^t(-1, s) = \widehat{w}^t_x(1, s) = 0, \ t - \tau \le s \le t, t \ge \tau, \\ \widehat{w}^t(0^-, s) = \widehat{w}^t(0^+, s), \ t - \tau \le s \le t, t \ge \tau, \\ \widehat{w}^t_x(0^+, s) - \widehat{w}^t_x(0^-, s) = u^*(s), \ t - \tau \le s \le t, t \ge \tau, \\ \widehat{w}^t(x, t - \tau) = \widehat{w}(x, t - \tau), \ \widehat{w}^t_s(x, t - \tau) \\ \quad = \widehat{w}_s(x, t - \tau), \ -1 \le x \le 1, t \ge \tau. \end{cases} \tag{34}$$

We consider system (32)–(34) in the state space $X = \mathcal{H}^3$. It is obvious that system (32)–(34) is equivalent to (35)–(37) for $t > \tau$ provided that $u^* \in L^2_{loc}(0, \infty)$ which will be clarified by (66) later in Lemma 2 of the appendix:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \ -1 < x < 1, t > \tau, \\ w(-1,t) = w_x(1,t) = 0, \ t \geq \tau, \\ w(0^-,t) = w(0^+,t), \ t \geq \tau, \\ w_x(0^+,t) - w_x(0^-,t) = k_2[w_t(0,t) \\ \quad +\varepsilon_s^t(0,t)], \ t \geq \tau, \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \ -1 \leq x \leq 1, \end{cases} \tag{35}$$

$$\begin{cases} \varepsilon_{ss}(x,s) = \varepsilon_{xx}(x,s), -1 < x < 1, 0 < s \leq t-\tau, t > \tau, \\ \varepsilon(-1,s) = \varepsilon_x(1,s) = 0, \ 0 \leq s \leq t-\tau, t \geq \tau, \\ \varepsilon(0^-,s) = \varepsilon(0^+,s), \ 0 \leq s \leq t-\tau, t \geq \tau, \\ \varepsilon_x(0^+,s) - \varepsilon_x(0^-,s) = k_1\varepsilon_s(0,s), 0 \leq s \leq t-\tau, t \geq \tau, \\ \varepsilon(x,0) = \widehat{w}_0(x) - w_0(x), \ \varepsilon_s(x,0) \\ \quad = \widehat{w}_1(x) - w_1(x), \ -1 \leq x \leq 1, \end{cases} \tag{36}$$

$$\begin{cases} \varepsilon^t_{ss}(x,s) = \varepsilon^t_{xx}(x,s), \ -1 < x < 1, t-\tau < s \leq t, t > \tau, \\ \varepsilon^t(-1,s) = \varepsilon^t_x(1,s) = 0, \ t-\tau \leq s \leq t, t \geq \tau, \\ \varepsilon^t(x,t-\tau) = \varepsilon(x,t-\tau), \ \varepsilon^t_s(x,t-\tau) \\ \quad = \varepsilon_s(x,t-\tau), \ -1 \leq x \leq 1, t \geq \tau, \end{cases} \tag{37}$$

where $\epsilon(x, s)$ and $\epsilon(x, s, t)$ are given by (21) and (29), respectively.

Theorem 4.1: *Let $k_1 > 0$, $k_2 > 0$, and $t > \tau$. Then for any $(w_0, w_1)^T \in \mathcal{H}$, $(\widehat{w}_0, \widehat{w}_1)^T \in \mathcal{H}$, there exists a unique solution of systems (35)-(37) such that $(w(\cdot,t), w_t(\cdot,t))^T \in C(\tau,\infty;\mathcal{H})$, $(\epsilon(\cdot, s), \epsilon_s(\cdot, s))^T \in C(0, t-\tau;\mathcal{H})$, $(\varepsilon^t(\cdot,s), \varepsilon^t_s(\cdot,s))^T \in C([t-\tau,t]\times(\tau,\infty);\mathcal{H})$. Moreover, for any $t > \tau$, $s \in [0, t-\tau]$ and $q \in (t-\tau, t]$, there exists a constant $C_{tsq} > 0$ such that*

$$\begin{aligned} &\left\|(w(\cdot,t), w_t(\cdot,t))^T\right\|_{\mathcal{H}} + \left\|(\varepsilon(\cdot,s), \varepsilon_s(\cdot,s))^T\right\|_{\mathcal{H}} \\ &\quad + \left\|(\varepsilon^t(\cdot,q), \varepsilon^t_s(\cdot,q))^T\right\|_{\mathcal{H}} \\ &\quad \leq C_{tsq}\left[\left\|(w_0,w_1)^T\right\|_{\mathcal{H}} + \left\|(\widehat{w}_0,\widehat{w}_1)^T\right\|_{\mathcal{H}}\right]. \end{aligned} \tag{38}$$

Proof: For any $(w_0, w_1)^T \in \mathcal{H}$, $(\widehat{w}_0, \widehat{w}_1)^T \in \mathcal{H}$, since $\mathbb{B}$ defined by (23) and (24) generates an exponentially stable C_0-semigroup on $\mathcal{H}$, there is a unique solution $(\varepsilon(\cdot,s), \varepsilon_s(\cdot,s))^T \in C(0, t-\tau;\mathcal{H})$ of (36) such that (25) holds true. Now for any given time $t > \tau$, write (37) as follows:

$$\frac{d}{ds}\begin{pmatrix} \varepsilon^t(\cdot,s) \\ \varepsilon^t_s(\cdot,s) \end{pmatrix} = \mathbb{A}\begin{pmatrix} \varepsilon^t(\cdot,s) \\ \varepsilon^t_s(\cdot,s) \end{pmatrix} \tag{39}$$

where $\mathbb{A}$ is defined by

$$\begin{cases} \mathbb{A}\left(f,g\right)^T = \left(g, f''\right)^T, \ \forall (f,g)^T \in D(\mathbb{A}) \\ D(\mathbb{A}) = \{(f,g)^T \in \mathcal{H} \cap (H^2((-1,0)\cup(0,1)) \\ \quad \times H^1((-1,0)\cup(0,1)))\}. \end{cases} \tag{40}$$

Then $\mathbb{A}$ is skew-adjoint in $\mathcal{H}$ and hence generates a conservative C_0-semigroup on $\mathcal{H}$. For any $(\epsilon(\cdot, t-\tau), \varepsilon_s(\cdot, t-\tau))^T \in \mathcal{H}$ determined by (36), there exists a unique solution $(\varepsilon^t(\cdot,s), \varepsilon^t_s(\cdot,s))^T \in C([t-\tau,t]\times(\tau,\infty);\mathcal{H})$ of (37) such that

$$\begin{aligned} &\left\|(\varepsilon^t(\cdot,s), \varepsilon^t_s(\cdot,s))^T\right\|_{\mathcal{H}} \\ &\quad = \left\|(\varepsilon(\cdot,t-\tau), \varepsilon_s(\cdot,t-\tau))^T\right\|_{\mathcal{H}}, \\ &\quad \forall s \in [t-\tau,t], t > \tau. \end{aligned} \tag{41}$$

■

Now we show well-posedness of (35) which can be written as that, for $\forall t > \tau$,

$$w_{tt}(\cdot,t) + Aw(\cdot,t) + k_2BB^*\left[w_t(\cdot,t) + \varepsilon^t_s(\cdot,t)\right] = 0 \tag{42}$$

where $B = \delta(x)$ and A: $D(A)(\subset L^2) \to L^2$ is defined as follows:

$$\begin{cases} Af = -f'', \forall f \in D(A), \\ D(A) = \{f \in H^2((-1,0)\cup(0,1)), f(-1) = 0, \\ \quad f'(1) = 0, f(0^-) = f(0^+), f'(0^-) = f'(0^+)\}. \end{cases}$$

By Corollary 1 of Guo and Luo (2002), (42) is well-posed, that is, for any $t > \tau$, there exists a unique solution of (42) such that $(w(\cdot,t), w_t(\cdot,t))^T \in C(\tau,\infty;\mathcal{H})$. Moreover, there exists a positive constant $C'_t > 0$ such that

$$\begin{aligned} &\|(w(\cdot,t), w_t(\cdot,t))^T\|^2_{\mathcal{H}} \\ &\quad \leq C'_t\left[\|(w(\cdot,\tau), w_t(\cdot,\tau))^T\|^2_{\mathcal{H}} + \int_\tau^t \varepsilon^{t\,2}_s(0,s)ds\right] \\ &\quad \leq C'_t\left[\|(w_0,w_1)^T\|^2_{\mathcal{H}} + \int_\tau^t \varepsilon^{t\,2}_s(0,s)ds\right]. \end{aligned} \tag{43}$$

On the other hand, from Lemma 2 of the appendix, we have that for $\forall t > \tau, k = 0, 1, \ldots,$

$$\int_t^\infty \varepsilon^{t\,2}_s(0,\rho)d\rho \leq \frac{2(2\omega + M^2)}{(\tau - 16k)\omega} \|(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T\|^2_{\mathcal{H}},$$

where ω, M are defined in (25). This together with (41), (43) and (25) gives the required result.

Now we show the asymptotic stability for non-smooth initial values.

Theorem 4.2: *Let $k_1 > 0$, $k_2 > 0$. Then for any $(w_0, w_1)^T \in \mathcal{H}$, $(\widehat{w}_0, \widehat{w}_1)^T \in \mathcal{H}$, the solution of (35) satisfies*

$$\lim_{t\to\infty} \| (w(\cdot,t), w_t(\cdot,t))^T \|_{\mathcal{H}} = 0. \tag{44}$$

Proof: Equation (35) can be written as follows:

$$\frac{d}{dt}\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t)\end{pmatrix} = \mathcal{A}_c \begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t)\end{pmatrix} + k_2\mathcal{B}\varepsilon_s^t(0,t), \tag{45}$$

where

$$\begin{cases} \mathcal{A}_c(f,g)^T = (g, f'')^T, \ \forall (f,g)^T \in D(\mathcal{A}_c), \\ D(\mathcal{A}_c) = \{(f,g)^T \in (H^2((-1,0)\cup(0,1)) \\ \quad \times H^1((-1,0)\cup(0,1))) \cap \mathcal{H}, \mid \\ \quad f(0^-) = f(0^+),\, f'(0^+) - f'(0^-) = k_2 g(0)\} \end{cases} \tag{46}$$

and $\mathcal{B}$ is defined in (10). A direct computation shows

$$\mathcal{B}\mathcal{A}_c^{-1}(\varphi,\psi)^T = -\varphi(0), \quad \forall (\varphi,\psi)^T \in \mathcal{H}, \tag{47}$$

which means $\mathcal{B}\mathcal{A}_c^{-1}$ is bounded. For the energy $E_0(t)$ of the system (35), a simple computation gives

$$\dot{E}_0(t) = -k_2 w_t^2(0,t), \tag{48}$$

which shows

$$k_2 \int_0^T | w_t(0,t) |^2 \, dt \le E_0(0), \tag{49}$$

for any $T > 0$. This inequality together with (47) illustrates that $\mathcal{B}$ is admissible for $e^{\mathcal{A}_c t}$. Therefore, there exists a unique solution to (45) such that $(w(\cdot,t), w_t(\cdot,t))^T \in C(\tau,\infty;\mathcal{H})$. The admissible of $\mathcal{B}$ implies that

$$\left\| \int_\tau^t e^{\mathcal{A}_c(t-s)}\mathcal{B}\varepsilon_s^t(0,s)ds \right\|_{\mathcal{H}} \le C_t \left\| \varepsilon_s^t(0,\cdot)\right\|_{L^2(\tau,t)} \le C_t t \left\| \varepsilon_s^t(0,\cdot)\right\|_{L^\infty(\tau,t)},$$

for some positive constants C_t independent of $\varepsilon_s^t(0,t)$.

On the other hand, it is known that $e^{\mathcal{A}_c t}$ is exponentially stable. Since $\mathcal{B}$ is admissible for $e^{\mathcal{A}_c t}$ with the control space $L^2_{\mathrm{loc}}(0,\infty)$, it is also admissible with control space $L^\infty_{\mathrm{loc}}(0,\infty)$. By Proposition 2.5 of Weiss (1989), we have

$$\left\| \int_{t_0}^t e^{\mathcal{A}_c(t-s)}\mathcal{B}\varepsilon_s^t(0,s)ds \right\|_{\mathcal{H}} \le \left\| \int_0^t e^{\mathcal{A}_c(t-s)}\mathcal{B}(0 \underset{t_0}{\diamondsuit} \varepsilon_s^t(0,s))ds \right\|_{\mathcal{H}} \le L_0 \left\| \varepsilon_s^t(0,\cdot)\right\|_{L^\infty(t_0,t)}$$

for some constants $L_0 > 0$ that is independent of $\varepsilon_s^t(0,t)$, and

$$(u \underset{\tau}{\diamondsuit} v)(t) = \begin{cases} u(t), \ 0 \le t \le \tau, \\ v(t), \ t > \tau. \end{cases}$$

Suppose that

$$\|e^{\mathcal{A}_c t}\|_{\mathcal{H}} \le M_0 e^{-\omega_0 t} \tag{50}$$

for some $M_0, \omega_0 > 0$.

Then we have

$$\begin{aligned} (w(\cdot,t), w_t(\cdot,t))^T &= e^{\mathcal{A}_c(t-\tau)}(w(\cdot,\tau), w_t(\cdot,\tau))^T \\ &+ \int_\tau^t e^{\mathcal{A}_c(t-\tau-s)}k_2\mathcal{B}\varepsilon_s^t(0,s)ds, \end{aligned} \tag{51}$$

which concludes that the the following inequality

$$\begin{aligned} &\left\|(w(\cdot,t), w_t(\cdot,t))^T\right\|_{\mathcal{H}} \le M_0 e^{-\omega_0(t-\tau)} \left\|(w_0,w_1)^T\right\|_{\mathcal{H}} \\ &+ \left\| \int_\tau^{t_0} e^{\mathcal{A}_c(t-\tau-s)}k_2\mathcal{B}\varepsilon_s^t(0,s)ds \right\|_{\mathcal{H}} \\ &+ \left\| \int_{t_0}^{t} e^{\mathcal{A}_c(t-\tau-s)}k_2\mathcal{B}\varepsilon_s^t(0,s)ds \right\|_{\mathcal{H}} \\ &\le M_0 e^{-\omega_0(t-\tau)} \left\|(w_0,w_1)^T\right\|_{\mathcal{H}} + k_2 M_0 e^{\omega_0\tau} \\ &\times \left\| \int_\tau^{t_0} e^{\mathcal{A}_c(t_0-s)}e^{\mathcal{A}_c(t-t_0)}\mathcal{B}\varepsilon_s^t(0,s)ds \right\|_{\mathcal{H}} \\ &+ k_2 M_0 e^{\omega_0\tau} \left\| \int_{t_0}^{t} e^{\mathcal{A}_c(t-s)}\mathcal{B}\varepsilon_s^t(0,s)ds \right\|_{\mathcal{H}} \\ &\le M_0 e^{-\omega_0(t-\tau)} \left\|(w_0,w_1)^T\right\|_{\mathcal{H}} + k_2 M_0^2 e^{\omega_0(\tau+t_0)} \cdot e^{-\omega_0 t} \\ &\times \left\| \int_\tau^{t_0} e^{\mathcal{A}_c(t_0-s)}\mathcal{B}\varepsilon_s^t(0,s)ds \right\|_{\mathcal{H}} \\ &+ k_2 M_0 e^{\omega_0\tau} \left\| \int_{t_0}^{t} e^{\mathcal{A}_c(t-s)}\mathcal{B}\varepsilon_s^t(0,s)ds \right\|_{\mathcal{H}} \\ &\le M_0 e^{-\omega_0(t-\tau)} \left\|(w_0,w_1)^T\right\|_{\mathcal{H}} \\ &+ k_2 M_0^2 e^{\omega_0(\tau+t_0)} \cdot e^{-\omega_0 t} C_{t_0} t_0 \left\|\varepsilon_s^t(0,\cdot)\right\|_{L^\infty(\tau,t_0)} \\ &+ k_2 M_0 e^{\omega_0\tau} L_0 \left\|\varepsilon_s^t(0,\cdot)\right\|_{L^\infty(t_0,t)}. \end{aligned} \tag{52}$$

Passing to the limit as $t \to \infty$ for the inequality (52), we finally obtain

$$\begin{aligned} &\lim_{t\to\infty} \|(w(\cdot,t), w_t(\cdot,t))^T\|_{\mathcal{H}} \\ &\quad \le k_2 M_0 e^{\omega_0\tau} L_0 \left\|\varepsilon_s^t(0,\cdot)\right\|_{L^\infty(t_0,t)}, \end{aligned} \tag{53}$$

which together with the following fact

$$\lim_{t\to\infty} \left\|\varepsilon_s^t(0,\cdot)\right\|_{L^\infty(t_0,t)} = 0$$

from (66) in Appendix gives the result of Theorem 4.2.■

Next we get the exponential stability of the closed-loop system for smooth initial values as the following theorem.

Theorem 4.3: *If $(w_0, w_1)^T \in \mathcal{H}$ and $(\widehat{w}_0, \widehat{w}_1)^T \in \mathcal{H}$ satisfy*

$$(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T \in D(\mathbb{B}),$$

where $\mathbb{B}$ is defined by (23), then the system (35) decays exponentially in the sense that

$$\|(w(\cdot,t), w_t(\cdot,t))^T\|_{\mathcal{H}} \le C_\tau e^{-\alpha t}\Big[\|(w_0, w_1)^T\|_{\mathcal{H}} + \sqrt{3+k_1^2}Me^{\omega\tau} \parallel \mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T \parallel_{\mathcal{H}}\Big] \quad (54)$$

for some $C_\tau > 0$ independent of initial values and $0 < \alpha < \min\{\omega, \omega_0\}$ where ω and ω_0 are defined in (25) and (50), respectively.

Proof: Now we also only consider the case of $16k \le \tau < 16k + 1$ like the Equations (73) and (74) in Lemma 2 of Appendix. If $(\varepsilon(\cdot, 0), \varepsilon_s(\cdot, 0))^T \in D(\mathbb{B})$ where $\mathbb{B}$ is defined by (23), then by (25),

$$\|\mathbb{B}e^{\mathbb{B}s}(\varepsilon(\cdot,0), \varepsilon_s(\cdot,0))^T\|_{\mathcal{H}} \le Me^{-\omega s}\|\mathbb{B}(\varepsilon(\cdot,0), \varepsilon_s(\cdot,0))^T\|_{\mathcal{H}}, \quad \forall s \ge 0, \quad (55)$$

that is,

$$\left(\int_{-1}^{1}[\varepsilon_{xx}^2(x,s) + \varepsilon_{xs}^2(x,s)]dx\right)^{1/2} \le Me^{-\omega s}\left(\int_{-1}^{1}[\varepsilon_{xx}^2(x,0) + \varepsilon_{xs}^2(x,0)]dx\right)^{1/2}, \quad \forall s \ge 0. \quad (56)$$

Since $\epsilon(-1, s) = \epsilon_x(1, s) = 0$ and $\epsilon_x(0^+, s) - \epsilon_x(0^-, s) = k_1\epsilon_s(0, s)$, it is easy to prove the following inequalities:

$$\begin{cases} \varepsilon_s^2(16k-\tau, s) \le \int_{-1}^{16k-\tau} \varepsilon_{xs}^2(x,s)dx, \quad \varepsilon_x^2(\tau-16k, s) \le \int_{\tau-16k}^{1} \varepsilon_{xx}^2(x,s)dx, \\ \varepsilon_s^2(\tau-16k, s) = \left|\varepsilon_s(0,s) + \int_0^{\tau-16k}\varepsilon_{xs}(x,s)dx\right|^2 \le 2\int_0^{\tau-16k}\varepsilon_{xs}^2(x,s)dx + 2\varepsilon_s^2(0,s), \\ \varepsilon_s^2(0,s) = \left|\int_{-1}^{0}\varepsilon_{xs}(x,s)dx\right|^2 \le \int_{-1}^{0}\varepsilon_{xs}^2(x,s)dx, \\ |\varepsilon_x(\tau-16k, s) - \varepsilon_x(16k-\tau, s)|^2 \\ = \left|\int_0^{\tau-16k}\varepsilon_{xx}(x,s)dx + \varepsilon_x(0^+,s) - \int_0^{16k-\tau}\varepsilon_{xx}(x,s)dx - \varepsilon_x(0^-,s)\right|^2 \\ = \left|\int_{16k-\tau}^{\tau-16k}\varepsilon_{xx}(x,s)dx + k_1\varepsilon_s(0,s)\right|^2 \le 2\left[\int_{-1}^{1}\varepsilon_{xx}^2(x,s)dx + k_1^2\varepsilon_s^2(0,s)\right], \\ \varepsilon_x^2(16k-\tau, s) = |\varepsilon_x(\tau-16k, s) - \varepsilon_x(16k-\tau, s) - \varepsilon_x(\tau-16k, s)|^2 \\ \le 2|\varepsilon_x(\tau-16k, s) - \varepsilon_x(16k-\tau, s)|^2 + 2\varepsilon_x^2(\tau-16k, s), \end{cases} \quad (57)$$

which combines with (68), (73) and (74) in Lemma 2 of Appendix indicate that

$$|\varepsilon_s^t(0,t)| \le 2\sqrt{3+k_1^2}Me^{-\omega(t-\tau)}\sqrt{\int_{-1}^{1}[\varepsilon_{xx}^2(x,0) + \varepsilon_{xs}^2(x,0)]dx} = 2\sqrt{3+k_1^2}Me^{-\omega(t-\tau)} \parallel \mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T \parallel_{\mathcal{H}}. \quad (58)$$

For the system (35) which is written as (45), suppose $0 < \alpha < \min\{\omega, \omega_0\}$ where ω and ω_0 are defined in (25) and (50), respectively, and let

$$Y(t) = e^{\alpha t}(w(\cdot,t), w_t(\cdot,t))^T. \quad (59)$$

Then

$$\frac{d}{dt}Y(t) = (\mathcal{A}_c + \alpha)Y(t) + k_2\mathcal{B}(0, e^{\alpha t}\varepsilon_s^t(0,t))^T. \quad (60)$$

Since $\mathcal{A}_c + \alpha$ and $\mathcal{B}$ satisfy assumptions $(H.1) - (H.4)$ of Theorem 7.4.1.1 of (Lasiecka and Triggiani, 2000, p.653), apply (7.4.1.3) of this theorem with $\varepsilon = 0$ to obtain

$$\|Y(t)\|_{\mathcal{H}} \le C_\tau\Big[\|(w(\cdot,\tau), w_t(\cdot,\tau))^T\|_{\mathcal{H}} + \|e^{\alpha\cdot}\varepsilon_s^t(0,\cdot)\|_{L^2(\tau,\infty)}\Big], \quad \forall t \ge \tau, \quad (61)$$

for some $C_\tau > 0$. Therefore,

$$\|(w(\cdot,t), w_t(\cdot,t))^T\|_{\mathcal{H}} \le C_\tau e^{-\alpha t}[\|(w(\cdot,\tau), \times w_t(\cdot,\tau))^T\|_{\mathcal{H}} + \|e^{\alpha\cdot}\varepsilon_s^t(0,\cdot)\|_{L^2(\tau,\infty)}], \quad \forall t \ge \tau. \tag{62}$$

Since by (58),

$$\begin{aligned}\left\|e^{\alpha\cdot}\varepsilon_s^t(0,\cdot)\right\|_{L^2(\tau,\infty)} &= \left(\int_\tau^\infty e^{2\alpha\rho}|\varepsilon_s^t(0,\rho)|^2 d\rho\right)^{1/2}\\ &\le 2\sqrt{3+k_1^2}Me^{\omega\tau}\ \|\mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T\|_{\mathcal{H}}\\ &\quad\times\left(\int_\tau^\infty e^{-2(\omega-\alpha)\rho}d\rho\right)^{1/2}\\ &\le \sqrt{\frac{2(3+k_1^2)}{\omega-\alpha}}Me^{\alpha\tau}\ \|\mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T\|_{\mathcal{H}},\end{aligned} \tag{63}$$

from (62) we have that

$$\begin{aligned}\left\|(w(\cdot,t), w_t(\cdot,t))^T\right\|_{\mathcal{H}} &\le C_\tau e^{-\alpha t}\Bigg[\left\|(w_0,w_1)^T\right\|_{\mathcal{H}} + \sqrt{\frac{2(3+k_1^2)}{\omega-\alpha}}Me^{\alpha\tau}\\ &\quad\times\left\|\mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T\right\|_{\mathcal{H}}\Bigg], \quad \forall t \ge \tau.\end{aligned} \tag{64}$$

■

5. Numerical simulation

In this section, we use the backward Euler method in the time domain and the Chebyshev spectral method in the space domain to give some numerical simulation results for the closed-loop system (35)–(37). Here we choose the space grid size $N = 40$, time step $dt = 0.0001$ and time span [0, 10]. Parameters and coefficients, respectively, are chosen to be $\tau = 0.5$, $k_1 = k_2 = 1$. For the initial values,

$$\begin{cases} w_0(x) = e^{-100x^2}, & -1 \le x \le 1,\\ w_1(x) = -e^{-100x^2}, & -1 \le x \le 1,\\ \varepsilon(x,0) = x, & -1 \le x \le 1,\\ \varepsilon_s(x,0) = -x, & -1 \le x \le 1,\end{cases} \tag{65}$$

we plot the displacement $w(x, t)$ and velocity $w_t(x, t)$ as the Figure 1 and 2 below respectively. It is seen that the displacement of the string is almost at rest after $t = 5$. That is to say, the predictor–observer-based scheme is also useful to make the pointwise control system converge for the string equation in which observation is subjected to a time delay.

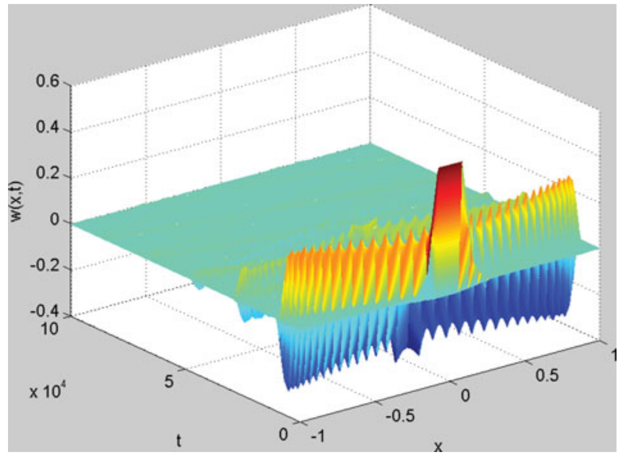

Figure 1. State of the closed-loop system.

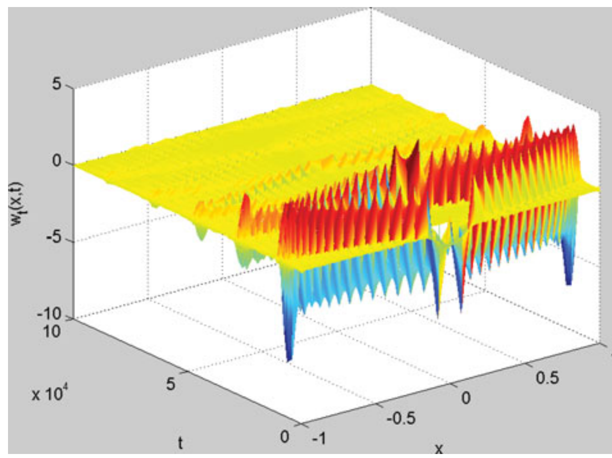

Figure 2. Velocity of the closed-loop system.

6. Conclusion

In a conclusion, this paper stabilises a string equation with delayed observation signal, where the feedback stabiliser is acting at the the middle joint of the string. Well-posedness of the open-loop system is obtained first. An observer is designed for the available observation, while a predictor is designed for the unavailable observation. Pointwise output feedback controller-based on the observer and predictor gives the closed-loop system which is exponentially stable for the smooth initial values and asymptotically stable for the non-smooth initial values. Simulation results demonstrate the stabilised effectivenss of the controller.

For the feedback stabilisation of the wave equation pointwise control system with time delay, an interesting further research problem would be the robustness against small changes of the actuators' locations or the robustness against perturbations in the time delay.

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Appendix

In this appendix, we present a lemma concerning the estimate $\int_t^\infty \varepsilon_s^{t2}(0,\rho)d\rho$ which is applied in proof of Theorem 4.1.

Lemma 2 *For any $(w_0, w_1)^T \in \mathcal{H}$, $(\widehat{w}_0, \widehat{w}_1)^T \in \mathcal{H}$, the solution of the system (37) satisfies the inequality as follows:*

$$\int_t^\infty \varepsilon_s^{t2}(0,\rho)d\rho \le \frac{2(2\omega+M^2)}{(\tau-16k)\omega} \times \|(\widehat{w}_0-w_0, \widehat{w}_1-w_1)^T\|_{\mathcal{H}}^2, \forall t>\tau, k=0,1,\ldots, \tag{66}$$

where ω, M are defined in (25) in the case of $16k \le \tau < 16k+1$.

Proof: For brevity in notation, let us consider the following wave equation:

$$\begin{cases} p_{\xi\xi}(x,\xi) - p_{xx}(x,\xi) = 0, \\ \quad -1<x<1, 0<\xi<\infty, \\ p(-1,\xi) = p_x(1,\xi) = 0, \quad 0\le\xi\le\tau, \\ p(x,0) = \varepsilon(x,t-\tau), \quad -1\le x\le 1, \\ p_\xi(x,0) = \varepsilon_s(x,t-\tau), \quad -1\le x\le 1. \end{cases} \tag{67}$$

Then

$$\varepsilon_s^t(0,t) = p_\xi(0,\tau). \tag{68}$$

Denote by

$$\widehat{f}(x,s) = \int_0^\infty e^{-s\rho} f(x,\rho)d\rho \tag{69}$$

as Laplace transform of the classical function $f(x,\xi)$ where x is a parameter.

Take Laplace transform for the system (A2) to obtain

$$\begin{cases} \dfrac{d^2\widehat{p}(x,s)}{dx^2} - x^2\widehat{p}(x,s) \\ \quad = -s\varepsilon(x,t-\tau) - \varepsilon_\rho(x,t-\tau), \\ \widehat{p}(-1,s) = \widehat{p}_x(1,s) = 0, \end{cases} \tag{70}$$

whose solution is found:

$$\begin{aligned} &\widehat{p}(x,s) \\ &= \frac{\cosh(sx)\cosh(s) - \sinh(sx)\sinh(s)}{s\cosh(2s)} \\ &\quad \times \int_{-1}^0 [s\varepsilon(\eta,t-\tau) + \varepsilon_\rho(\eta,t-\tau)]\sinh(s(1+\eta))d\eta \\ &\quad + \frac{\cosh(sx)\sinh(s) + \sinh(sx)\cosh(s)}{s\cosh(2s)} \\ &\quad \times \int_0^1 [s\varepsilon(\eta,t-\tau) + \varepsilon_\rho(\eta,t-\tau)]\cosh(s(1-\eta))d\eta \\ &\quad - \frac{1}{s}\int_0^x [s\varepsilon(\eta,t-\tau) + \varepsilon_\rho(\eta,t-\tau)]\sinh(s(x-\eta))d\eta. \end{aligned} \tag{71}$$

By $\widehat{p_\xi}(x,s) = s\widehat{p}(x,s) - \varepsilon(x,t-\tau)$ and (A6), a simple computation gives

$$\begin{aligned} &\widehat{p_\xi}(0,s) \\ &= \left[\int_{-1}^0 \big(s\varepsilon(\eta,t-\tau) + \varepsilon_\rho(\eta,t-\tau)\big) \right. \\ &\quad \times \sinh(s(1+\eta))d\eta\cosh(s) \\ &\quad + \int_0^1 \big(s\varepsilon(\eta,t-\tau) + \varepsilon_\rho(\eta,t-\tau)\big) \\ &\quad \left. \times \cosh(s(1-\eta))d\eta\sinh(s)\right]\frac{1}{\cosh(2s)} - \varepsilon(0,t-\tau) \\ &= \int_{-1}^0 \varepsilon_\rho(\eta,t-\tau)\frac{\sinh(s(1+\eta))\cosh(s)}{\cosh(2s)}d\eta \\ &\quad + \int_0^1 \varepsilon_\rho(\eta,t-\tau)\frac{\cosh(s(1-\eta))\sinh(s)}{\cosh(2s)}d\eta \\ &\quad - \int_{-1}^0 \varepsilon_x(\eta,t-\tau)\frac{\sinh(s(1+\eta))\cosh(s)}{\cosh(2s)}d\eta \\ &\quad \times + \int_0^1 \varepsilon_x(\eta,t-\tau)\frac{\sinh(s(1-\eta))\sinh(s)}{\cosh(2s)}d\eta. \end{aligned} \tag{72}$$

Using the inverse Laplace transform formulae (see Oberhettinger & Baddi, 1973), we have that when k is a non-negative integer, there are two cases as follows:

if $16k + 4m_1 + 2m_2 \le \tau < 16k + 4m_1 + 2m_2 + 1$,

$$\begin{aligned} &2p_\xi(0,\tau)\\ &\quad = (-1)^{m_1}\varepsilon_s(16k+4m_1+2m_2-\tau, t-\tau)\\ &\qquad + (-1)^{m_1}\varepsilon_s(\tau-16k-4m_1-2m_2, t-\tau)\\ &\qquad + (-1)^{m_1+1}\varepsilon_x(16k+4m_1+2m_2-\tau, t-\tau)\\ &\qquad + (-1)^{m_1+m_2}\varepsilon_x(\tau-16k-4m_1-2m_2, t-\tau), \end{aligned} \tag{73}$$

and if $16k + 4m_1 + 2m_2 + 1 \le \tau < 16k + 4m_1 + 2m_2 + 2$,

$$\begin{aligned} &2p_\xi(0,\tau)\\ &\quad = (-1)^{m_1+m_2}\varepsilon_s(16k+4m_1+2m_2+2-\tau, t-\tau)\\ &\qquad + (-1)^{m_1+1}\varepsilon_s(\tau-16k-4m_1-2m_2-2, t-\tau)\\ &\qquad + (-1)^{m_1+m_2+1}\varepsilon_x(16k+4m_1+2m_2+2-\tau, t-\tau)\\ &\qquad + (-1)^{m_1}\varepsilon_x(\tau-16k-4m_1-2m_2-2, t-\tau), \end{aligned} \tag{74}$$

for all of $m_1 = 0, 1, 2, 3$ and $m_2 = 0, 1$.

We only consider the case that $16k \le \tau < 16k + 1$ since other cases can be treated similarly. Let $16k \le \tau < 16k + 1$, then $0 \le \tau - 16k < 1$ and $-1 < 16k - \tau \le 0$. For $t > \tau$ and $s \in [0, t - \tau]$, let

$$\begin{aligned} \rho(s) = &\int_0^{\tau-16k} x\varepsilon_x(x,s)\varepsilon_s(x,s)dx\\ &+ \int_{16k-\tau}^0 x\varepsilon_x(x,s)\varepsilon_s(x,s)dx. \end{aligned} \tag{75}$$

Then we have $|\rho(s)| \le \frac{1}{2}\|(\varepsilon(\cdot,s), \varepsilon_s(\cdot,s))^T\|^2_{\mathcal{H}}$ and

$$\begin{aligned} \dot\rho(s) = &\frac{\tau-16k}{2}\big[\varepsilon_s^2(16k-\tau,s) + \varepsilon_x^2(16k-\tau,s)\\ &+ \varepsilon_s^2(\tau-16k,s) + \varepsilon_x^2(\tau-16k,s)\big]\\ &- \frac{1}{2}\int_0^{\tau-16k}[\varepsilon_x^2(x,s)+\varepsilon_s^2(x,s)]dx\\ &- \frac{1}{2}\int_{16k-\tau}^0[\varepsilon_x^2(x,s)+\varepsilon_s^2(x,s)]dx. \end{aligned}$$

Integrate above over $[0, T]$ for any $T > 0$ to give

$$\begin{aligned} &\frac{\tau-16k}{2}\int_0^T\big[\varepsilon_s^2(16k-\tau,s)+\varepsilon_x^2(16k-\tau,s)\\ &\quad+\varepsilon_s^2(\tau-16k,s)+\varepsilon_x^2(\tau-16k,s)\big]ds\\ &= \rho(T)-\rho(0)+\frac{1}{2}\int_0^T\bigg(\int_0^{\tau-16k}[\varepsilon_x^2(x,s)\\ &\quad+\varepsilon_s^2(x,s)]dx+\int_{16k-\tau}^0[\varepsilon_x^2(x,s)+\varepsilon_s^2(x,s)]dx\bigg)ds. \end{aligned} \tag{76}$$

Using (A11) and (25), we can easily get that as follows:

$$\begin{aligned} &\int_0^\infty\big[\varepsilon_s^2(16k-\tau,s)+\varepsilon_x^2(16k-\tau,s)\\ &\quad+\varepsilon_s^2(\tau-16k,s)+\varepsilon_x^2(\tau-16k,s)\big]ds\\ &\quad\le \frac{2\omega+M^2}{(\tau-16k)\omega}\|(\varepsilon(\cdot,0),\varepsilon_s(\cdot,0))^T\|^2_{\mathcal{H}}\\ &\quad= \frac{2\omega+M^2}{(\tau-16k)\omega}\|(\widehat{w}_0-w_0,\widehat{w}_1-w_1)^T\|^2_{\mathcal{H}} \end{aligned} \tag{77}$$

which together with (A3), (A8) and (A9) yields (A1). ■