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Stabilization of a Non-homogeneous Rotating Body-Beam System with the Torque and Nonlinear Distributed Controls*

GUO Yaping · WANG Junmin

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Abstract This paper considers the stabilization of non-homogeneous rotating body-beam system with the torque and nonlinear distributed controls. To stabilize the system, the authors propose the torque and nonlinear distributed controls applied on the disk and flexible beam respectively. As long as the angular velocity of the disk does not exceed the square root of the first eigenvalue of the related self-adjoint positive definite operator, the authors show that the torque and nonlinear distributed control laws suppress the system vibrations, in the sense that the beam vibrations are forced to decay exponentially to zero and the body rotates with a desired angular velocity.

Keywords Non-homogeneous, nonlinear distributed control, stabilization.

1 Introduction

Flexible structures, which arise from control of numerous mechanical systems such as spacecraft[1], flexible marine risers[2] and robots arms[3], can be modeled as a coupled system of a partial differential equation and an ordinary differential equation (PDE-ODE) system. One typical example of flexible structures is the rotating disk-beam system which consists of a disk and a flexible beam. Assume that a beam (B) clamped at the center of a disk (D) and free at the other end. The disk rotates freely about its axis with a nonuniform angular velocity and the motion of the beam is confined to a plane perpendicular to the disk (see Figure 1). Moreover, the disk is supposed to rotate without friction, while the beam is clamped at the left end $x = 0$, constrained to the $x - y$ plane and all the deflections are assumed to be parallel to the y-axis.

GUO Yaping
School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081*, China; School of Mathematical Sciences, Shanxi University, Taiyuan* 030006*, China.* Email: guoyaping0904@163.com.
WANG Junmin
School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081*, China.*
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The rotating body-beam system is shown in Figure 1, which can be described by the following system:

$$\begin{cases} \rho(x)u_{tt} + (EI(x)u_{xx})_{xx} + \Theta(t) = \rho(x)\omega^2(t)u, & x \in (0,1),\ t > 0, \\ u(0,t) = u_x(0,t) = 0, & t \geq 0, \\ \dfrac{d}{dt}\Big\{\Big(I_d + \displaystyle\int_0^1 \rho(x)u^2 dx\Big)\omega(t)\Big\} = \Gamma(t), & t > 0, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad \omega(0) = \omega_0, \end{cases} \tag{1}$$

where the positive constants $EI(x)$, $\rho(x)$ and I_d are respectively the flexural rigidity, the mass per unit length of the beam and the body's moment of inertia, respectively. In addition, $u(x,t)$ denotes the transverse displacement of the beam at time t and position x; $\omega(t)$ is the angular velocity; $\Theta(t)$ is the control exerted on the beam and $\Gamma(t)$ is the torque control to be applied on the disk.

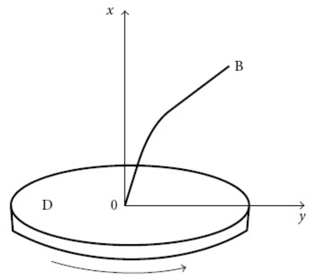

Figure 1 The body-beam system

In the past decades, the stabilization problem of the rotating body-beam system has attracted more attentions[4–6]. In [1], Baillieul and Levi have proved that the body-beam system, with structural damping $\Theta(t) = c_1 u_{xxxxt}(x,t)$ $(c_1 > 0)$, has a finite number of rotating equilibrium states. By taking into account of the effect of viscous damping $\Theta(t) = c_2 u_t(x,t)$ $(c_2 > 0)$, the torque control is proposed to stabilize the body-beam system in [7], where the exponential stability is proved using Lyapunov analysis. At least one boundary control (usually force or moment) is presented in the feedback law in addition of the torque control, which are used to stabilizes the body-beam system in [8].

We would like to mention that the nonlinear controls have been proposed in some works related to the rotating body-beam system. A class of nonlinear boundary controls have been provided in [9], namely, $u_{xx} = -f(u_{xt}(1,t))$, $u_{xxx} = g(u_t(1,t))$, an exponential stability result has been established. A nonlinear torque control exerted on the body, and a direct strain control exerted on the beam are presented in [10], where the exponential stability of the body-beam system is proved by Riesz basis method.

In this paper, we propose a feedback law which consists of a nonlinear distributed control applied on the flexible beam and torque control exerted on the body,

$$\Theta(t) = \mathcal{F}(u_t), \quad \Gamma(t) = -\gamma(\omega(t) - \widehat{\omega}), \quad \gamma > 0.$$

Our aim is to answer the question whether the solution of the following system tends to a desired state, where the beam position is neutral (perpendicular to the disk) and the angular velocity of disk is constant. The non-homogeneous rotating disk-beam system with the torque and nonlinear distributed controls as

$$\begin{cases} \rho(x)u_{tt} + (EI(x)u_{xx})_{xx} + \mathcal{F}(u_t) = \rho(x)\omega^2(t)u, & x \in (0,1), t > 0, \\ u(0,t) = u_x(0,t) = u_{xx}(1,t) = (EI(x)u_{xx})_x(1,t) = 0, & t \geq 0, \\ \dfrac{d}{dt}\left\{\left(I_d + \displaystyle\int_0^1 \rho(x)u^2 dx\right)\omega(t)\right\} = \mathit{\Gamma}(t), & t > 0, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad \omega(0) = \omega_0, \end{cases} \tag{2}$$

For nonlinear distributed control $\mathcal{F}(u_t)$, the following hypotheses are verified:
(HI) The function $\mathcal{F} \in C^0(\mathbb{R})$ is nondecreasing continuous and satisfies $\mathcal{F}(0) = 0$,

$$s\mathcal{F}(s) \geq 0, \quad \forall s \in \mathbb{R}; \tag{3}$$

(HII) There exist positive constants M_1, M_2 and $1 < p$ such that

$$M_1|s| \leq |\mathcal{F}(s)| \leq M_2|s|, \quad \forall s \in \mathbb{R}. \tag{4}$$

The paper is organized as follows. In Section 2, we set notations, reformulate the above system into an evolution system; In Section 3, the well-posedness of the decoupled subsystem is established by the nonlinear maximal monotone theories. Furthermore, we consider the exponential stability of the subsystem by the multiplier method. In Section 4, the exponential stability of the global closed loop system is discussed.

2 Set Up the System Operator

In this paper, the flexural rigidity $EI(x)$ and the mass per unit length of the beam $\rho(x)$ are supposed to be positive smooth functions. Moreover, we assume that

$$0 < \rho_0 < \rho(x) \in C[0,1], \quad 0 < EI_0 < EI(x) \in C^4[0,1]. \tag{5}$$

Let us introduce the following spaces:

$$H_0^n(0,1) = \{u \in H^n(0,1); u(0) = u_x(0) = 0\}, \quad n \in \mathbb{N},$$
$$\mathcal{H} = H_0^2(0,1) \times L^2(0,1),$$

where the space $L^2(0,1)$ is the Hilbert space of square summable functions equipped with the inner product

$$\langle f, g\rangle_{L^2(0,1)} = \int_0^1 \rho(x) f g dx.$$

The inner product of H_0^2 can be taken as

$$\langle f, g\rangle_{H_0^2(0,1)} = \int_0^1 EI(x) f_{xx} g_{xx} dx.$$

The norm in $\mathcal{H}$ is induced by the following inner product:

$$\|(y,z)^{\mathrm{T}}\|_{\mathcal{H}}^2 = \|y(x,t)\|_{H_0^2(0,1)}^2 - \widehat{\omega}^2\|y(x,t)\|_{L^2(0,1)}^2 + \|z(x)\|_{L^2(0,1)}^2, \tag{6}$$

which is the usual one of the space $H^2(0,1)\times L^2(0,1)$ provided that $|\widehat{\omega}| < 2\sqrt{3EI_0/\|\rho\|_\infty}$, where $\|\rho\|_\infty = \|\rho\|_{L^\infty(0,1)}$.

We consider the system (2) in the state space $\widetilde{\mathcal{H}} = \mathcal{H}\times\mathbb{R}$, with inner product

$$\|(\varPhi,\omega)^{\mathrm{T}}\|_{\widetilde{\mathcal{H}}}^2 = \|\varPhi\|_{\mathcal{H}}^2 + |\omega|^2, \tag{7}$$

where $\varPhi = (y,z)\in\mathcal{H}$.

Define the unbounded nonlinear operator $\widetilde{\mathcal{A}}$ in $\widetilde{\mathcal{H}}$

$$\widetilde{\mathcal{A}} = \mathrm{diag}(\mathcal{A}_{\widehat{\omega}},\ 0) \tag{8}$$

with

$$D(\mathcal{A}_{\widehat{\omega}}) = \{(y,z)\in H_0^4(0,1)\times H_0^2(0,1); y_{xx}(1) = (EI(x)y_{xx})_x(1) = 0\}, \tag{9}$$

and $\phi = (y,z)^{\mathrm{T}}\in\mathcal{D}(\mathcal{A}_{\widehat{\omega}})$,

$$\mathcal{A}_{\widehat{\omega}}(y,z)^{\mathrm{T}} = \Big(-z, \frac{1}{\rho(x)}(EI(x)y_{xx})_{xx} + \frac{1}{\rho(x)}\mathcal{F}(z) - \widehat{\omega}^2 y\Big)^{\mathrm{T}}. \tag{10}$$

Next, we introduce the following nonlinear operator, for any $\phi(t) = (y,z)^{\mathrm{T}}\in\mathcal{H}$,

$$\boldsymbol{J}(\phi(t),\omega(t),t) = \Big(0, (\omega^2(t)-\widehat{\omega}^2)y, \frac{-\gamma(\omega(t)-\widehat{\omega}) - 2\omega(t)\int_0^1\rho(x)yz\,dx}{I_d + \int_0^1\rho(x)y^2dx}\Big)^{\mathrm{T}}. \tag{11}$$

With these notations, the system (2) can be formally written into the following abstract nonlinear evolutionary equation:

$$\begin{pmatrix}\dot{\phi}(t)\\ \dot{\omega}(t)\end{pmatrix} = -\widetilde{\mathcal{A}}\begin{pmatrix}\phi(t)\\ \omega(t)\end{pmatrix} + \boldsymbol{J}\begin{pmatrix}\phi(t)\\ \omega(t)\end{pmatrix} \tag{12}$$

with initial conditions

$$\phi(0) = \phi_0, \quad \omega(0) = \omega_0,$$

where $\phi(t) = (u,u_t)^{\mathrm{T}}$, $\phi_0 = (u_0,u_1)^{\mathrm{T}}$.

In particular, we consider the following system in the space $\mathcal{H}$:

$$\begin{cases}\dot{\phi}(t) = -\mathcal{A}_{\widehat{\omega}}\phi(t),\\ \phi(0) = \phi_0,\end{cases} \tag{13}$$

which is the subsystem of (12).

3 Well-posedness and Stabilization of System (13)

In this section, we consider the exponential stability of System (13). Consider the norm of the state which is just the energy of System (13):

$$E(t) = \frac{1}{2}\left\{\int_0^1 EI(x)|u_{xx}(x,t)|^2 - \rho(x)\widehat{\omega}^2|u(x,t)|^2 + \rho(x)|u_t(x)|^2 dx\right\}. \tag{14}$$

Differentiating E with respect to time t, we get

$$\begin{aligned}\dot{E}(t) &= \int_0^1 \left(EI(x)u_{xx}u_{xxt} - \rho(x)\widehat{\omega}^2 uu_t + \rho(x)u_t u_{tt}\right)dx \\ &= -\int_0^1 u_t\mathcal{F}(u_t)dx \le 0, \quad \forall t \ge 0.\end{aligned} \tag{15}$$

First, we prove the well-posedness of the system (13) using the following result.

Lemma 3.1 *The operator $\mathcal{A}_{\widehat{\omega}}$ given by* (9)–(10) *is maximal monotone on $\mathcal{H}$ with the domain $\mathcal{D}(\mathcal{A}_{\widehat{\omega}})$ that is dense in $\mathcal{H}$.*

Proof For any $\phi = (u,v)^{\mathrm{T}} \in \mathcal{D}(\mathcal{A}_{\widehat{\omega}})$, $\psi = (y,z)^{\mathrm{T}} \in \mathcal{D}(\mathcal{A}_{\widehat{\omega}})$, we have

$$\langle \mathcal{A}_{\widehat{\omega}}\phi - \mathcal{A}_{\widehat{\omega}}\psi, \phi - \psi\rangle_{\mathcal{H}} = \int_0^1 \{\mathcal{F}(v) - \mathcal{F}(z)\}(v-z)dx \ge 0. \tag{16}$$

This means that $\mathcal{A}_{\widehat{\omega}}$ is monotone. To prove maximal monotonicity, it is enough to show that

$$\mathcal{R}(I + \mathcal{A}_{\widehat{\omega}}) = \mathcal{H}. \tag{17}$$

For a given $U = (g,h)^{\mathrm{T}} \in \mathcal{H}$, solve $(I + \mathcal{A}_{\widehat{\omega}})V = U$, $V = (y,z)^{\mathrm{T}} \in \mathcal{D}(\mathcal{A}_{\widehat{\omega}})$ to get $z = y - g$ and y satisfies following equations:

$$\begin{cases}(EI(x)y_{xx})_{xx} + \mathcal{F}(y-g) - \rho(x)(\widehat{\omega}^2 - 1)y = \rho(x)(h+g), \\ y(0) = y_x(0) = y_{xx}(1) = (EI(x)y_{xx})_x(1) = 0.\end{cases} \tag{18}$$

If $y \in \mathcal{H}_0^4(0,1)$ is a solution of (18), then multiplying by $\varphi \in \mathcal{H}_0^2(0,1)$ both sides of the first equation of (18) and integrating from 0 to 1 with respect to x, we have

$$a(y,\varphi) + \int_0^1 \mathcal{F}(y-g)\varphi dx - G(\varphi) = 0, \tag{19}$$

where the bilinear functional $a(y,\varphi)$ is defined and coercive on $\mathcal{H}_0^2(0,1)$, the bounded functional $G(\cdot)$ is defined on $\mathcal{H}_0^2(0,1)$ as

$$a(y,\varphi) = \int_0^1 EI(x)y_{xx}\varphi_{xx}dx + (1-\widehat{\omega}^2)\int_0^1 \rho(x)y\varphi dx,$$

and $G(\varphi) = \int_0^1 \rho(x)(h+g)\varphi dx$. As in [11], let

$$H(\varphi) = \frac{1}{2}a(\varphi,\varphi) + \int_0^1 K(\varphi)dx - G(\varphi),$$

where $K(\varphi) = \int_0^\varphi \mathcal{F}(s-g)ds$. From (3), we deduced that the function $H(\varphi)$ is convex, coercive and continuous in $H_0^2(0,1)$. Hence, by a minimization theorem (Proposition 38.15, p.155, [12]), there exists a function y in $H_0^2(0,1)$ such that $H(y) = \inf_{\varphi \in H_0^2(0,1)} H(\varphi)$. This implies that the function

$$\mathfrak{Y} : \lambda \to \mathfrak{Y}(\lambda) = H(y + \lambda\varphi)$$

admits a minimum at $\lambda = 0$ and thus, $\frac{d}{d\lambda}(H(y + \lambda\varphi))|_{\lambda=0} = 0$. This means that for all $\varphi \in H_0^2(0,1)$,

$$a(y,\varphi) + \int_0^1 \mathcal{F}(y-g)\varphi dx - G(\varphi) = 0, \tag{20}$$

or $y \in H_0^2(0,1)$ satisfies Equation (18). On the other hand, we know that the system (18) is a regular elliptic boundary value problem, hence, from the classical elliptic theory[13], we deduce that $y \in H_0^4(0,1)$. For $g \in H_0^2(0,1)$, we see that $z = y - g \in H_0^2(0,1)$, and $(y,z) \in \mathcal{D}(\mathcal{A}_{\widehat{\omega}})$ satisfies the equation (17). Thus the maximality of the operator $\mathcal{A}_{\widehat{\omega}}$ is proved.

Finally, it remains to prove that the domain $\mathcal{D}(\mathcal{A}_{\widehat{\omega}})$ is dense in $\mathcal{H}$. If there exists $y \in \mathcal{H}$, such that

$$\langle y, z\rangle_{\mathcal{H}} = 0, \quad \forall z \in \mathcal{D}(\mathcal{A}_{\widehat{\omega}}).$$

Suppose $y = (I + \mathcal{A}_{\widehat{\omega}})y_0, y_0 \in \mathcal{D}(\mathcal{A}_{\widehat{\omega}})$, then

$$\langle y_0, y_0\rangle_{\mathcal{H}} \le \langle y_0, y_0\rangle_{\mathcal{H}} + \langle \mathcal{A}_{\widehat{\omega}} y_0, y_0\rangle_{\mathcal{H}} = \langle (I + \mathcal{A}_{\widehat{\omega}})y_0, y_0\rangle_{\mathcal{H}} = \langle y, y_0\rangle_{\mathcal{H}} = 0,$$

which implies that $y = y_0 = 0$. Thus, $\mathcal{D}(\mathcal{A}_{\widehat{\omega}})$ is dense in $\mathcal{H}$. ▮

Since the operator $\mathcal{A}_{\widehat{\omega}}$ is maximal monotone with the dense domain $\mathcal{D}(\mathcal{A}_{\widehat{\omega}})$ in the energy space $\mathcal{H}$, using the method of Theorem 3.1 (see p.54, [14]) to deal with the evolution equation (13), we obtain the following result.

Theorem 3.2 1) *For any initial data* $(u_0, u_1) \in \mathcal{D}(\mathcal{A}_{\widehat{\omega}})$, *the system* (13) *admits a unique strong solution* $(u, u_t) \in \mathcal{D}(\mathcal{A}_{\widehat{\omega}})$ $(t \ge 0)$. *Moreover, the function* $t \to \|\mathcal{A}_{\widehat{\omega}}\phi(t)\|_{\mathcal{H}}$ *is nonincreasing.* 2) *For any initial data* $(u_0, u_1) \in \overline{\mathcal{D}(\mathcal{A}_{\widehat{\omega}})} = \mathcal{H}$, *the system* (13) *admits a unique mild solution* $(u, u_t) = T(t)(u_0, u_1) \in \mathcal{H}$ $(t \ge 0)$, *where* $T(t) = \mathrm{e}^{-\mathcal{A}_{\widehat{\omega}} t}$ *is the strongly continuous semigroup of contractions on* $\mathcal{H}$ *generated by* $-\mathcal{A}_{\widehat{\omega}}$ *on* $\mathcal{H}$.

Next, we establish the uniform decay of the energy $E(t)$ of the system (13) under hypotheses (HI) and (HII). We assume that the initial data belongs to the domain $\mathcal{D}(\mathcal{A}_{\widehat{\omega}})$, the solution of Equation (13) has the regularity properties expressed by Theorem 3.2. All the decay estimates of the energy established in that case can be easily extended to weak solutions using denseness of the domain $\mathcal{D}(\mathcal{A}_{\widehat{\omega}})$ in the space $\mathcal{H}$, and the contraction of the semigroup $\{T(t)\}$. Suppose

$$E_\varepsilon(t) = E(t) - \varepsilon\phi(t), \tag{21}$$

where the constant ε is to be determined, and

$$\phi(t) = \int_0^1 \rho(x) x u_t u dx. \tag{22}$$

Calculating the time derivative of $\phi(t)$ along the solution of (2) and using the boundary conditions of (13) yield

$$\begin{aligned}
\dot{\phi}(t) &= \int_0^1 \rho(x)(xu_t^2 + xu_{tt}u)dx \\
&= \int_0^1 \rho(x)xu_t^2 dx + \int_0^1 xu\left(-(EI(x)u_{xx})_{xx} - \mathcal{F}(u_t) + \rho(x)\omega^2 u\right)dx \\
&= \int_0^1 \rho(x)xu_t^2 dx + \int_0^1 xu\left(-\mathcal{F}(u_t) + \rho(x)\omega^2 u\right)dx \\
&\quad -2\int_0^1 EI(x)u_x u_{xx}dx - \int_0^1 xEI(x)u_{xx}u_{xx}dx \\
&\le \int_0^1 x\rho(x)u_t^2 dx + \int_0^1 xu\left(-\mathcal{F}(u_t) + \rho(x)\omega^2 u\right)dx \\
&\quad + \int_0^1 EI(x)u_x^2 dx + \int_0^1 EI(x)u_{xx}^2 dx - EI\int_0^1 xu_{xx}u_{xx}dx \\
&= \int_0^1 x\rho(x)u_t^2 dx + \int_0^1 xu\left(-\mathcal{F}(u_t) + \rho(x)\omega^2 u\right)dx \\
&\quad + \int_0^1 EI(x)u_x^2 dx + \int_0^1 (1-x)EI(x)u_{xx}^2 dx.
\end{aligned}$$

Indeed for any $u \in H_0^2(0,1)$, $u = \int_0^x (x-\xi)u_{\xi\xi}^2 d\xi$. Then using the Cauchy inequality, we get

$$|u|^2 \le \int_0^x (x-\xi)^2 d\xi \int_0^x |u_{\xi\xi}|^2 d\xi \le \frac{x^3}{3}\int_0^1 |u_{\xi\xi}|^2 d\xi.$$

Thus, we have

$$\begin{aligned}
\dot{\phi}(t) &\le \int_0^1 x\rho(x)u_t^2 dx + \int_0^1 \mathcal{F}^2(u_t)dx + \int_0^1 u^2 dx \\
&\quad + \int_0^1 x\rho(x)\omega^2 u^2 dx + \int_0^1 EI(x)u_x^2 dx + \int_0^1 (1-x)EI(x)u_{xx}^2 dx \\
&\le \int_0^1 x\rho(x)u_t^2 dx + M_2\int_0^1 u_t\mathcal{F}(u_t)dx + (1+\omega^2)\int_0^1 x\rho(x)u^2 dx + \int_0^1 2EI(x)u_x^2 dx \\
&\le \int_0^1 \rho(x)u_t^2 dx + M_2\int_0^1 u_t\mathcal{F}(u_t)dx \\
&\quad + \int_0^1 \left(\frac{\rho(x)}{EI(x)} + \frac{EI_0}{\|\rho\|_\infty}\frac{\rho(x)}{EI(x)}\right)EI(x)u_{xx}^2 dx + \int_0^1 2EI(x)u_{xx}^2 dx \\
&\le \int_0^1 \rho(x)u_t^2 dx + M_2\int_0^1 u_t\mathcal{F}(u_t)dx \\
&\quad + \int_0^1 \left(\frac{\|\rho\|_\infty}{EI_0} + 1\right)EI(x)u_{xx}^2 dx + \int_0^1 2EI(x)u_{xx}^2 dx,
\end{aligned}$$

which implies that there exist the positive constants c_1 and c_2 satisfying

$$\dot{\phi}(t) \le c_1 E(t) - c_2\dot{E}(t). \tag{23}$$

It is easy to show that there exists a positive constant c_3 such that

$$\phi(t) \le c_3 E(t), \tag{24}$$

from which we have $(1 - c_3\varepsilon)E(t) \le E_\varepsilon(t) \le (1 + c_3\varepsilon)E(t)$. If we choose ε is small enough such that $\varepsilon \le c_3^{-1}$. From (23) and (24), we calculate the derivative of the perturbed energy

$$\dot{E}_\varepsilon(t) = \dot{E}(t) - \varepsilon\dot{\phi}(t) = -c_1\varepsilon E(t) + (1 + c_2\varepsilon)\dot{E}(t) \le -\frac{c_1\varepsilon}{1 + c_3\varepsilon}E_\varepsilon(t). \tag{25}$$

Thus,

$$E_\varepsilon(t) \le \exp\left(-\frac{c_1\varepsilon}{1 + c_3\varepsilon}t\right)E_\varepsilon(0) \le (1 + c_3\varepsilon)\exp\left(-\frac{c_1\varepsilon}{1 + c_3\varepsilon}t\right)E(0),$$

which means $E(t) \le \frac{1+c_3\varepsilon}{1-c_3\varepsilon}\exp(-\frac{c_1\varepsilon}{1+c_3\varepsilon}t)E(0)$, we deduce that the energy decays exponentially. Hence, there exist positive constants $M = \sqrt{\frac{1+c_3\varepsilon}{1-c_3\varepsilon}}$, $c = \frac{c_1\varepsilon}{2(1+c_3\varepsilon)}$, such that

$$\|\phi(t)\|_{\mathcal{H}} = E^{\frac{1}{2}}(t) \le M\mathrm{e}^{-ct}\|\phi(0)\|_{\mathcal{H}}, \quad \forall t \ge 0.$$

We obtain the following result.

Theorem 3.3 *Assume that* $|\omega| < 2\sqrt{3EI/\rho}$, (HI) *and* (HII) *hold. For any initial condition* $\phi_0 = (u_0, u_1) \in \mathcal{H}$, *the corresponding solution* $\phi(t) = (u, u_t) = T(t)\phi_0 \in \mathcal{H}$ $(t \ge 0)$ *of the system* (13) *satisfies*

$$\|\phi(t)\|_{\mathcal{H}} \le M\mathrm{e}^{-ct}\|\phi(0)\|_{\mathcal{H}}, \quad \forall t \ge 0, \tag{26}$$

where M, c *are positive constants.*

4 Well-Posedness and Stabilization of System

In this section, we establish the exponential stability of the closed-loop system (12) under hypotheses (HI) and (HII) .

Theorem 4.1 *For any initial data* $(\phi_0, \omega_0)^{\mathrm{T}} \in \widetilde{\mathcal{H}}$, *the system* (12) *admits a unique solution* $(\phi(t), \omega(t))^{\mathrm{T}}$. *Moreover,* $\|(\phi(t), \omega(t))^{\mathrm{T}}\|_{\widetilde{\mathcal{H}}}$ *is uniformly bounded for all* $t \ge 0$.

Proof We will show that the global system (12) is well posed on $[0, \infty)$. We have proved in Lemma 3.1 that the operator $\mathcal{A}_{\widehat{\omega}}$ is maximal monotone with dense domain $\mathcal{D}(\mathcal{A}_{\widehat{\omega}})$ in $\mathcal{H}$. From (8), we deduce that the operator $\widetilde{\mathcal{A}}$ is maximal monotone with dense domain $\mathcal{D}(\widetilde{\mathcal{A}})$ in $\widetilde{\mathcal{H}}$. The (11) yields the nonlinear operator $\boldsymbol{J}$ is Lipschitz continuous on $\mathbb{R}$. Using the M. Pierre result (see P.126, [15]), the system (12) exists a local solution $(\phi(t), \omega(t))^T$, i.e, there is a time interval $[0, T]$ (where T is a positive constant) on which a unique solution of (12) exists.

Consider the Lyapunov function $V : \widetilde{\mathcal{H}} \to \mathbb{R}^+$

$$V(\phi(t), \omega(t)) = \frac{1}{2}\left\{I_d\{\omega(t) - \widehat{\omega}\}^2 + \{\omega(t) - \widehat{\omega}\}^2\|u\|^2_{L^2(0,1)}\right\} + \frac{1}{2}\left\{\|u\|^2_{H_0^2(0,1)} + \|u_t\|^2_{L^2(0,1)} - \widehat{\omega}^2\|u\|^2_{L^2(0,1)}\right\}. \tag{27}$$

Differentiating $V(\phi(t),\omega(t))$ along solution of (12) gives

$$\begin{aligned}\dot V &= \{\omega(t)-\widehat\omega\}\,\dot\omega(t)\left\{I_d+\|u\|^2_{L^2(0,1)}\right\}+\{\omega(t)-\widehat\omega\}^2\langle u,u_t\rangle_{L^2(0,1)}\\ &\quad+\langle u,u_t\rangle_{H_0^2(0,1)}+\langle u_t,u_{tt}\rangle_{L^2(0,1)}-\widehat\omega^2\langle u,u_t\rangle_{L^2(0,1)}\\ &=\{\omega(t)-\widehat\omega\}\,\Gamma(t)-\int_0^1 u_t\mathcal{F}(u_t)dx. \end{aligned}\tag{28}$$

The hypothesis (HI) yields

$$\dot V(\phi(t),\omega(t))=-\gamma\{\omega(t)-\widehat\omega\}^2-\int_0^1 u_t\mathcal{F}(u_t)dx\le 0,\tag{29}$$

thus, $V(\phi(t),\omega(t))$ is nonincreasing on its domain. It also implies that $\|(\phi(t),\omega(t))^{\rm T}\|_{\widetilde{\mathcal H}}$ is uniformly bounded for all $t\ge 0$. From Proposition 1 in [9], we obtain that for any initial condition, the system (12) has a unique global solution. ∎

Theorem 4.2 *Assume that* $|\widehat\omega|<2\sqrt{3EI/\rho}$, (HI) *and* (HII) *hold. For any initial condition* $(\phi_0,\widehat\omega_0)\in\widetilde{\mathcal H}$ *with* $\phi_0=(u_0,u_1)$, *the corresponding solution* $(\phi(t),\omega(t))\in\widetilde{\mathcal H}$ *of the system* (12) *exponentially tends to the equilibrium point* $(0,\widehat\omega)$.

Proof Let $X(t)=(\phi(t),\omega(t))$ be the global solution of the system (12), with initial date $X_0=(\phi_0,\omega_0)\in\mathcal D(\widetilde{\mathcal A})$, $\delta>0$ and $t_k=t_0+k\delta$. By (29), the Lyapunov function $V(\phi(t),\omega(t))$ is nonincreasing and converges to a positive constant l as $t\to\infty$. Hence, the integral $\int_0^{+\infty}\gamma(\omega(t)-\widehat\omega)^2dt$ is finite. $\int_0^{+\infty}(\omega(t)-\widehat\omega)^2dt$ and its derivative are bounded, the Barbalat's lemma implies that $\lim_{t\to\infty}(\omega(t)-\widehat\omega)=0$. Hence, for any $\varepsilon>0$, there exists a T_0 large enough such that

$$|\omega(t)-\widehat\omega|<\varepsilon,\quad \text{for } t\ge T_0.\tag{30}$$

In addition, one can show that $\omega(t)-\widehat\omega\in L^2([0,\infty);\mathbb R)\cap L^\infty([0,\infty);\mathbb R)$.

Let $\mu(t)=|\omega^2(t)-\widehat\omega^2|$, we have

$$\lim_{\xi\to\infty}\int_\xi^{\xi+\delta}\mu(\tau)d\tau=0,\quad\forall\delta>0.\tag{31}$$

According to (31), we choose δ and t_0 as follows:

$$\delta>\frac1c(\ln M+2c_0),\qquad\int_\xi^{\xi+\delta}\mu(\tau)d\tau\le\frac{c_0}{{\rm e}^{c_0\delta}},\quad\forall\xi\ge t_0\ge T_0,\tag{32}$$

where $c_0>\max(1,c)$.

Since the operator $\widetilde{\mathcal A}$ is accretive, from [16], we know that, for $t\in[t_{k-1},t_k],k\ge1$,

$$\|X(t)-S(t-t_{k-1})X(t_{k-1})\|_{\widetilde{\mathcal H}}\le\int_{t_0}^t\|\boldsymbol J(X(\tau)\|_{\widetilde{\mathcal H}}d\tau,\tag{33}$$

where $\{S(t-t_{k-1})(\cdot)\}_{t\in[t_{k-1},t_k]}$ is the semigroup of nonlinear contractions generated by $\widetilde{\mathcal A}$. So, we have

$$\|X(t)\|_{\widetilde{\mathcal H}}\le\|S(t-t_{k-1})X(t_{k-1})\|_{\widetilde{\mathcal H}}+\int_{t_0}^t\|\boldsymbol J(X(\tau)\|_{\widetilde{\mathcal H}}d\tau,\tag{34}$$

As mentioned in Theorem 3.3, the semigroup $\mathrm{e}^{-\mathcal{A}_{\widehat{\omega}}t}$ satisfies $\|\mathrm{e}^{-\mathcal{A}_{\widehat{\omega}}t}\|_{L(\mathcal{H})} \leq M\mathrm{e}^{-ct}$, where $L(\mathcal{H})$ is the operator norm, M and c are constants. From (11) and (34), we have

$$\|\phi(t)\|_{\mathcal{H}} \leq \|\mathrm{e}^{-\mathcal{A}_{\widehat{\omega}}(t-t_{k-1})}\phi(t_{k-1})\|_{\mathcal{H}} + \int_{t_0}^{t} |\omega^2(t) - \widehat{\omega}^2| \|\phi(\tau)\|_{\mathcal{H}} d\tau. \tag{35}$$

Seting $y_k(t) = \mathrm{e}^{c(t-t_{k-1})}\|\phi(t)\|_{\widetilde{\mathcal{H}}}$, we get

$$y_k(t) \leq M y_k(t_{k-1}) + \mathrm{e}^{c\delta} \int_{t_{k-1}}^{t} \mu(\tau) y_k(\tau) d\tau. \tag{36}$$

Applying the Gronwall's lemma to the above inequality, we have

$$y_k(t) \leq M y_k(t_{k-1}) \exp\left(\mathrm{e}^{c\delta} \int_{t_{k-1}}^{t} \mu(\tau) d\tau\right), \quad t \in [t_{k-1}, t_k]. \tag{37}$$

Hence, $y_k(t_k) \leq M y_k(t_{k-1})\alpha_k$, where $\alpha_k = \exp\left(\mathrm{e}^{c\delta}\int_{t_{k-1}}^{t_k} \mu(\tau)d\tau\right)$. By (32), (35) and (37), it implies that $\|\phi(t)\|_{\mathcal{H}} \leq M\|\phi(t_k)\|_{\mathcal{H}} \exp\left(-c(t-t_k) + \alpha_{k+1}\right)$, for $t \in [t_k, t_{k+1}]$, and

$$\begin{aligned}
\|\phi(t_k)\|_{\mathcal{H}} &\leq M^k \exp\Bigg(-k\delta + \sum_{j=1}^{k} \alpha_j\Bigg) \|\phi_0\|_{\mathcal{H}} \\
&= M^k \exp\left(-k\delta + \mathrm{e}^{c\delta} \int_{t_0}^{t_k} \mu(\tau) d\tau\right) \|\phi_0\|_{\mathcal{H}} \\
&\leq \exp\left(k \ln M - k\delta + \mathrm{e}^{c\delta} \int_{t_0}^{t_k} \mu(\tau) d\tau\right) \|\phi_0\|_{\mathcal{H}} \\
&\leq \exp\left(k \ln M - k\delta + \mathrm{e}^{c\delta} \frac{c_0}{\mathrm{e}^{c_0\delta}}\right) \|\phi_0\|_{\mathcal{H}} \\
&\leq \exp\left(-c_0 k\right) \|\phi_0\|_{\mathcal{H}}.
\end{aligned} \tag{38}$$

Hence,

$$\begin{aligned}
\|\phi(t)\|_{\mathcal{H}} &\leq M\|\phi(t_k)\|_{\mathcal{H}} \exp\left(-c(t-t_k) + \alpha_{k+1}\right) \\
&\leq M \exp\left(-c_0 k\right) \|\phi_0\|_{\mathcal{H}} \exp\left(-c(t-t_k) + \alpha_{k+1}\right) \\
&\leq M \exp\left(-c_0 k - c(t-t_k) + c_0\right) \|\phi_0\|_{\mathcal{H}} \\
&\leq M \exp\left(-c_0(k-1) - c(t-t_k)\right) \|\phi_0\|_{\mathcal{H}}.
\end{aligned} \tag{39}$$

Thereafter, we show that for any initial condition, $\phi(t)$ tends exponentially to zero in $\mathcal{H}$ as $t \to \infty$. Now, return to the differential equation $\omega(t)$ in (12), the exponential convergence of $\omega(t)$ towards $\widehat{\omega}$ can be established analogously to [7]. Hence, we show that the solution of the closed-loop system exponentially tends to the equilibrium point $(0, \widehat{\omega})$. ∎

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