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Stabilization of a pendulum in dynamic boundary feedback with a memory type heat equation

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This paper addresses a dynamic feedback stabilization of an interconnected pendulum system with a memory type heat equation, where the kernel memory is an exponential polynomial. By introducing some new variables, the time-variant system is transformed into a time-invariant one. The detailed spectral analysis is presented. Remarkably, the resolvent of the closed-loop system operator is not compact anymore. The residual spectrum is shown to be empty and the continuous spectrum consists of finite isolated points. Furthermore, it is shown that there is a sequence of generalized eigenfunctions, which forms a Riesz basis for the Hilbert state space. This deduces the spectrum-determined growth condition for the C_0-semigroup, and the exponential stability is then followed. Finally, some numerical simulations are presented to show the effectiveness of this feedback control design.

Keywords: heat equation with memory; spectrum; asymptotic analysis; Riesz basis; exponential stability.

1. Introduction

The control and dynamics of the pendulum system have been studied in the past decades, especially in the field of robotics and intelligent vehicles. A basic problem is controlling the position of a singlelink rotational joint by using a motor placed at the pivot, mathematically that is an inverted pendulum to which one can apply a torque as an external force (Sontag, 1998). The motion of the inverted pendulum with an external torque can be formulated as the following second-order linearized differential equation:

$$\ddot{y}(t) - \frac{g}{l}y(t) = u(t), \tag{1.1}$$

where y denotes the angular displacement from the inverted equilibrium, g denotes the acceleration due to gravity, l is the length of the pendulum and $u(t)$ denotes the value of the external torque at time t. Suh & Bien (1979) introduced a proportional-minus-delay (PMD) controller for the stabilization of the pendulum system, which is very popular until now. The so-called PMD controller is given by

$$u(t) = ay(t) + by(t-\tau). \tag{1.2}$$

Let $v(x,t) = y(t + \tau(x-1))$, and since the time delay itself is a dynamical system, by the new variable, system (1.1) can be written as follows:

$$\begin{cases} \ddot{y}(t) + ky(t) = bv(0,t), & t > 0,\ k,b \in \mathbb{R}, \\ \tau v_t(x,t) = v_x(x,t), & x \in (0,1),\ t \geqslant 0, \\ v(1,t) = y(t), \end{cases} \tag{1.3}$$

where $k = -g/l - a$ and the partial differential equation (PDE) part is considered as the controller and the original control plant ordinary differential equation (ODE) is connected with PDE through boundary output of the PDE. According to Atay (1999) or Wang *et al.* (2011), the system (1.3) is exponentially stable if k, b satisfy

$$\min\{k - n^2\pi^2, (n+1)^2\pi^2 - k\} > (-1)^n b > 0 \tag{1.4}$$

for some non-negative integer n. On the other hand, in Zhao & Wang (2014), the PDE in (1.3) is replaced by a heat equation, which is a major control strategy, and the coupled system is given as follows:

$$\begin{cases} \ddot{y}(t) + ky(t) = bv(0,t), & t > 0, \\ v_t(x,t) = v_{xx}(x,t), & x \in (0,1),\ t \geqslant 0, \\ v_x(0,t) = b\dot{y}(t), \\ v(1,t) = 0. \end{cases} \tag{1.5}$$

It is shown that system (1.5) is exponentially stable under a heat PDE compensator, with the coefficients $k > 0$ and $b \neq 0$.

However, it is indicated that the coupling system does not take the memory effect into account, which may exist in some materials particularly in low temperature. Moreover, the fact that the thermal disturbance at one point affects the whole elastic body instantly, which is implied by (1.5), is not physically acceptable (Fatori & noz Rivera, 2001). In Gurtin & Pipkin (1968), the authors indicated that a heat equation, which is parabolic, the speed of propagation of thermal disturbance is infinite, and in the same paper, they proposed a general theory for heat conduction with finite propagation speed. The linearized Gurtin–Pipkin heat equation is described by

$$\theta_t(x,t) = \int_{\infty}^{t} k(t-s)\theta_{xx}(x,s)\,\mathrm{d}s,$$

where the kernel k is supposed to be a positive non-increasing function of its variable. Instead of the memory from infinity, this type of heat equation with memory starting from the starting point was considered in Pandolfi (2005):

$$\theta_t(x,t) = \int_{0}^{t} k(t-s)\theta_{xx}(x,s)\,\mathrm{d}s.$$

The cosine operator approach was used to study its well-posedness. In Wang *et al.* (2009), the same heat equation under the Dirichlet boundary condition was considered, with a special type of kernel

$$k(t) = \sum_{j=1}^{N} a_j^2\, e^{-b_j t}, \quad 0 < a_j, b_j \in \mathbb{R}, \quad 1 \leqslant N \in \mathbb{N}.$$

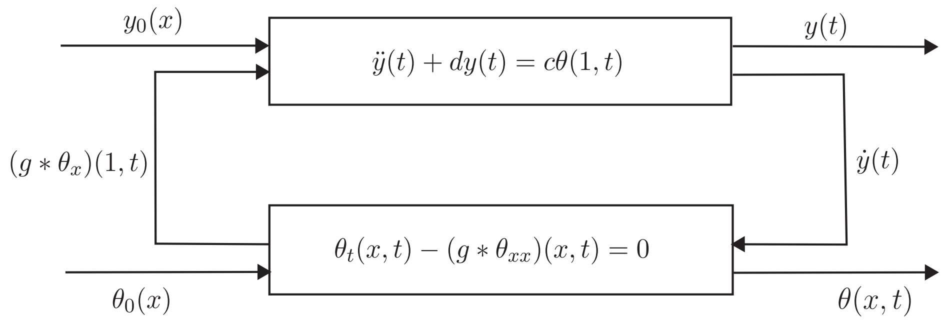

FIG. 1. Interconnected system of ODE-heat equation with memory.

It is shown that this system achieve strongly exponential stability, though the resolvent of the system operator is not compact.

Hence, with these inspiration, it is natural to raise a question of whether a heat equation with memory can be regarded as a compensator to stabilize the pendulum, so we study the following interconnected system (see Fig. 1):

$$\begin{cases} \ddot{y}(t) + dy(t) = c\theta(1,t), & t > 0, \\ \theta_t(x,t) - (g * \theta_{xx})(x,t) = 0, & x \in (0,1), \\ \theta(0,t) = 0, & t \geqslant 0, \\ (g * \theta_x)(1,t) = -c\dot{y}(t), & t \geqslant 0, \\ y(0) = y_0, \quad \dot{y}(0) = y_1, & \\ \theta(x,0) = \theta_0(x), & x \in [0,1], \end{cases} \tag{1.6}$$

where $d > 0$, $c \neq 0$. The sign '*' denotes the convolution product:

$$(g * l)(x,t) = \int_0^t g(t-s) l(x,s) \, \mathrm{d}s.$$

The kernel being taken as an exponential type

$$g(t) = \sum_{j=1}^{N} a_j^2 e^{-b_j t}, \quad 0 < a_j, b_j \in \mathbb{R}, \ \ 1 \leqslant N \in \mathbb{N}.$$

For simplicity, we assume that

$$0 < b_1 < b_2 < \cdots < b_N. \tag{1.7}$$

Our study is aim to understand the dynamic behaviour of (1.6) and (1.7), particularly in large time behaviour. This paper is organized as follows. In Section 2, we introduce some new variables, so that the system (1.6) is reduced to be a time-invariant system. And then we formulate the system into an abstract evolution equation. Section 3 is devoted to the spectral analysis of the system. Riesz basis property and exponential stability of the system are given in Section 4. In Section 5, these analytic results are confirmed by numerical simulations.

2. System operator setup

Introduce

$$\phi_j(x,t) = \int_0^t a_j\, e^{-b_j(t-s)}\theta_x(x,s)\,\mathrm{d}s, \quad j = 1,2,\ldots,N,$$

we have $(\phi_j)_t = a_j\theta_x - b_j\phi_j$. Then, equation (1.6) can be changed into

$$\begin{cases} \ddot{y}(t) + dy(t) = c\theta(1,t), & t > 0, \\ \theta_t(x,t) - \left[\displaystyle\sum_{j=1}^N a_j\phi_j(x,t)\right]_x = 0, & x \in (0,1), \\ (\phi_j)_t(x,t) = a_j\theta_x(x,t) - b_j\phi_j(x,t), & t > 0, \\ \theta(0,t) = 0, \quad \displaystyle\sum_{j=1}^N a_j\phi_j(1,t) = -c\dot{y}(t), & t \geqslant 0, \\ y(0) = y_0, \quad \dot{y}(0) = y_1, & \\ \theta(x,0) = \theta_0(x), \quad \phi_j(x,0) = 0, & j = 1,2,\ldots,N. \end{cases} \tag{2.1}$$

The energy function for (2.1) is given by

$$E(t) = \frac{1}{2}\left[dy^2(t) + \dot{y}^2(t)\right] + \frac{1}{2}\int_0^1 \left[\theta^2(x,t) + \sum_{j=1}^N \phi_j^2(x,t)\right]\mathrm{d}x. \tag{2.2}$$

A direct computation shows that

$$\begin{aligned} \dot{E}(t) &= dy(t)\dot{y}(t) + \dot{y}(t)\ddot{y}(t) + \int_0^1 \left[\theta\theta_t + \sum_{j=1}^N \phi_j(\phi_j)_t\right](x,t)\,\mathrm{d}x \\ &= c\dot{y}(t)\theta(1) + \int_0^1 \theta\left(\sum_{j=1}^N a_j\phi_j\right)_x \mathrm{d}x + \int_0^1 \sum_{j=1}^N \phi_j(a_j\theta_x - b_j\phi_j)\,\mathrm{d}x \\ &= c\dot{y}(t)\theta(1) + \theta\left(\sum_{j=1}^N a_j\phi_j\right)\Bigg|_0^1 - \sum_{j=1}^N b_j\int_0^1 \phi_j^2\,\mathrm{d}x \\ &= -\sum_{j=1}^N b_j\int_0^1 \phi_j^2\,\mathrm{d}x \leqslant 0. \end{aligned}$$

So $E(t)$ is non-increasing.

We consider system (1.6) in the energy state space

$$\mathscr{H} = \mathbb{C} \times \mathbb{C} \times L^2(0,1) \times (L^2(0,1))^N,$$

with the norm induced by the following inner product:

$$\begin{aligned}\langle X, Y\rangle &= d\langle u,\tilde{u}\rangle_{\mathbb{C}} + \langle v,\tilde{v}\rangle_{\mathbb{C}} + \int_0^1 w\bar{\tilde{w}}\,\mathrm{d}x + \sum_{j=1}^{N}\int_0^1 h_j\overline{\tilde{h}_j}\,\mathrm{d}x \\ &= du\bar{\tilde{u}} + v\bar{\tilde{v}} + \int_0^1 w\bar{\tilde{w}}\,\mathrm{d}x + \sum_{j=1}^{N}\int_0^1 h_j\overline{\tilde{h}_j}\,\mathrm{d}x \\ &\quad \forall\ X=(u,v,w,h_1,\ldots,h_N)\in\mathscr{H},\quad Y=(\tilde{u},\tilde{v},\tilde{w},\tilde{h}_1,\ldots,\tilde{h}_N)\in\mathscr{H}.\end{aligned} \tag{2.3}$$

Now, define the system operator $\mathscr{A} : D(\mathscr{A})(\subset\mathscr{H})\to\mathscr{H}$ by

$$\begin{cases}\mathscr{A}\begin{pmatrix}u\\ v\\ w\\ h_1\\ \vdots\\ h_N\end{pmatrix}^{\top} = \begin{pmatrix}v\\ -du+cw(1)\\ \left[\sum_{j=1}^{N}a_jh_j(x)\right]'\\ a_1w'(x)-b_1h_1(x)\\ \vdots\\ a_Nw'(x)-b_Nh_N(x)\end{pmatrix}^{\top}, \\ D(\mathscr{A}) = \left\{\begin{pmatrix}u\\ v\\ w\\ h_1\\ \vdots\\ h_N\end{pmatrix}^{\top} \,\middle|\, \begin{array}{c} w(0)=0,\quad w\in H^1(0,1),\\ \sum_{j=1}^{N}a_jh_j(1)=-cv,\\ h_j\in L^2(0,1), j=1,\ldots,N,\\ \sum_{j=1}^{N}a_jh_j(x)\in H^1(0,1).\end{array}\right\}.\end{cases} \tag{2.4}$$

Then (1.6) can be formulated as an abstract evolution equation in $\mathscr{H}$:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}Z(t)=\mathscr{A}Z(t),\\ Z(0)=Z_0,\end{cases} \tag{2.5}$$

where $Z(t)=(y(t),\dot{y}(t),\theta(\cdot,t),\phi_1(\cdot,t),\ldots,\phi_N(\cdot,t))$ is the state variable and $Z_0=(y_0,y_1,\theta_0(x),0,\ldots,0)$ is the initial value.

THEOREM 2.1 Let $\mathscr{A}$ be defined by (2.4). Then $\mathscr{A}^{-1}$ exists and hence $0\in\rho(\mathscr{A})$, the resolvent set of $\mathscr{A}$. Moreover, $\mathscr{A}$ is dissipative and thus $\mathscr{A}$ generates a C_0-semigroup of contractions $e^{\mathscr{A}t}$ on $\mathscr{H}$.

Proof. Let $\tilde{Z} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{h}_1, \ldots, \tilde{h}_N) \in \mathscr{H}$. Solve $\mathscr{A}Z = \tilde{Z}$ for $Z = (u, v, w, h_1, \ldots, h_N) \in D(\mathscr{A})$, that is

$$\begin{cases} v = \tilde{u}, \quad -du + cw(1) = \tilde{v}, \\ \left[\sum_{j=1}^{N} a_j h_j(x)\right]' = \tilde{w}(x), \\ a_j w'(x) - b_j h_j(x) = \tilde{h}_j(x), \quad j = 1, 2, \ldots, N, \\ w(0) = 0, \quad \sum_{j=1}^{N} a_j h_j(1) = -cv. \end{cases} \tag{2.6}$$

From (2.6), we have

$$u = -\frac{1}{d}\tilde{v} + \frac{c}{d}w(1), \quad v = \tilde{u} \tag{2.7}$$

and

$$h_j(x) = \frac{a_j}{b_j}w'(x) - \frac{1}{b_j}\tilde{h}_j(x), \quad j = 1, 2, \ldots, N. \tag{2.8}$$

Multiplying a_j to both sides of equality (2.8), adding them from 1 to N with respect to j, we have

$$H(x) := \sum_{j=1}^{N} a_j h_j(x) = \sum_{j=1}^{N} \frac{a_j^2}{b_j} w'(x) - \sum_{j=1}^{N} \frac{a_j}{b_j}\tilde{h}_j(x). \tag{2.9}$$

Considering boundary condition $H(1) = \sum_{j=1}^{N} a_j h_j(1) = -c\tilde{u}$, we integrate both sides of the third equation of (2.6) from x to 1 with respect to x, to obtain

$$H(x) = H(1) - \int_x^1 \tilde{w}(s)\, \mathrm{d}s,$$

that is

$$\sum_{j=1}^{N} \frac{a_j^2}{b_j} w'(x) = \sum_{j=1}^{N} \frac{a_j}{b_j}\tilde{h}_j(x) - c\tilde{u} - \int_x^1 \tilde{w}(\tau)\, \mathrm{d}\tau. \tag{2.10}$$

Integrating both sides of (2.10) in x from 0 to x, and using $w(0) = 0$, we obtain

$$w(x) = \frac{1}{T}\left[\int_0^x \sum_{j=1}^{N} \frac{a_j}{b_j}\tilde{h}_j(s)\, \mathrm{d}s - c\tilde{u}x - \int_0^x \int_s^1 \tilde{w}(\tau)\, \mathrm{d}\tau\, \mathrm{d}s\right], \tag{2.11}$$

where $T = \sum_{j=1}^{N} a_j^2/b_j > 0$. Collecting (2.7), (2.8) and (2.11), we get the unique solution Z to equation (2.6). Hence, $\mathscr{A}^{-1}$ exists, or $0 \in \rho(\mathscr{A})$.

Now, we show that $\mathscr{A}$ is dissipative in $\mathscr{H}$. Given $Z = (u, v, w, h_1, \ldots, h_N) \in D(\mathscr{A})$, we have

$$\langle \mathscr{A}Z, Z\rangle = \left\langle \begin{pmatrix} v \\ -du + cw(1) \\ \left[\sum_{j=1}^{N} a_j h_j(x)\right]' \\ a_1 w'(x) - b_1 h_1(x) \\ \vdots \\ a_N w'(x) - b_N h_N(x) \end{pmatrix}^{\top}, \begin{pmatrix} u \\ v \\ w(x) \\ h_1(x) \\ \vdots \\ h_N(x) \end{pmatrix}^{\top} \right\rangle$$

$$= dv\bar{u} + [-du + cw(1)]\bar{v} + \int_0^1 \left[\sum_{j=1}^{N} a_j h_j(x)\right]' \bar{w}(x)\,\mathrm{d}x + \int_0^1 \sum_{j=1}^{N} (a_j w'(x) - b_j h_j(x))\bar{h}_j(x)\,\mathrm{d}x$$

$$= dv\bar{u} - du\bar{v} + cw(1)\bar{v} + \bar{w}(x)\left[\sum_{j=1}^{N} a_j h_j(x)\right]\Bigg|_0^1 - \int_0^1 \left[\sum_{j=1}^{N} a_j h_j(x)\right] \overline{w'}(x)\,\mathrm{d}x$$

$$+ \int_0^1 \left[\sum_{j=1}^{N} a_j \bar{h}_j(x)\right] w'(x)\,\mathrm{d}x - \sum_{j=1}^{N} b_j \int_0^1 |h_j(x)|^2\,\mathrm{d}x,$$

and hence

$$\mathrm{Re}\langle \mathscr{A}Z, Z\rangle = -\sum_{j=1}^{N} b_j \int_0^1 |h_j(x)|^2\,\mathrm{d}x \leqslant 0. \tag{2.12}$$

So $\mathscr{A}$ is dissipative and generates a C_0-semigroup $e^{\mathscr{A}t}$ of contractions on $\mathscr{H}$ by the Lumer–Philips theorem (Pazy, 1983, p. 14). □

3. Spectral analysis of the system

In this section, we analyse the spectrum of $\mathscr{A}$. First, we consider the eigenvalue problem. Suppose $\mathscr{A}Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (u, v, w, h_1, \ldots, h_N) \in D(\mathscr{A})$. Then

$$\begin{cases} v = \lambda u, \quad -du + cw(1) = \lambda v, \\ \left[\sum_{j=1}^{N} a_j h_j(x)\right]' = \lambda w(x), \\ a_j w'(x) - b_j h_j(x) = \lambda h_j(x), \quad j = 1, 2, \ldots, N, \\ w(0) = 0, \quad \sum_{j=1}^{N} a_j h_j(1) = -cv. \end{cases} \tag{3.1}$$

Lemma 3.1 Let $\mathscr{A}$ be defined by (2.4). Then for each $\lambda \in \sigma_p(\mathscr{A})$, we have $\mathrm{Re}\lambda < 0$.

Proof. By Theorem 2.1, since $\mathscr{A}$ is dissipative, we have for each $\lambda \in \sigma(\mathscr{A})$, $\mathrm{Re}\lambda \leqslant 0$. So we only need to show that there is no eigenvalue on the imaginary axis. Let $\lambda = \pm i\mu^2 \in \sigma_p(\mathscr{A})$ with $\mu \in \mathbb{R}^+$ and $0 \neq Z = (u, v, w, h_1, \ldots, h_N) \in D(\mathscr{A})$ be its associated eigenfunction of $\mathscr{A}$. Then by (2.12), we have

$$\mathrm{Re}\langle \mathscr{A}Z, Z\rangle = -\sum_{j=1}^{N} b_j \int_0^1 |h_j(x)|^2\,\mathrm{d}x = 0.$$

Hence $h_j(x) = 0$, $j = 1, 2, \ldots, N$. Then from (3.1), we have $w(x) = u = v = 0$. Therefore, $Z = (u, v, w, h_1, \ldots, h_N) = 0$. There is no eigenvalue on the imaginary axis. The proof is complete. □

PROPOSITION 3.1 Let $\mathscr{A}$ be defined by (2.4). Then $\lambda = -b_j \in \rho(\mathscr{A})$, $j = 1, 2, \ldots, N$, where $\rho(\mathscr{A})$ is the resolvent set of $\mathscr{A}$.

Proof. We only give the proof for $\lambda = -b_1$ because other cases can be treated similarly. Let $\lambda = -b_1$ and $Z = (u, v, w, h_1, \ldots, h_N) \in D(\mathscr{A})$, $G = (f, g, k, \varphi_1, \ldots, \varphi_N) \in \mathscr{H}$. Solve $(-b_1 I - \mathscr{A})Z = G$, that is,

$$\begin{cases} -b_1 u - v = f, \quad -b_1 v + du - cw(1) = g, \\ -b_1 w(x) - \left[\sum_{j=1}^{N} a_j h_j(x)\right]' = k(x), \\ -a_1 w'(x) = \varphi_1(x), \\ -b_1 h_j(x) - [a_j w'(x) - b_j h_j(x)] = \varphi_j(x), \quad j = 2, \ldots, N, \\ w(0) = 0, \quad \sum_{j=1}^{N} a_j h_j(1) = -cv. \end{cases} \tag{3.2}$$

From the forth equation of (3.2) and the boundary condition $w(0) = 0$, we obtain

$$w(x) = -\frac{1}{a_1}\int_0^x \varphi_1(\tau)\,\mathrm{d}\tau. \tag{3.3}$$

Substituting (3.3) into the first and second equation of (3.2), respectively, we obtain

$$u = \frac{1}{b_1^2 + d}[g + cw(1) - b_1 f] = \frac{1}{b_1^2 + d}\left[g - \frac{c}{a_1}\int_0^1 \varphi_1(\tau)\,\mathrm{d}\tau - b_1 f\right]$$

and

$$v = \frac{-b_1}{b_1^2 + d}[g + cw(1) - b_1 f] - f = \frac{-b_1}{b_1^2 + d}\left[g - \frac{c}{a_1}\int_0^1 \varphi_1(\tau)\,\mathrm{d}\tau - b_1 f\right] - f.$$

From the fifth and forth equations of (3.2), we have

$$h_j(x) = \frac{1}{b_j - b_1}[\varphi_j(x) + a_j w'(x)] = \frac{1}{b_j - b_1}\left[\varphi_j(x) - \frac{a_j}{a_1}\varphi_1(x)\right], \quad j = 2, \ldots, N.$$

Now, we calculate $h_1(x)$. Noting the last boundary condition, we integrate both sides of the third equation of (3.2) from x to 1, then

$$\begin{aligned} h_1(x) &= \frac{1}{a_1}\int_x^1 b_1 w(\tau) + k(\tau)\,\mathrm{d}\tau - \frac{1}{a_1}\sum_{j=2}^{N} a_j h_j(x) - \frac{c}{a_1}v \\ &= \frac{-b_1}{a_1^2}\int_x^1\int_0^\tau \varphi_1(s)\,\mathrm{d}s\,\mathrm{d}\tau + \frac{1}{a_1}\int_x^1 k(\tau)\,\mathrm{d}\tau - \frac{1}{a_1}\sum_{j=2}^{N}\left[\frac{a_j}{b_j-b_1}\left(\varphi_j(x) - \frac{a_j}{a_1}\varphi_1\right)\right] \\ &\quad + \frac{cb_1}{a_1(b_1^2+d)}\left[g - \frac{c}{a_1}\int_0^1 \varphi_1(\tau)d\tau - b_1 f\right] + \frac{c}{a_1}f. \end{aligned}$$

So $Z = (u, v, w, h_1, \dots, h_N)$ is uniquely determined. Hence, $(-b_1 I - \mathscr{A})^{-1}$ exists and is bounded. Similar manner can be applied to $\lambda = -b_j,\ j = 2, 3, \dots, N$. Therefore, $-b_j \in \rho(\mathscr{A}),\ j = 1, 2, \dots, N$, we complete the proof. □

When $\lambda \neq -b_j,\ j = 1, \dots, N$, it follows from (3.1) that

$$\begin{cases} v = \lambda u, \quad u = \dfrac{c}{\lambda^2 + d} w(1), \\ h_j(x) = \dfrac{a_j}{\lambda + b_j} w'(x), \quad j = 1, 2, \dots, N, \end{cases} \tag{3.4}$$

and w satisfies

$$\begin{cases} \displaystyle\sum_{j=1}^{N} \frac{a_j^2}{\lambda + b_j} w''(x) = \lambda w(x), \\ w(0) = 0, \quad \displaystyle\sum_{j=1}^{N} \frac{a_j^2}{\lambda + b_j} w'(1) = -\frac{c^2\lambda}{\lambda^2 + d} w(1). \end{cases} \tag{3.5}$$

LEMMA 3.2 Let $\mathscr{A}$ be defined by (2.4) and

$$\Delta = \left\{ \lambda \in \mathbb{C} \,\middle|\, \sum_{j=1}^{N} \frac{a_j^2}{\lambda + b_j} = 0 \right\}. \tag{3.6}$$

Then

$$\Delta \cap \sigma_p(\mathscr{A}) = \emptyset,$$

where $\sigma_p(\mathscr{A})$ stands for the point spectrum of $\mathscr{A}$.

Proof. Obviously, $0 \notin \Delta$. Suppose $\lambda_0 \in \Delta$ is an eigenvalue of $\mathscr{A}$, and $0 \neq Z = (u, v, w, h_1, \dots, h_N) \in D(\mathscr{A})$ satisfies $\mathscr{A}Z = \lambda_0 Z$. Then, (3.5) becomes

$$w(x) \equiv 0.$$

This together with (3.4) yields $u = v = 0,\ h_j(x) = 0,\ j = 1, 2, \dots, N$, which contradicts to $Z \neq 0$. The proof is complete. □

LEMMA 3.3 Let $\mathscr{A}$ be defined as in (2.4) and let $\varDelta$ be given by (3.6). Then

$$\varDelta = \{\lambda_{c,k} \in (-b_{k+1}, -b_k) \subset \mathbb{R},\ k = 1, 2, \ldots, N-1\}. \tag{3.7}$$

Proof. Since $-b_j \notin \varDelta,\ j = 1, 2, \ldots, N, p(\lambda) = 0$ is equivalent to $q(\lambda) = 0$, where

$$p(\lambda) = \sum_{j=1}^{N} \frac{a_j^2}{\lambda + b_j}, \quad q(\lambda) = p(\lambda) \prod_{j=1}^{N} (\lambda + b_j). \tag{3.8}$$

Thus, the elements of $\varDelta$ are zeros of $q(\lambda)$. However, $q(\lambda)$ is a $(N-1)$th-order polynomial, and hence there are at most $N-1$ number of zeros of $p(\lambda)$. Now we find all these zeros. Note that for any $j = 1, 2, \ldots, N-1$, when j is even, $q(-b_j) > 0$, $q(-b_{j+1}) < 0$. By the continuity of $q(\lambda)$, there exists a solution $q(\lambda) = 0$ in $(-b_{j+1}, -b_j)$. This complete the proof by (1.7). □

We need the Lemma 3.3 of Wang *et al.* (2009).

LEMMA 3.4 Suppose that $\lambda \neq 0$ and $\lambda \notin \varDelta$. Then,

$$\lambda \prod_{j=1}^{N} (\lambda + b_j) = a(\lambda^2 + b\lambda + c) q(\lambda) + h(\lambda), \tag{3.9}$$

where $q(\lambda)$ is given by (3.8)

$$a = \left(\sum_{j=1}^{N} a_j^2 \right)^{-1}, \quad b = a \sum_{j=1}^{N} a_j^2 b_j, \quad c = a \sum_{j=1}^{N} a_j^2 b_j^2 + b^2, \tag{3.10}$$

and $h(\lambda)$ is a residual polynomial in λ with order $N-2$.

By Lemmas 3.2 and 3.4, the eigenvalue problem (3.5) is equivalent to the following problem:

$$\begin{cases} w''(x) - a[\lambda^2 + b\lambda + c + a^{-1} h(\lambda) q^{-1}(\lambda)] w(x) = 0, \\ w(0) = 0, \\ (\lambda^2 + d) w'(1) = -ac^2 [\lambda^2 + b\lambda + c + a^{-1} h(\lambda) q^{-1}(\lambda)] w(1) := K(\lambda) w(1). \end{cases} \tag{3.11}$$

where a, b and c are constants given by (3.10), $q(\lambda)$ is a polynomial in λ with order $N-1$ given by (3.8) and $h(\lambda)$ is given by (3.9):

$$h(\lambda) = \lambda \prod_{j=1}^{N} (\lambda + b_j) - a(\lambda^2 + b\lambda + c) q(\lambda).$$

Therefore,

$$\begin{cases} q(\lambda) \neq 0 \quad \text{for any } \lambda \neq 0 \text{ and } \lambda \notin \varDelta, \\ h(\lambda) q^{-1}(\lambda) = \mathscr{O}(\lambda^{-1}) \quad \text{as } |\lambda| \to \infty. \end{cases} \tag{3.12}$$

LEMMA 3.5 Suppose that $\lambda \neq 0$ and $\lambda \notin \Delta$. Then, for $x \in [0, 1]$,

$$e^{\sqrt{a}\lambda x}, \quad e^{-\sqrt{a}\lambda x} \tag{3.13}$$

are linearly independent fundamental solutions of $w''(x) - a\lambda^2 w(x) = 0$, and

$$w''(x) - a[\lambda^2 + b\lambda + c + a^{-1}h(\lambda)q^{-1}(\lambda)]w(x) = 0, \quad \text{as } |\lambda| \to \infty,$$

has the following asymptotic fundamental solutions:

$$\begin{cases} w_1(x) = e^{\sqrt{a}\lambda x}[w_{10}(x) + w_{11}(x)\lambda^{-1}] + \mathscr{O}(\lambda^{-2}), \\ w_2(x) = e^{-\sqrt{a}\lambda x}[w_{20}(x) + w_{21}(x)\lambda^{-1}] + \mathscr{O}(\lambda^{-2}), \end{cases} \tag{3.14}$$

where

$$\begin{cases} w_{10}(x) = e^{(1/2)\sqrt{a}bx}, \quad w_{11}(x) = -\frac{1}{8}\sqrt{a}(b^2 - 4c)xe^{(1/2)\sqrt{a}bx}, \\ w_{20}(x) = e^{(-1/2)\sqrt{a}bx}, \quad w_{21}(x) = \frac{1}{8}\sqrt{a}(b^2 - 4c)xe^{(-1/2)\sqrt{a}bx}. \end{cases} \tag{3.15}$$

Proof. The first claim is trivial. We only need to show that (3.14) is the asymptotic fundamental solutions of $w''(x) - a[\lambda^2 + b\lambda + c + a^{-1}h(\lambda)q^{-1}(\lambda)]w(x) = 0$. This can be done along the same way of Birkhoff (1908) and Naimark (1967). Here, we present briefly a simple calculation to (3.14).

Let

$$\tilde{w}_1(x, \lambda) := e^{\sqrt{a}\lambda x}\left[w_{10}(x) + \frac{w_{11}(x)}{\lambda}\right], \quad \tilde{w}_2(x, \lambda) := e^{-\sqrt{a}\lambda x}\left[w_{20}(x) + \frac{w_{21}(x)}{\lambda}\right], \tag{3.16}$$

where $w_{i0}(x)$ and $w_{i1}(x)$ are some functions to be determined, and

$$D(w) = w''(x) - a[\lambda^2 + b\lambda + c + a^{-1}h(\lambda)q^{-1}(\lambda)]w(x), \tag{3.17}$$

where $h(\lambda)q^{-1}(\lambda) = \mathscr{O}(\lambda^{-1})$. Substitute $\bar{w}_1(x, \lambda)$ and $\bar{w}_2(x, \lambda)$ into $D(w)$, respectively, to yield

$$\begin{aligned} e^{-\sqrt{a}\lambda x}D(\tilde{w}_1(x, \lambda)) &= a\lambda^2\left[w_{10}(x) + \frac{w_{11}(x)}{\lambda}\right] + 2\sqrt{a}\lambda\left[w'_{10}(x) + \frac{w'_{11}(x)}{\lambda}\right] + \left[w''_{10}(x) + \frac{w''_{11}(x)}{\lambda}\right] \\ &\quad - a\left[w_{10}(x) + \frac{w_{11}(x)}{\lambda}\right][\lambda^2 + b\lambda + c + a^{-1}h(\lambda)q^{-1}(\lambda)] \\ &= \lambda[2\sqrt{a}w'_{10}(x) - abw_{10}(x)] \\ &\quad + [2\sqrt{a}w'_{11}(x) - abw_{11}(x) + w''_{10}(x) - acw_{10}(x)] + \lambda^{-1}F_1(x, \lambda) \end{aligned}$$

and

$$e^{\sqrt{a}\lambda x}D(\tilde{w}_2(x,\lambda)) = a\lambda^2\left[w_{20}(x)+\frac{w_{21}(x)}{\lambda}\right] - 2\sqrt{a}\lambda\left[w'_{20}(x)+\frac{w'_{21}(x)}{\lambda}\right] + \left[w''_{20}(x)+\frac{w''_{21}(x)}{\lambda}\right]$$
$$- a\left[w_{20}(x)+\frac{w_{21}(x)}{\lambda}\right][\lambda^2+b\lambda+c+a^{-1}h(\lambda)q^{-1}(\lambda)]$$
$$= -\lambda[2\sqrt{a}w'_{20}(x)+abw_{20}(x)]$$
$$- [2\sqrt{a}w'_{21}(x)+abw_{21}(x)-w''_{20}(x)+acw_{20}(x)]+\lambda^{-1}F_2(x,\lambda),$$

where

$$F_i(x,\lambda) = w''_{i1}(x) - acw_{i1}(x) - [\lambda w_{i0}(x)+w_{i1}(x)]h(\lambda)q^{-1}(\lambda), \quad i=1,2,$$

in which

$$|[\lambda w_{i0}(x)+w_{i1}(x)]h(\lambda)q^{-1}(\lambda)| \leqslant M, \quad |F_i(x,\lambda)| \leqslant M \quad \forall x\in[0,1]$$

for some positive constant M. Thus, letting the coefficients of λ^1 and λ^0 be zero gives

$$2\sqrt{a}w'_{10}(x) - abw_{10}(x) = 0, \quad 2\sqrt{a}w'_{20}(x) + abw_{20}(x) = 0,$$

and

$$2\sqrt{a}w'_{11}(x) - abw_{11}(x) + w''_{10}(x) - acw_{10}(x) = 0,$$
$$2\sqrt{a}w'_{21}(x) + abw_{21}(x) - w''_{20}(x) + acw_{20}(x) = 0.$$

Now, use the conditions $w_{i0}(0)=1$, $w_{i1}(0)=0$, $i=1,2$, to obtain

$$w_{10}(x) = e^{(1/2)\sqrt{a}bx}, \quad w_{11}(x) = -\tfrac{1}{8}\sqrt{a}(b^2-4c)xe^{(1/2)\sqrt{a}bx},$$
$$w_{20}(x) = e^{(-1/2)\sqrt{a}bx}, \quad w_{21}(x) = \tfrac{1}{8}\sqrt{a}(b^2-4c)xe^{(-1/2)\sqrt{a}bx}.$$

These are (3.15). When $|\lambda|$ is large enough, we obtain the linearly independent asymptotic fundamental solutions of $w''(x) - a[\lambda^2+b\lambda+c+a^{-1}h(\lambda)q^{-1}(\lambda)]w(x)=0$ given by (3.14) (see Birkhoff, 1908):

$$w_1(x) = e^{\sqrt{a}\lambda x}[w_{10}(x)+w_{11}(x)\lambda^{-1}] + \mathcal{O}(\lambda^{-2})$$

and

$$w_2(x) = e^{-\sqrt{a}\lambda x}[w_{20}(x)+w_{21}(x)\lambda^{-1}] + \mathcal{O}(\lambda^{-2}).$$

The proof is complete. □

Let $\lambda \neq 0$ and $\lambda \notin \Delta$, and let

$$w(x) = ew_1(x) + fw_2(x),$$

where $w_i(x)$, $i=1,2$ are given by (3.14). Then, substitute the above equality into the boundary conditions of (3.11) to obtain

$$\Delta(\lambda)[e,f]^\top = 0,$$

where

$$\Delta(\lambda) = \begin{bmatrix} 1 & 1 \\ (\lambda^2+d)w'_1(1) - K(\lambda)w_1(1) & (\lambda^2+d)w'_2(1) - K(\lambda)w_2(1) \end{bmatrix}. \tag{3.18}$$

Hence, (3.11) has non-trivial solution if and only if

$$\det(\Delta(\lambda)) = 0, \tag{3.19}$$

and the eigenvalues of (3.11) are the zeros of (3.19). Note that

$$\begin{aligned}\det(\Delta(\lambda)) &= (\lambda^2+d)[w_2'(1)-w_1'(1)] - K(\lambda)[w_2(1)-w_1(1)]\\ &= -(\lambda^2+d)\, e^{-\sqrt{a}(\lambda+b/2)}\left\{\sqrt{a}(\lambda+p)+\left[\tfrac{1}{2}\sqrt{a}b - p(1-\tfrac{1}{2}\sqrt{a}b)\lambda^{-1}\right]\right\}\\ &\quad -(\lambda^2+d)\, e^{\sqrt{a}(\lambda+b/2)}\left\{\sqrt{a}(\lambda-p)-\left[\tfrac{1}{2}\sqrt{a}b - p\left(1+\tfrac{1}{2}\sqrt{a}b\right)\lambda^{-1}\right]\right\}\\ &\quad - K(\lambda)\left[e^{-\sqrt{a}(\lambda+b/2)}(1+p\lambda^{-1}) - e^{\sqrt{a}(\lambda+b/2)}(1-p\lambda^{-1}) + \mathscr{O}(\lambda^{-2})\right]\\ &= 0,\end{aligned} \tag{3.20}$$

where

$$p := \frac{\sqrt{a}}{8}(b^2-4c).$$

When $\lambda \neq 0$, multiply $1/\lambda^3$ to both sides of (3.20), we obtain

$$-\sqrt{a}e^{-\sqrt{a}(\lambda+b/2)} - \sqrt{a}\, e^{\sqrt{a}(\lambda+b/2)} + \mathscr{O}(\lambda^{-1}) = 0. \tag{3.21}$$

Finally, since the solutions of $e^{2\sqrt{a}(\lambda+b/2)}+1=0$ are

$$\tilde{\lambda}_n = -\frac{b}{2} + \frac{\sqrt{a}}{a}\left(n-\frac{1}{2}\right)\pi \mathrm{i}, \quad n \in \mathbb{Z}.$$

Applying Rouché's theorem, we get the solutions of (3.21)

$$\lambda_n = -\frac{b}{2} + \frac{\sqrt{a}}{a}\left(n-\frac{1}{2}\right)\pi \mathrm{i} + \mathscr{O}(n^{-1}), \quad n \in \mathbb{Z}.$$

THEOREM 3.1 The eigenvalue of (3.11) have the following asymptotic expressions:

$$\lambda_n = -\frac{b}{2} + \frac{\sqrt{a}}{a}\left(n-\frac{1}{2}\right)\pi \mathrm{i} + \mathscr{O}(n^{-1}), \quad n \in \mathbb{Z}, \tag{3.22}$$

in particular,

$$\operatorname{Re}\lambda_n \to -\frac{1}{2}b = -\frac{1}{2}\frac{\sum_{i=1}^N a_i^2 b_i}{\sum_{i=1}^N a_i^2} < 0, \quad \text{as } |n| \to \infty, \tag{3.23}$$

that is, $\operatorname{Re}\lambda = -\frac{1}{2}b$ is the asymptote of the eigenvalues λ_n given by (3.22). Here, b is given by (3.10). Furthermore, the corresponding eigenfunctions $w_n(x)$, $n \in \mathbb{Z}$ have the asymptotic expressions

$$w_n(x) = \sin(n-\tfrac{1}{2})\pi x + \mathscr{O}(n^{-1}) \tag{3.24}$$

and

$$\lambda_n^{-1} w_n'(x) = -\mathrm{i}\sqrt{a}\cos(n-\tfrac{1}{2})\pi x + \mathscr{O}(n^{-1}). \tag{3.25}$$

Moreover, $\{w_n(x), n \in \mathbb{Z}\}$ and $\{\lambda_n^{-1} w_n'(x), n \in \mathbb{Z}\}$ are approximately normalized in $L^2(0,1)$, in the sense that there exist positive constants c_1 and c_2 independent of n such that for $n \in \mathbb{Z}$,

$$c_1 \leqslant \|w_n\|_{L^2}, \quad \|\lambda_n^{-1} w_n'\|_{L^2} \leqslant c_2. \tag{3.26}$$

Proof. Equation (3.22) has been proved. Only proofs for (3.24–3.26) are needed. Since $\lambda \neq 0$, $\lambda \notin \Delta$, in view of (3.18), (3.22), Lemma 3.5 and some facts in linear algebra, the eigenfunction w corresponding to the eigenvalue λ is given by

$$\begin{aligned} w(\lambda, x) &= \det \begin{bmatrix} 1 & 1 \\ w_1(x) & w_2(x) \end{bmatrix} \\ &= w_2(x) - w_1(x) = e^{(-1/2)\sqrt{a}bx} e^{-\sqrt{a}\lambda x} - e^{(1/2)\sqrt{a}bx} e^{\sqrt{a}\lambda x} + \mathscr{O}(\lambda^{-1}), \end{aligned}$$

$$\begin{aligned} w'(\lambda, x) &= \det \begin{bmatrix} 1 & 1 \\ w_1'(x) & w_2'(x) \end{bmatrix} \\ &= -\left(\frac{1}{2}\sqrt{a}b + \sqrt{a}\lambda\right)\left(e^{(-1/2)\sqrt{a}bx} e^{-\sqrt{a}\lambda x} + e^{(1/2)\sqrt{a}bx} e^{\sqrt{a}\lambda x}\right) + \mathscr{O}(\lambda^{-1}). \end{aligned}$$

Owing to the fact of (3.22) that

$$\tfrac{1}{2}\sqrt{a}b + \sqrt{a}\lambda_n = (n - \tfrac{1}{2})\pi \mathrm{i} + \mathscr{O}(n^{-1}), \quad n \in \mathbb{Z}.$$

Equations (3.24) and (3.25) are thus proved by taking

$$w_n(x) = \tfrac{1}{2}\mathrm{i} w(\lambda_n, x).$$

Finally,

$$\|w_n\|_{L^2} = \int_0^1 \sin^2\left(n - \frac{1}{2}\right)\pi x \, \mathrm{d}x + \mathscr{O}(n^{-1}) = \frac{1}{2} + \mathscr{O}(n^{-1})$$

and

$$\|\lambda_n^{-1} w_n'\|_{L^2} = \int_0^1 \left(-\mathrm{i}\sqrt{a}\cos\left(n - \frac{1}{2}\right)\pi x\right)\overline{\left(-\mathrm{i}\sqrt{a}\cos\left(n - \frac{1}{2}\right)\pi x\right)} \mathrm{d}x + \mathscr{O}(n^{-1}) = \frac{a}{2} + \mathscr{O}(n^{-1}).$$

These give (3.26). The proof is complete. □

By Lemma 3.2, the eigenvalue problem (3.5) is equivalent to the following problem:

$$\begin{cases} w''(x) = s^2(\lambda) w(x), \\ w(0) = 0, \quad p(\lambda) w'(1) = R(\lambda) w(1), \end{cases} \tag{3.27}$$

where $p(\lambda)$ is given by (3.8), $s(\lambda) = \sqrt{\lambda / p(\lambda)}$ and $R(\lambda) = -c^2 \lambda / (\lambda^2 + d)$. Hence,

$$w(x) = e^{s(\lambda)x} - e^{-s(\lambda)x}. \tag{3.28}$$

By boundary condition $p(\lambda)w'(1) = R(\lambda)w(1)$, and for any $\lambda_c \in \Delta$, when $\lambda \to \lambda_c$, $\mu = \lambda - \lambda_c \to 0$, (3.27) has non-trivial solution if and only if

$$e^{2s(\lambda)} + 1 + \frac{2R(\lambda)}{p(\lambda)s(\lambda) - R(\lambda)} = e^{2s(\mu)} - 1 + \mathscr{O}(\mu^2) = 0. \tag{3.29}$$

The above equation is achieved from the fact that, when $\lambda \to \lambda_c$,

$$\begin{aligned} p(\lambda) &= \sum_{j=1}^{N} \frac{a_j^2}{\lambda + b_j} = \sum_{j=1}^{N} \frac{a_j^2}{\lambda_c + b_j} \frac{1}{1 + (\lambda - \lambda_c)/(\lambda_c + b_j)} \\ &= -\mu \sum_{j=1}^{N} a_j^2 \left[\frac{1}{(\lambda_c + b_j)^2} - \frac{\mu}{(\lambda_c + b_j)^3} + \mathscr{O}(\mu^2) \right], \end{aligned}$$

and

$$\begin{aligned} \frac{2R(\lambda)}{p(\lambda)s(\lambda) - R(\lambda)} &= \frac{-2c^2\lambda/(\lambda^2 + d)}{p(\lambda)\sqrt{\lambda/p(\lambda)} + c^2\lambda/(\lambda^2 + d)} = \frac{-2c^2}{(\lambda^2 + d)\sqrt{p(\lambda)/\lambda} + c^2} \\ &= \frac{-2}{1 + \mathscr{O}(\mu^2)} = -2[1 - \mathscr{O}(\mu^2)]. \end{aligned}$$

From (3.29), $e^{2s(\mu)} - 1 = 0$ yield

$$s(\mu) = n\pi \mathrm{i}, \quad \text{i.e.} \quad \frac{\lambda}{p(\lambda)} = -n^2\pi^2, \quad n = 1, 2, \ldots. \tag{3.30}$$

We also have

$$\begin{aligned} \frac{\lambda}{p(\lambda)} &= \frac{\lambda_c + \mu}{p(\lambda)} = -\frac{1}{\mu} \frac{\lambda_c(1 + \mu/\lambda_c)}{\sum_{i=1}^{N}(a_i^2/(\lambda_c + b_i)^2)\left[1 - \frac{\sum_{i=1}^{N} a_i^2/(\lambda_c + b_i)^3}{\sum_{i=1}^{N} a_i^2/(\lambda_c + b_i)^2}\mu + \mathscr{O}(\mu^2)\right]} \\ &= -\frac{\lambda_c(1 + \mu/\lambda_c)}{\mu\Lambda}\left[1 + \frac{\tilde{\Lambda}}{\Lambda}\mu\right] + \mathscr{O}(\mu) = -\frac{1}{\mu}\frac{\lambda_c}{\Lambda}\left[1 + \left(\frac{1}{\lambda_c} + \frac{\tilde{\Lambda}}{\Lambda}\right)\mu\right] + \mathscr{O}(\mu), \end{aligned} \tag{3.31}$$

where

$$\Lambda = \sum_{j=1}^{N} \frac{a_j^2}{(\lambda_c + b_j)^2}, \quad \tilde{\Lambda} = \sum_{j=1}^{N} \frac{a_j^2}{(\lambda_c + b_j)^3}.$$

This together with (3.30), we have

$$-\frac{1}{\mu}\frac{\lambda_c}{\Lambda}\left[1 + \left(\frac{1}{\lambda_c} + \frac{\tilde{\Lambda}}{\Lambda}\right)\mu\right] + \mathscr{O}(\mu) = -n^2\pi^2, \quad n \to \infty.$$

Thus

$$\mu_n = \frac{1}{n^2\pi^2}\frac{\lambda_c}{\Lambda} + \mathscr{O}(n^{-3}), \quad n \to \infty,$$

or

$$\lambda_n = \lambda_c + \mu_n = \lambda_c + \frac{1}{n^2\pi^2}\frac{\lambda_c}{\Lambda} + \mathscr{O}(n^{-3}), \quad n \to \infty.$$

We summarize these results as Proposition 3.2.

PROPOSITION 3.2 Let $\mathscr{A}$ be defined by (2.4) and λ be an eigenvalue of $\mathscr{A}$, satisfying $\lambda \neq -b_j$, $j = 1, \ldots, N$. Then the eigenfunction corresponding to λ is of the form

$$\left(\frac{cw(1)}{\lambda^2 + d}, \frac{c\lambda w(1)}{\lambda^2 + d}, w(x), \frac{a_1}{\lambda + b_1}w'(x), \ldots, \frac{a_N}{\lambda + b_N}w'(x)\right),$$

where $w(x) = \sin(n - \frac{1}{2})\pi x + \mathscr{O}(n^{-1})$, for some $n \in \mathbb{N}^+$. Moreover,

(i) When $|\lambda| \to \infty$, the eigenvalues $\{\lambda_{n0}, \bar{\lambda}_{n0}\}$ of $\mathscr{A}$ have the following asymptotic expressions:

$$\lambda_{n0} = -\frac{b}{2} + \frac{\sqrt{a}}{a}\left(n - \frac{1}{2}\right)\pi i + \mathscr{O}(n^{-1}), \quad n \in \mathbb{Z}. \tag{3.32}$$

The corresponding eigenfunctions $(cw(1)/(\lambda^2 + d),\ c\lambda w(1)/(\lambda^2 + d),\ w(x), (a_1/(\lambda + b_1))w'(x), \ldots, (a_N/(\lambda + b_N))w'(x))$, satisfy

$$w_n(x) = \sin(n - \tfrac{1}{2})\pi x + \mathscr{O}(n^{-1}). \tag{3.33}$$

(ii) For any $1 \leqslant k \leqslant N - 1$, there is a sequence of eigenvalues $\{\lambda_{n,k}\}$ of $\mathscr{A}$, which have the following asymptotic expressions:

$$\lambda_{n,k} = \lambda_{c,k} + \frac{1}{n^2\pi^2}\frac{\lambda_{c,k}}{\Lambda_k} + \mathscr{O}(n^{-3}), \quad n \to \infty, \tag{3.34}$$

where

$$\Lambda_k = \sum_{j=1}^{N} \frac{a_j^2}{(\lambda_{c,k} + b_j)^2}.$$

Furthermore, the corresponding eigenfunctions $(cw(1)/(\lambda^2 + d), c\lambda w(1)/(\lambda^2 + d), w(x), (a_1/(\lambda + b_1))w'(x), \ldots, (a_N/(\lambda + b_N))w'(x))$, satisfy (3.33).

The following result is a direct consequence of the above analysis.

THEOREM 3.2 Let $\mathscr{A}$ be defined as in (2.4). Then

(i) $\mathscr{A}$ has the eigenvalues

$$\{\lambda_{n0}, \bar{\lambda}_{n0}, \lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N-1},\ n \in \mathbb{N}^+\},$$

where λ_{n0} and $\lambda_{n,k}$, $k = 1, 2, \ldots, N - 1$ have the asymptotic expressions (3.32) and (3.34), respectively.

(ii) When $|\lambda| \to \infty$, the eigenfunctions corresponding to λ_{n0} and $\bar{\lambda}_{n0}$ are given by

$$W_{n0}(x) = \Big(0, 0, \sin\big(n - \tfrac{1}{2}\big)\pi x,\ -\mathrm{i}\sqrt{a}a_1 \cos\big(n - \tfrac{1}{2}\big)\pi x, \ldots, -\mathrm{i}\sqrt{a}a_N \cos\big(n - \tfrac{1}{2}\big)\pi x\Big) + \Big(\mathscr{O}(n^{-2}), \mathscr{O}(n^{-1}), \mathscr{O}(n^{-1}), \mathscr{O}(n^{-1}), \ldots, \mathscr{O}(n^{-1})\Big)$$

and

$$\bar{W}_{n0}(x) = \Big(0, 0, \sin\big(n - \tfrac{1}{2}\big)\pi x,\ i\sqrt{a}a_1 \cos\big(n - \tfrac{1}{2}\big)\pi x, \ldots, \mathrm{i}\sqrt{a}a_N \cos\big(n - \tfrac{1}{2}\big)\pi x\Big) + \Big(\mathscr{O}(n^{-2}), \mathscr{O}(n^{-1}), \mathscr{O}(n^{-1}), \mathscr{O}(n^{-1}), \ldots, \mathscr{O}(n^{-1})\Big), \tag{3.35}$$

for $n \to \infty$.

(iii) When $\lambda_{n,k} \to \lambda_{c,k}, n \to \infty$, the eigenfunctions corresponding to $\lambda_{n,k}$ are given by

$$W_{n,k}(x) = \left(0, 0, 0, \frac{a_1}{\lambda_{n,k} + b_1} \cos\left(n - \frac{1}{2}\right)\pi x, \ldots, \frac{a_N}{\lambda_{n,k} + b_N} \cos\left(n - \frac{1}{2}\right)\pi x\right) + \mathscr{O}(n^{-1}) \tag{3.36}$$

for $k = 1, 2, \ldots, N - 1$.

In order to investigate the residual and continuous spectrum of $\mathscr{A}$, we need the adjoint operator $\mathscr{A}^*$.

LEMMA 3.6 Let $\mathscr{A}$ be defined by (2.4). Then

$$\mathscr{A}^* \begin{pmatrix} u \\ v \\ w \\ h_1 \\ \vdots \\ h_N \end{pmatrix}^{\top} = -\begin{pmatrix} v \\ -du + cw(1) \\ \left(\sum_{j=1}^{N} a_j h_j\right)' \\ a_1 w' + b_1 h_1 \\ \vdots \\ a_N w' + b_N h_N \end{pmatrix}^{\top}, \tag{3.37}$$

with

$$D(\mathscr{A}^*) = \left\{ \begin{pmatrix} u \\ v \\ w \\ h_1 \\ \vdots \\ h_N \end{pmatrix}^{\top} \middle| \begin{array}{c} w(0) = 0, \\ \sum_{j=1}^{N} a_i h_i(1) = -cv, \\ \sum_{j=1}^{N} a_j h_j(x) \in H^1(0,1), \\ h_i \in L^2(0,1), \quad i = 1, \ldots, N. \end{array} \right\}. \tag{3.38}$$

THEOREM 3.3 Let $\mathscr{A}$ be defined by (2.4). Then $\sigma_r(\mathscr{A}) = \emptyset$, where $\sigma_r(\mathscr{A})$ denotes the set of residual spectrum of $\mathscr{A}$.

Proof. Since $\lambda \in \sigma_r(\mathscr{A})$, $\bar{\lambda} \in \sigma_p(\mathscr{A}^*)$, the proof will be accomplished if we can show that $\sigma_p(\mathscr{A}) = \sigma_p(\mathscr{A}^*)$. This is because obviously the eigenvalues of $\mathscr{A}$ are symmetric on the real axis. From (3.37), the eigenvalue problem $\mathscr{A}^*Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (u, v, w, h_1, \ldots, h_N) \in D(\mathscr{A}^*)$ we have

$$\begin{cases} -v = \lambda u, \quad du - cw(1) = \lambda v, \\ -\left(\sum_{i=1}^{N} a_j h_j\right)' = \lambda w, \\ -[a_j w'(x) + b_j h_j(x)] = \lambda h_j(x), \quad j = 1, \ldots, N, \\ w(0) = 0, \quad \sum_{i=1}^{N} a_j h_j(1) = -cv. \end{cases} \tag{3.39}$$

Equation (3.39) is the same as (3.1) by setting $\tilde{h}_j(x) = -h_j(x), j = 1, 2, \ldots, N$ and $\tilde{v} = -v$. Hence, $\mathscr{A}^*$ has the same eigenvalues with $\mathscr{A}$. The proof is complete. □

THEOREM 3.4 Let $\mathscr{A}$ be defined as in (2.4) and let Δ be given by (3.6). Then

$$\sigma_c(\mathscr{A}) = \Delta, \tag{3.40}$$

where $\sigma_c(\mathscr{A})$ is the set of the continuous spectrum of $\mathscr{A}$.

Proof. Let $\lambda \notin \sigma_p(\mathscr{A})$ and $\lambda \neq -b_j$. For any $G = (f, g, k, \varphi_1, \ldots \varphi_n) \in \mathscr{H}$, solve $(\lambda I - \mathscr{A})[u, v, w, h_1, \ldots, h_N] = G$, that is

$$\begin{cases} \lambda u - v = f, \quad \lambda v + du - cw(1) = g, \\ \lambda w(x) - \left[\sum_{j=1}^{N} a_j h_j(x)\right]' = k(x), \\ \lambda h_j(x) - [a_j w'(x) - b_j h_j(x)] = \varphi_j(x), \quad j = 1, 2, \ldots, N, \\ w(0) = 0, \quad \sum_{j=1}^{N} a_j h_j(1) = -cv. \end{cases} \tag{3.41}$$

Step I: We first show $\Delta \subseteq \sigma_c(\mathscr{A})$. From (3.41), we obtain

$$h_j = \frac{a_j}{\lambda + b_j} w'(x) + \frac{1}{\lambda + b_j}\varphi_j(x), \quad \sum_{j=1}^{N} a_j h_j(x) = \sum_{j=1}^{N} \frac{a_j^2}{\lambda + b_j} w'(x) + \sum_{j=1}^{N} \frac{a_j}{\lambda + b_j}\varphi_j(x). \tag{3.42}$$

Let

$$H(x) = \sum_{j=1}^{N} a_j h_j(x), \quad \Phi(\lambda, x) = \sum_{j=1}^{N} \frac{a_j}{\lambda + b_j}\varphi_j(x). \tag{3.43}$$

We claim that $\varDelta \subseteq \sigma_c(\mathscr{A})$. In fact, when $\lambda \in \varDelta$, it has

$$\sum_{j=1}^{N} \frac{a_j^2}{\lambda + b_j} = 0.$$

By (3.42) and (3.43), we have

$$H(x) = \Phi(\lambda, x).$$

Since $H(x) \in H^1(0,1)$, the above identity holds true unless $\Phi(\lambda, x) \in H^1(0,1)$. This shows that $\lambda \notin \rho(\mathscr{A})$, or

$$\varDelta \subseteq \sigma_c(\mathscr{A}),$$

by Lemma 3.2 and Theorem 3.3.

Step II: Now we show that $\sigma_c(\mathscr{A}) \subseteq \varDelta$, or equivalently, any $\lambda \notin \sigma_p(\mathscr{A}) \cup \varDelta$ belongs to $\rho(\mathscr{A})$. To do this, suppose that $\lambda \notin \sigma_p(\mathscr{A}) \cup \varDelta$. We write (3.41) as

$$\begin{cases} H'(x) = \lambda w(x) - k(x), \\ w'(x) = \mu(\lambda) H(x) - \mu(\lambda) \Phi(\lambda, x), \\ w(0) = 0, \quad (\lambda^2 + d) H(1) + c^2 \lambda w(1) = cdf - c\lambda g, \end{cases} \tag{3.44}$$

where

$$\mu(\lambda) = \left(\sum_{j=1}^{N} \frac{a_j^2}{\lambda + b_j} \right)^{-1} \quad \text{for any } \lambda \notin \sigma_p(\mathscr{A}) \cup \varDelta.$$

Hence, the first two equations (3.44) can be rewritten as the following system of the first-order differential equations:

$$\begin{cases} \begin{bmatrix} H \\ w \end{bmatrix}' = A(\lambda) \begin{bmatrix} H \\ w \end{bmatrix} - \begin{bmatrix} k \\ \mu(\lambda)\Phi \end{bmatrix}, \\ w(0) = 0, \quad (\lambda^2 + d) H(1) + c^2 \lambda w(1) = cdf - c\lambda g, \end{cases} \tag{3.45}$$

where

$$A(\lambda) = \begin{bmatrix} 0 & \lambda \\ \mu(\lambda) & 0 \end{bmatrix}.$$

Note that

$$e^{A(\lambda)x} = \begin{bmatrix} a_{11}(\lambda, x) & a_{12}(\lambda, x) \\ a_{21}(\lambda, x) & a_{22}(\lambda, x) \end{bmatrix},$$

where

$$\begin{cases} a_{11}(\lambda, x) = \cosh\left(\sqrt{\lambda \mu(\lambda)}\, x\right), \quad a_{12}(\lambda, x) = \dfrac{\sqrt{\lambda}}{\sqrt{\mu(\lambda)}} \sinh\left(\sqrt{\lambda \mu(\lambda)}\, x\right), \\ a_{21}(\lambda, x) = \dfrac{\sqrt{\mu(\lambda)}}{\sqrt{\lambda}} \sinh\left(\sqrt{\lambda \mu(\lambda)}\, x\right), \quad a_{22}(\lambda, x) = \cosh\left(\sqrt{\lambda \mu(\lambda)}\, x\right). \end{cases}$$

By $w(0) = 0$, the general solution of (3.45) is given by

$$\begin{aligned}\begin{bmatrix} H(x) \\ w(x) \end{bmatrix} &= e^{A(\lambda)x} \begin{bmatrix} H(0) \\ w(0) \end{bmatrix} - \int_0^x e^{A(\lambda)(x-\tau)} \begin{bmatrix} k(\tau) \\ \mu(\lambda)\Phi(\lambda,\tau) \end{bmatrix} \mathrm{d}\tau \\ &= \begin{bmatrix} a_{11}(\lambda,x)H(0) \\ a_{21}(\lambda,x)H(0) \end{bmatrix} \\ &\quad - \int_0^x \begin{bmatrix} a_{11}(\lambda,x-\tau)k(\tau) + a_{12}(\lambda,x-\tau)\mu(\lambda)\Phi(\lambda,\tau) \\ a_{21}(\lambda,x-\tau)k(\tau) + a_{22}(\lambda,x-\tau)\mu(\lambda)\Phi(\lambda,\tau) \end{bmatrix} \mathrm{d}\tau,\end{aligned}$$

that is

$$\begin{cases} H(x) = a_{11}(\lambda,x)H(0) - \xi_1(\lambda,x), \\ w(x) = a_{21}(\lambda,x)H(0) - \xi_2(\lambda,x), \end{cases} \tag{3.46}$$

where

$$\xi_j(\lambda,x) = \int_0^x \left[a_{j1}(\lambda,x-\tau)k(\tau) + a_{j2}(\lambda,x-\tau)\mu(\lambda)\Phi(\lambda,\tau)\right] \mathrm{d}\tau, \quad j = 1,2.$$

Put the boundary condition of $(\lambda^2 + d)H(1) + c^2\lambda w(1) = cdf - c\lambda g$ into (3.46), we have

$$(\lambda^2 + d)[a_{11}(\lambda,1)H(0) - \xi_1(\lambda,1)] + c^2\lambda[a_{21}(\lambda,1)H(0) - \xi_2(\lambda,1)] = cdf - c\lambda g,$$

then

$$[(\lambda^2 + d)a_{11}(\lambda,1) + c^2\lambda a_{21}(\lambda,1)]H(0) = (\lambda^2 + d)\xi_1(\lambda,1) + c^2\lambda\xi_2(\lambda,1) + cdf - c\lambda g.$$

The coefficient of $H(0)$ equals

$$\begin{aligned}&(\lambda^2 + d)a_{11}(\lambda,1) + c^2\lambda a_{21}(\lambda,1) \\ &\quad = \frac{1}{\sqrt{\lambda\mu(\lambda)}}[(\lambda^2 + d)\sqrt{\lambda\mu(\lambda)}\cosh(\sqrt{\lambda\mu(\lambda)}) + c^2\lambda\mu(\lambda)\sinh(\sqrt{\lambda\mu(\lambda)})] \neq 0,\end{aligned}$$

since $\lambda \notin \sigma_p(\mathscr{A})$ and by simple calculation of the eigenvalue problem (3.5). So w is uniquely determined by (3.46) and $w' \in L^2(0,1)$. Once $w'(x)$ is known, the $h_j(x)$, u and v are also uniquely determined by (3.42) and (3.41). Hence, $(\lambda I - \mathscr{A})^{-1}$ exists and is bounded. Therefore, $\lambda \in \rho(\mathscr{A})$. The proof is complete. □

4. Riesz basis property and exponential stability

In this section, let us first recall some notation. For a closed operator A, in a Hilbert space H, a non-zero element $\phi \in H$ is called a generalized eigenvector of A, corresponding to an eigenvalue λ of A, if there is an integer $\nu \geqslant 1$ such that $(\lambda I - A)^\nu \phi = 0$. If $\nu = 1$, then ϕ is an eigenvector. A sequence $\{\phi_n\}_{n=1}^\infty$ in H is called a Riesz basis for H if there exists an orthonormal basis $\{e_n\}_{n=1}^\infty$ in H and a linear bounded invertible operator T such that

$$T\phi_n = e_n, \quad n = 1, 2, \ldots .$$

Now, we study the Riesz basis property for system (2.5). To do this, we need the following result mentioned in Guo & Zhang (2012) (see also Guo & Zwart, 2001).

THEOREM 4.1 Let A be a densely closed linear operator in a Hilbert space H with isolated eigenvalues $\{\lambda_i\}_1^\infty$ and $\sigma_r(A)=\emptyset$. Let $\{\phi_n\}_1^\infty$ be a Riesz basis for H. Suppose that there are $N_0 \geqslant 1$ and a sequence of generalized eigenvectors $\{\psi_n\}_{N_0}^\infty$ of A such that

$$\sum_{n=N_0}^{\infty} \|\psi_n - \phi_n\|_H^2 < \infty. \tag{4.1}$$

Then there exist $M\,(\geqslant N_0)$ number of generalized eigenvectors $\{\psi_{n0}\}_1^M$ such that $\{\psi_{n0}\}_1^M \bigcup \{\psi_n\}_{M+1}^\infty$ forms a Riesz basis for H.

THEOREM 4.2 Let $\mathscr{A}$ be defined by (2.4). Then

(i) There is a sequence of generalized eigenfunctions of $\mathscr{A}$, which forms a Riesz basis for the state space $\mathscr{H}$.

(ii) All eigenvalues with large modulus are algebraically simple.

Therefore, for the semigroup $e^{\mathscr{A}t}$, the spectrum-determined growth condition holds true: $s(\mathscr{A})=\omega(\mathscr{A})$, where $s(\mathscr{A})=\sup\{\mathrm{Re}\lambda \mid \lambda\in\sigma(\mathscr{A})\}$ is the spectral bound of $\mathscr{A}$ and $\omega(\mathscr{A})=\lim_{t\to\infty}(1/t)\ln\|e^{\mathscr{A}t}\|$ is the growth order of $e^{\mathscr{A}t}$.

Proof. For any $n\in\mathbb{N}^+$, set

$$\begin{cases} e_1=(1,0,0,\dots,0), \quad e_2=(0,1,0,\dots,0),\\ \psi_{n0}(x)=\left(0,0,\sin\left(n-\dfrac{1}{2}\right)\pi x,\ -\mathrm{i}\sqrt{a}a_1\cos\left(n-\dfrac{1}{2}\right)\pi x,\dots,-\mathrm{i}\sqrt{a}a_N\cos\left(n-\dfrac{1}{2}\right)\pi x\right),\\ \bar{\psi}_{n0}(x)=\left(0,0,\sin\left(n-\dfrac{1}{2}\right)\pi x,\ \mathrm{i}\sqrt{a}a_1\cos\left(n-\dfrac{1}{2}\right)\pi x,\dots,\mathrm{i}\sqrt{a}a_N\cos\left(n-\dfrac{1}{2}\right)\pi x\right),\\ \psi_{n,k}(x)=\left(0,0,0,\dfrac{a_1}{\lambda_{n,k}+b_1},\dots,\dfrac{a_N}{\lambda_{n,k}+b_N}\right)\cos\left(n-\dfrac{1}{2}\right)\pi x, \quad k=1,2,\dots,N-1. \end{cases} \tag{4.2}$$

We need to prove

$$\{e_1,e_2,\psi_{n0},\bar{\psi}_{n0},\psi_{n,1},\psi_{n,2},\dots,\psi_{n,N-1}\}_1^\infty \tag{4.3}$$

form a Riesz basis in $\mathscr{H}$. Since we know that

$$\begin{cases} e_1=(1,0,0,\dots,0), \quad e_2=(0,1,0,\dots,0),\\ \phi_{n0}(x)=\left(0,0,\sqrt{2}\sin\left(n-\frac{1}{2}\right)\pi x,0,\dots,0\right),\\ \phi_{n,1}(x)=\left(0,0,0,\sqrt{2}\cos\left(n-\frac{1}{2}\right)\pi x,0,\dots,0\right),\\ \phi_{n,2}(x)=\left(0,0,0,0,\ \sqrt{2}\cos\left(n-\frac{1}{2}\right)\pi x,\dots,0\right),\\ \qquad\vdots\\ \phi_{n,N}(x)=0,0,0,0,\dots,0,\sqrt{2}\cos\left(n-\frac{1}{2}\right)\pi x), \end{cases} \tag{4.4}$$

forms an orthonormal basis in $\mathscr{H}$. Then there exist an invertible matrix T_n, such that

$$\{e_1, e_2, \psi_{n0}, \bar{\psi}_{n0}, \psi_{n,1}, \psi_{n,2}, \dots, \psi_{n,N-1}\}_1^\infty T_n = \{e_1, e_2, \phi_{n0}, \phi_{n,1}, \phi_{n,2}, \dots, \phi_{n,N}\}_1^\infty,$$

where

$$T_n = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mathrm{i}\sqrt{a}a_1 & -\mathrm{i}\sqrt{a}a_1 & \dfrac{a_1}{\lambda_{n,1}+b_1} & \dfrac{a_1}{\lambda_{n,2}+b_1} & \cdots & \dfrac{a_1}{\lambda_{n,N-1}+b_1} \\ 0 & 0 & \mathrm{i}\sqrt{a}a_2 & -\mathrm{i}\sqrt{a}a_2 & \dfrac{a_2}{\lambda_{n,1}+b_2} & \dfrac{a_2}{\lambda_{n,2}+b_2} & \cdots & \dfrac{a_2}{\lambda_{n,N-1}+b_2} \\ \cdots & \cdots & \cdots & \cdots & & & & \\ 0 & 0 & \mathrm{i}\sqrt{a}a_N & -\mathrm{i}\sqrt{a}a_N & \dfrac{a_N}{\lambda_{n,1}+b_N} & \dfrac{a_N}{\lambda_{n,2}+b_N} & \cdots & \dfrac{a_N}{\lambda_{n,N-1}+b_N} \end{bmatrix}.$$

Since $b_j \neq b_k$, $\lambda_{n,j} \neq \lambda_{n,k}$, $1 \leqslant j < k \leqslant N$, a direct computation shows

$$\det T_n = -4\mathrm{i}\sqrt{a} \left(\prod_{i=1}^N a_i \right) \frac{\prod_{1 \leqslant i < j \leqslant N} (b_i - b_j) \prod_{1 \leqslant i < j \leqslant N} (\lambda_{n,i} - \lambda_{n,j})}{\prod_{i=1}^{N-1} \left[\prod_{j=1}^N (\lambda_{n,i} + b_j) \right]} \neq 0,$$

where a is given by (3.10). Then the two branches of vectors $\{e_1, e_2, \psi_{n0}, \bar{\psi}_{n0}, \psi_{n,1}, \psi_{n,2}, \dots, \psi_{n,N-1}\}_1^\infty$ and $\{e_1, e_2, \phi_{n0}, \phi_{n,1}, \phi_{n,2}, \dots, \phi_{n,N}\}_1^\infty$ are equivalent. Hence,

$$\{e_1, e_2, \psi_{n0}, \bar{\psi}_{n0}, \psi_{n,1}, \psi_{n,2}, \dots, \psi_{n,N-1}\}_1^\infty$$

forms a Riesz basis for $\mathscr{H}$. By (4.2) and Theorem 3.2, there exists an $N_0 \in \mathbb{N}^+$, such that

$$\sum_{n=N_0}^\infty \left(\|W_{n0} - \psi_{n0}\|_{\mathscr{H}}^2 + \|\bar{W}_{n0} - \bar{\psi}_{n0}\|_{\mathscr{H}}^2 + \sum_{k=1}^{N-1} \|W_{n,k} - \psi_{n,k}\|_{\mathscr{H}}^2 \right) = \mathscr{O}(n^{-2}) < \infty. \tag{4.5}$$

So by Theorem 4.1, we conclude that the generalized eigenfunctions of $\mathscr{A}$ form a Riesz basis in $\mathscr{H}$, then (i) and hence (ii) hold true. The proof is complete. □

Now we establish the exponential stability of the system (2.4).

THEOREM 4.3 Let $\mathscr{A}$ be defined by (2.4). Then the spectrum-determined growth condition $\omega(\mathscr{A}) = s(\mathscr{A})$ holds true for the C_0-semigroup $e^{\mathscr{A}t}$ generated by $\mathscr{A}$. Moreover, the system (2.4) is exponentially stable, i.e., there exist two positive constants M and ω such that the C_0-semigroup $e^{\mathscr{A}t}$ satisfies

$$\|e^{\mathscr{A}t}\| \leqslant Me^{-\omega t}, \tag{4.6}$$

for some $M, \omega > 0$.

Proof. The spectrum-determined growth condition follows from Theorem 4.2. By Lemmas 3.1 and 3.3, Theorems 3.3 and 3.4 for each $\lambda \in \sigma(\mathscr{A})$, we have $\operatorname{Re}\lambda < 0$. This, together with (3.22) and the spectrum-determined growth condition, shows that $e^{\mathscr{A}t}$ is exponentially stable. □

5. Numerical applications

In this section, numerical simulations are carried out for the system (2.4) with Matlab software. By using the finite difference method, we can obtain the approximate solution of the system. In Fig. 2, we show the simulation results for (2.4) with $c = 1$, $d = 1$, and with the initial conditions $y_0 = -2$, $y_1 = 0$. Figure 3 presents the stability convergence of the system with different coefficients and initial conditions, where $c = -1.5$, $d = 8$, and $y_0 = 2$, $y_1 = 4$.

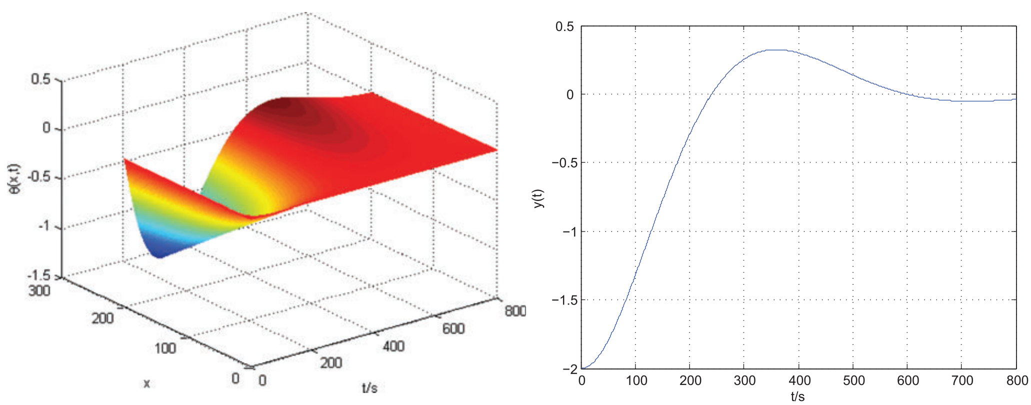

FIG. 2. The solution of $\theta(x,t)$ and $y(t)$ when $c = 1, d = 1, y_0 = -2, y_1 = 0$.

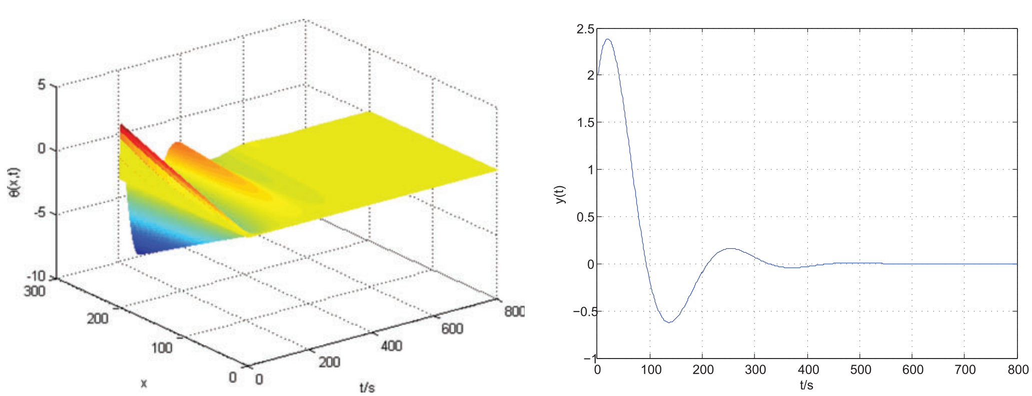

FIG. 3. The solution of $\theta(x,t)$ and $y(t)$ when $c = -1.5, d = 8, y_0 = 2, y_1 = 4$.

6. Concluding remarks

In this paper, we show that the pendulum system is stabilized by compensating with a memory type heat equation. This kind of control design is quite different from the previous PMD controller and the latest control design based on a backstepping method. First, compared with system (1.3) with condition (1.4), the system (2.4) (or (1.6)) only requires $d > 0$, $c \neq 0$, and under different initial conditions the system (2.4) is always exponentially stable. It largely relaxes the restrictions on the parameters d, c. Secondly, by the Riesz basis approach, not using the traditional Lyapunov function, we show that there is a sequence of generalized eigenfunctions which forms a Riesz basis for the state space of the closed-loop system, and then the spectrum-determined growth condition and the exponential stability are established.

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