Asymptotic behavior on a kind of parabolic Monge–Ampère equation

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Abstract

In this paper, we apply level set and nonlinear perturbation methods to obtain the asymptotic behavior of the solution to a kind of parabolic Monge–Ampère equation at infinity. The Jörgens–Calabi–Pogorelov theorem for parabolic and elliptic Monge–Ampère equation can be regarded as special cases of our result. © 2015 Elsevier Inc. All rights reserved.

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1. Introduction and main results

In affine geometry, a well known theorem of Jörgens \((n = 2 \ [11])\), Calabi \((n \leq 5 \ [4])\) and Pogorelov \((n \geq 2 \ [14])\) asserts that a convex improper affine hypersurface is an elliptic paraboloid. This theorem can also be stated as follows: any classical convex solution of the elliptic Monge–Ampère equation

\[
\det(D^2 u) = 1 \quad \text{in } \mathbb{R}^n
\]

must be a quadratic polynomial.

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Along the lines of affine geometry, a simpler and more analytic proof was given by Cheng and Yau in [6]. Jost and Xin also give another proof of this result in [12]. Caffarelli proved that Jörgens–Calabi–Pogorelov theorem remains valid for viscosity solution in [2].

In [8], Gutiérrez and Huang established Jörgens–Calabi–Pogorelov theorem to the following kind of parabolic Monge–Ampère equation

$$-u_t \det(D^2u) = 1 \quad \text{in } \mathbb{R}^{n+1}_- := \mathbb{R}^n \times (-\infty, 0], \quad -M_1 \leq u_t \leq -M_2,$$

where \( M_1 \) and \( M_2 \) are two positive constants. In [17], Xiong and Bao extended Jörgens–Calabi–Pogorelov theorem to more general parabolic Monge–Ampère equations of the form

$$u_t = \rho \left( \log(\det(D^2u)) \right) \quad \text{in } \mathbb{R}^{n+1}_-,$$

which covers the results in [8].

In [3], Caffarelli and Li obtained the asymptotic behavior of convex viscosity solutions of

$$\det(D^2u) = f(x) \quad \text{in } \mathbb{R}^n,$$

where \( f \in C^0(\mathbb{R}^n) \) satisfies

$$0 \leq \inf_{\mathbb{R}^n} f \leq \sup_{\mathbb{R}^n} f < \infty,$$

support \((f - 1)\) is bounded.

Recently, Zhang, Wang and Bao [18] have extended the above result to the following parabolic Monge–Ampère equation

$$-u_t \det(D^2u) = f(x, t) \quad \text{in } \mathbb{R}^{n+1}_-.$$

In this paper, we investigate classical solutions to the parabolic Monge–Ampère equation

$$u_t - \log(\det(D^2u)) = f(x, t) \quad \text{in } \mathbb{R}^{n+1}_-, \quad (1)$$

such that there exist two constants \( c_0 \) and \( C_0 \) with

$$c_0 \leq u_t \leq C_0 \quad \text{in } \mathbb{R}^{n+1}_-, \quad (2)$$

where \( f \in C^0(\mathbb{R}^{n+1}_-) \) and there exists \( a \in \mathbb{R} \) such that

support \((f - a)\) is bounded. \( \quad (3) \)

In the following theorem, we obtain the asymptotic behavior of solutions of (1) and (2) under the hypothesis (3).
Theorem 1.1. Let $n \geq 4$ and $f \in C^0(\mathbb{R}_+^{n+1})$ satisfy (3). Assume that $u \in C^{2,1}(\mathbb{R}_+^{n+1})$ is a solution of (1) and (2) which is convex in $x$. Then there exist an $n \times n$ symmetric positive definite matrix $A$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^1$ satisfying

$$u(x,t) = \tau t + \frac{x^T Ax}{2} + b^T x + c, \quad x \in \mathbb{R}^n, t \leq t^*$$ (4)

$$u(x,t) = \tau t + \frac{x^T Ax}{2} + b^T x + c + O(|x|^{2-n}), \quad |x| \to \infty, t^* \leq t \leq 0,$$ (5)

where $\tau := \log \det A + a$ and $t^* := \sup \{t \leq 0 : f(x,s) = a, \forall x \in \mathbb{R}^n, s \leq t\} > -\infty$.

By taking $a = 0$ and $f \equiv 0$ in Theorem 1.1, we can obtain the following theorem.

Corollary 1.2. Let $n \geq 4$ and $u \in C^{2,1}(\mathbb{R}_+^{n+1})$ be a solution to

$$u_t = \log \left( \det (D^2 u) \right) \quad (x, t) \in \mathbb{R}_+^{n+1},$$ (6)

such that there exist two constants $c_0$ and $C_0$ with

$$c_0 \leq u_t \leq C_0 \quad (x, t) \in \mathbb{R}_+^{n+1},$$ (7)

under the assumption that $u$ is convex in $x$. Then there exist an $n \times n$ symmetric positive definite matrix $A$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^1$ such that

$$u(x,t) = \tau t + \frac{x^T Ax}{2} + b^T x + c, \quad (x, t) \in \mathbb{R}_+^{n+1},$$

where $\tau := \log \det A$.

Remark 1.1. It should be remarked that Xiong and Bao have obtained Corollary 1.2 for $n \geq 2$ and $u \in C^{4,2}(\mathbb{R}_+^{n+1})$ in [17].

Now we consider an elliptic Monge–Ampère equation

$$\det (D^2 u) = \exp \left\{ \frac{1}{2} Du \cdot x - u \right\} \quad \text{in} \ \mathbb{R}^n.$$ (8)

Since an entire solution to (8) is a self-shrinking solution to Lagrangian mean curvature flow in pseudo-Euclidean space, we can obtain an analogous result with the Jörgens–Calabi–Pogorelov theorem on (8) below.

Corollary 1.3. Let $n \geq 4$ and $u \in C^2(\mathbb{R}^n)$ be a convex solution to (8) satisfying

$$\limsup_{|x| \to +\infty} \frac{1}{2} Du(x) \cdot x - u(x) < \infty.$$ (9)
Then there exist an $n \times n$ symmetric positive definite matrix $A$ and $c \in \mathbb{R}^1$ such that

$$u(x) = \frac{x^T Ax}{2} + c, \quad x \in \mathbb{R}^n.$$  

**Remark 1.2.** By replacing the condition (9) with

$$\lim\inf_{|x| \to +\infty} |x|^2 D^2 u(x) > 2(n - 1)I,$$

Huang and Wang obtained Corollary 1.3 for $n \geq 2$ in [10]. Independently, Chau, Chen and Yuan obtained the same result by using a different method in [5].

**Proof of Corollary 1.3.** First of all, we define

$$v(x,t) = (1-t)u\left(\frac{x}{\sqrt{1-t}}\right) \quad \text{in} \quad \mathbb{R}^{n+1}.$$  

Then we have that $v$ is a solution to

$$v_t = \log\left(\det(D^2v)\right) \quad \text{in} \quad \mathbb{R}^{n+1}.$$  

Due to (9), we can also deduce that $|v_t|$ is bounded in $\mathbb{R}^{n+1}_+$. By Corollary 1.2, there exist an $n \times n$ symmetric positive definite matrix $A$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^1$ and $\tau \in \mathbb{R}^1$ satisfying

$$\tau = \log\det A,$$

such that

$$v(x,t) = \tau t + \frac{x^T Ax}{2} + b^T x + c \quad \text{in} \quad \mathbb{R}^{n+1}.$$  

Lastly, by taking $t = 0$, we have

$$u(x) = \frac{x^T Ax}{2} + b^T x + c \quad \text{in} \quad \mathbb{R}^n.$$  

Since $u$ is the solution to (8), it is easy to check that $b = 0$.  

Our paper is organized as follows. In Section 2, we will state some preliminaries and simplifications. In Section 3, the proof of Theorem 1.1 is given by being divided into five steps.
2. Preliminaries and simplifications

We begin this section by introducing some notations. We denote by $D^2u(x, t)$ the matrix of second derivatives of $u$ with respect to $x$ and $Du(x, t)$ the gradient of $u$ with respect to $x$.

Let $u \in C^{2, 1}(\mathbb{R}^{n+1})$ be a solution of (1) and (2) which is convex in $x$. Without loss of generality, we can assume that $C_0 < 0$. Indeed, by defining

$$
\tilde{u}(x, t) = u(x, t) - (1 + C_0)t \quad \text{in } \mathbb{R}^{n+1},
$$

we can see that $\tilde{u} \in C^{2, 1}(\mathbb{R}^{n+1})$ is a solution to the equation

$$
\tilde{u}_t - \log \left( \det(D^2\tilde{u}) \right) = \tilde{f}(x, t) \quad \text{in } \mathbb{R}^{n+1},
$$

such that there exist two positive constants $\tilde{c}_0 = c_0 - C_0 - 1$ and $\tilde{C}_0 = -1$ with

$$
\tilde{c}_0 \leq \tilde{u}_t \leq \tilde{C}_0 \quad \text{in } \mathbb{R}^{n+1},
$$

where $\tilde{f} = f - (C_0 + 1), \tilde{a} = a - (C_0 + 1), \tilde{t}^* = t^*$ and

\[ \text{support } (\tilde{f} - \tilde{a}) \text{ is bounded.} \]

Once we have proved that for such $\tilde{u}$, there exist an $n \times n$ symmetric positive definite matrix $A$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^1$ and $\tilde{\tau} \in \mathbb{R}^1$ satisfying

$$
\tilde{\tau} - \log \det A = \tilde{a}
$$

such that

$$
\tilde{u}(x, t) = \tilde{\tau}t + \frac{x^T A x}{2} + b^T x + c, \quad x \in \mathbb{R}^n, t \leq \tilde{t}^*,
$$

$$
\tilde{u}(x, t) = \tilde{\tau}t + \frac{x^T A x}{2} + b^T x + c + O(|x|^{2-n}), \quad |x| \to \infty, \tilde{t}^* \leq t \leq 0,
$$

then by denoting $\tau = 1 + C_0 + \tilde{\tau}$ and $t^* = 1 + C_0 + \tilde{t}^*$, we have

$$
u(x, t) = \tau t + \frac{x^T A x}{2} + b^T x + c, \quad x \in \mathbb{R}^n, t \leq t^*,
$$

$$
u(x, t) = \tau t + \frac{x^T A x}{2} + b^T x + c + O(|x|^{2-n}), \quad |x| \to \infty, t^* \leq t \leq 0.
$$

We say a function $u : \mathbb{R}^{n+1} \to \mathbb{R}$, $(x, t) \mapsto u(x, t)$, is called parabolically convex if it is continuous, convex in $x$ and non-increasing in $t$. By this definition, it is easy to see that the assumption $C_0 < 0$ yields that $u$ is actually parabolically convex.
Throughout the paper, we will always assume that
\begin{equation}
  u(0, 0) = 0, \quad Du(0, 0) = 0,
\end{equation}
and
\begin{equation}
  u \geq 0 \quad \text{in} \quad \mathbb{R}^{n+1}.
\end{equation}
In fact, let \( \hat{u}(x, t) = u(x, t) - u(0, 0) - Du(0, 0) \cdot x \). Then we have \( \hat{u}(0, 0) = 0, \quad D\hat{u}(0, 0) = 0 \), and
\begin{equation}
  \hat{u}_t - \log\left(\det(D^2\hat{u})\right) = f(x, t), \quad c_0 \leq \hat{u}_t \leq C_0 \quad \text{in} \quad \mathbb{R}^{n+1},
\end{equation}
which show that assumption (12) is reasonable. By (12) and the definition of parabolically convex function we can get that
\begin{equation*}
  \hat{u}(x, t) \geq \hat{u}(x, 0) \geq \hat{u}(0, 0) = 0, \quad \forall (x, t) \in \mathbb{R}^{n+1}.
\end{equation*}
This completes the proof of assumption (13).

Let \( D \subset \mathbb{R}^{n+1}_- \) be a bounded set and \( t \leq 0 \); then we denote
\begin{equation*}
  D(t) := \{ x \in \mathbb{R}^n : (x, t) \in D \},
\end{equation*}
and \( t_0 = \inf\{ t : D(t) \neq \emptyset \} \). The parabolic boundary of the bounded domain \( D \) is defined by
\begin{equation*}
  \partial_p D := \left( \overline{D(t_0)} \times \{ t_0 \} \right) \cup \left( \bigcup_{t \in \mathbb{R}} (\partial D(t) \times \{ t \}) \right),
\end{equation*}
where \( \overline{D(t_0)} \) denotes the closure of \( D(t_0) \) and \( \partial D(t) \) denotes the boundary of \( D(t) \). We say that the set \( D \subset \mathbb{R}^{n+1}_- \) is a bowl-shaped domain if \( D(t) \) is convex for each \( t \) and \( D(t_1) \subset D(t_2) \) for \( t_1 \leq t_2 \).

At the end of this section, we will list three lemmas that are needed throughout this paper. The proof of Lemma 2.1 and Lemma 2.2 can be found in [18]. And we will only prove Lemma 2.3.

Throughout the paper, we will always denote
\begin{equation*}
  B_r(0) := \{ x \in \mathbb{R} : |x| < r \},
\end{equation*}
\begin{equation*}
  P_r(A, \tau, \bar{x}) := \{ (x, t) \in \mathbb{R}^{n+1}_- : \frac{1}{2}(x - \bar{x})^T A(x - \bar{x}) + \tau t < r \},
\end{equation*}
where \( r > 0, \tau < 0, \bar{x} \in \mathbb{R}^n \) and \( A \) is an \( n \times n \) symmetric positive definite matrix.

**Lemma 2.1.** Let \( n \geq 1 \) and \( U \) be an \( (n+1) \times (n+1) \) real upper-triangular matrix. Assume that the diagonals of \( U \) are nonnegative and for some \( 0 < \epsilon < 1 \),
\begin{equation}
  (1 - \epsilon)P_1(I, -1, 0) \subset U(P_1(I, -1, 0)) \subset (1 + \epsilon)P_1(I, -1, 0).
\end{equation}
Then
\[ \|U - I\| \leq C(n)\sqrt{\epsilon}. \] (15)

**Lemma 2.2.** Let \( n \geq 3 \) and \( A = (a_{ij}(x,t)) \) be a real \( n \times n \) symmetric positive definite matrix with \n\[ |a_{ij}(x,t) - \delta_{ij}| \leq C(|x|^2 + |t|)^{-\frac{\epsilon}{4}}, \quad (x,t) \in \mathbb{R}^{n+1}, \] (16)
and \( a_{ij} \in C^{1+a, \frac{1+a}{2}}(\mathbb{R}^{n+1}) \), where \( \epsilon, \alpha \in (0,1) \) are constants. Then there exists a positive solution \( u \in C^\infty(\mathbb{R}^{n+1}) \) of
\[ u_t - a_{ij}D_{iju} = e^t \left( (1 + |x|^2)^{\frac{3-n}{2}} + n(n-2)(1 + |x|^2)^{-\frac{n+2}{2}} \right) \geq 0, \quad (x,t) \in \mathbb{R}^{n+1}, \] (17)
satisfying
\[ 0 \leq u(x,t) \leq C(n)e^t (1 + |x|^2)^{\frac{3-n}{2}}, \quad (x,t) \in \mathbb{R}^{n+1}. \] (18)

**Lemma 2.3.** Let \( g \in C^{2,1}(\mathbb{R}^{n+1} \setminus P_1(I, \tau, 0)) \) satisfy
\[ g_s - \log \left( \det(I + D^2g) \right) = 0, \quad I + D^2g > 0, \quad c_0 < g_s + \tau < C_0, \quad (y,s) \in \mathbb{R}^{n+1} \setminus P_1(I, \tau, 0), \]
\[ |g(y,s)| \leq \beta \left( \frac{|y|^2}{2} + \tau s \right)^{\frac{3-n}{2}}, \quad (y,s) \in \mathbb{R}^{n+1} \setminus P_1(I, \tau, 0), \] (19)
where \( \tau, c_0, C_0 < 0 \) and \( \beta, \epsilon > 0 \). Then there exists some constant \( r = r(n, \beta, \epsilon) \geq 2 \) such that
\[ |D^2g(y,s)| + |g_s(y,s)| \leq C \left( \frac{|y|^2}{2} + \tau s \right)^{-\frac{n}{2}}, \quad (y,s) \in \mathbb{R}^{n+1} \setminus P_\tau(I, \tau, 0), \]
where \( C \) depends on \( n, \beta, \epsilon, c_0 \) and \( C_0 \).

**Proof.** For \( (y,s) \in \partial P_{R^2}(I, \tau, 0), R > 2, \) let
\[ \eta_R(z,\iota) := \left( \frac{4}{R^4} \right)^2 \left( \frac{1}{2} |y + R^4 z|^2 - (s + \frac{R^2}{16} l) + g(y + \frac{R}{4} z, s + \frac{R^2}{16} l) \right), \quad (z,\iota) \in P_{\frac{R}{2}}(I, \tau, 0). \]
Since
\[ \frac{1}{2} |y + R^4 z|^2 + \tau (s + \frac{R^2}{16} l) = \frac{1}{2} (|y|^2 + \frac{R^2}{2} y^T z + \frac{R^2}{16} |z|^2) + \tau (s + \frac{R^2}{16} l) \]
\[ \geq \frac{1}{2} |y|^2 + \tau s + \frac{R}{4} y^T z \]
\[ \geq R^2 - \frac{R}{4} |y||z| \]
\[ \geq R^2 - \frac{R}{4}(\sqrt{2}R)(\frac{3}{\sqrt{2}}) \]
\[ = \frac{R^2}{4} \]
\[ > 1, \]

we can see that \( \eta_R \) is well defined in \( P_{\frac{9}{4}}(I, \tau, 0) \).

By the decay hypothesis (19) on \( g \), we have

\[ \eta_R(z, \iota) \leq \frac{16}{R^2} R^2 + \frac{9}{4} + \frac{4}{R}(\sqrt{2}R)(\frac{3}{\sqrt{2}}) + \left( \frac{4}{R} \right)^2 \beta \left( \frac{1}{2} |y + \frac{R}{4} z|^2 + \tau(s + \frac{R^2}{16} \iota) \right)^{-\frac{1}{2}} \]
\[ \leq \frac{121}{4} + \frac{162 \epsilon \beta}{R^{2+\epsilon}}, \]

we can take \( r_1 \) satisfying \( \frac{162 \epsilon \beta}{r_1^{2+\epsilon}} = 1 \), then if \( R \geq \max\{r_1, 2\} \), we have

\[ \eta_R(z, \iota) \leq \frac{125}{4}. \]

Similarly, we also have

\[ \eta_R(z, \iota) \geq \frac{16}{R^2} R^2 + \frac{4}{R} y \cdot z - \left( \frac{4}{R} \right)^2 \beta \left( \frac{1}{2} |y + \frac{R}{4} z|^2 + \tau(s + \frac{R^2}{16} \iota) \right)^{-\frac{1}{2}} \]
\[ \geq 16 - \left( \frac{4}{R} \right)(\sqrt{2}R)(\frac{3}{\sqrt{2}}) - \frac{162 \epsilon \beta}{R^{2+\epsilon}} \]
\[ = 4 - \frac{162 \epsilon \beta}{R^{2+\epsilon}}. \]

Then if \( R \geq \max\{r_1, 2\} \), we have

\[ \eta_R(z, \iota) \geq 3. \]

In conclusion, there exists some \( r = r(n, \beta, \epsilon) = \max\{r_1, 2\} \geq 1 \) such that for \( (y, s) \in \partial P_{R^2}(I, \tau, 0) \), \( R > r \),

\[ 3 \leq \eta_R(z, \iota) \leq \frac{125}{4}, \quad (z, \iota) \in P_{\frac{9}{4}}(I, \tau, 0). \]

Since \( \eta_R \) satisfies

\[ \eta_{R_1} - \log \left( \text{det}(D^2\eta_R) \right) = \tau, \quad D^2\eta_R > 0, \quad c_0 < \eta_{R_1} < C_0, \quad (z, \iota) \in P_{\frac{9}{4}}(I, \tau, 0), \]

by the interior estimates of Pogorelov, Evans–Krylov for parabolic equations, we have

\[ \|\eta_R\|_{C^{2,1}([P_{\frac{9}{4}}(I, \tau, 0)])} \leq C, \quad C^{-1} I \leq D^2\eta_R \leq CI \text{ in } P_1(I, \tau, 0). \]
Here and in the following, $C \geq 1$ denotes some constant depending on $n$, $\tau$, $c_0$ and $C_0$ unless otherwise stated.

Clearly,

$$g_R(z, \iota) := \left(\frac{4}{R}\right)^2 g\left(y + \frac{R}{4} z, s + \frac{R^2}{16} \iota\right)$$

satisfies

$$g_R - a_{ij} D_{ij} g_R = 0, \quad \text{in } P_{\frac{3}{4}}(I, \tau, 0),$$

and

$$\|g_R\|_{C^{2,1}(P_1(I, \tau, 0))} \leq C, \quad C^{-1} I \leq I + D^2 g_R \leq CI \quad \text{in } P_1(I, \tau, 0), \quad (20)$$

where $(a_{ij}(z, \iota)) = \int_0^1 (I + \theta D^2 g_R)^{-1} d\theta$ satisfies, in view of (20), that

$$\|a_{ij}\|_{C^{2+a, \frac{2+a}{2} + \alpha}(P_1(I, \tau, 0))} \leq C, \quad C^{-1} I \leq (a_{ij}) \leq CI \quad \text{in } P_1(I, \tau, 0).$$

By interior Schauder theory (see [13]) and (19),

$$|D^2 g_R(0, 0)| + |g_R(0, 0)| \leq C \|g_R\|_{L^\infty(P_1(I, \tau, 0))} \leq CR^{-\varepsilon}.$$ 

The result of Lemma 2.3 follows. \qed

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into the following five steps. And in this section we will always assume that $u \in C^{2,1}(\mathbb{R}^{n+1})$ be a parabolically convex solution of (1) and (2) with the assumption $C_0 < 0$ without loss of generality, which also satisfies normalizations (12) and (13).

3.1. Normalization of level sets and solutions

Given $H > 0$, let the level set of $u$ be defined as

$$Q_H = \{(x, t) \in \mathbb{R}^{n+1} : u(x, t) < H\},$$

and for every $t \leq 0$, we define

$$Q_H(t) = \{x \in \mathbb{R}^n : u(x, t) < H\}.$$ 

Let $x_H$ and $E$ denote the mass center of $Q_H(0)$ and the ellipsoid of minimum volume containing $Q_H(0)$ with center $x_H$ respectively. By a normalization lemma of John-Cordoba and Gallegos (see [7]), there exists some affine transformation

$$T_H(x) = a_H x + b_H,$$ 

(21)
where $a_H$ is an $n \times n$ matrix and $b_H \in \mathbb{R}^n$ satisfying

$$\det(a_H) = 1,$$

$$T_H(E) = B_R(0), \quad \text{for some } R = R(H) > 0,$$  \hspace{1cm} (22)  \hspace{1cm} (23)

and

$$B_{\alpha_nR}(0) \subset T_H(Q_H(0)) \subset B_R(0),$$  \hspace{1cm} (24)

where $\alpha_n = n^{-\frac{3}{2}}$.

The following proposition gives us the relationship between $R$ and $H$.

**Proposition 3.1.** There exists some constant $C$ depending on $n$, $c_0$, $C_0$, $\sup_{\mathbb{R}^{n+1}} f$ and $\inf_{\mathbb{R}^{n+1}} f$ such that

$$C^{-1} \leq \frac{H}{R^2} \leq C.$$  

**Proof.** Since

$$u_t(x, 0) - \log\left(\det D^2u(x, 0)\right) = f(x, 0), \quad x \in \mathbb{R}^n,$$

we have

$$\det D^2u(x, 0) = \exp\{u_t(x, 0) - f(x, 0)\}.$$

Let $M_1 = \exp\{\inf_{\mathbb{R}^n}(u_t(x, 0) - f(x, 0))\}$, then

$$M_1 \leq \det D^2u(x, 0).$$

Now we consider the function

$$w_1(y) = M_1^{-\frac{1}{n}} u(T_H^{-1}(y), 0), \quad y \in \mathbb{R}^n.$$

Then,

$$\det D^2w_1 \geq 1 \quad \text{in } B_{\alpha_nR}(0) \quad \text{and} \quad w_1 \leq M_1^{-\frac{1}{n}} H \quad \text{on } \partial B_{\alpha_nR}(0).$$

And we also consider the comparison function

$$\nu_1(y) = \frac{1}{2}(|y|^2 - \alpha_n^2 R^2) + M_1^{-\frac{1}{n}} H, \quad y \in \mathbb{R}^n.$$
It is obvious that $v_1$ is a $C^2$ convex function satisfying

$$\det D^2 v_1 = 1 \quad \text{in} \quad B_{\alpha_n R}(0) \quad \text{and} \quad v_1 = M_1^{-\frac{1}{n}} H \quad \text{on} \quad \partial B_{\alpha_n R}(0).$$

By comparison principle, we have $w_1 \leq v_1$ in $B_{\alpha_n R}(0)$. In particular,

$$0 \leq w_1(0) \leq -\frac{1}{2} \alpha_n^2 R^2 + M_1^{-\frac{1}{n}} H.$$

It follows that

$$\frac{1}{2} \alpha_n^2 M_1^{\frac{1}{n}} \leq \frac{H}{R^2}.$$

Similarly, we can obtain

$$\frac{H}{R^2} \leq \frac{1}{2} M_2^\frac{1}{n}, \quad \text{for} \quad M_2 = \exp \{ \sup_{\mathbb{R}^n} (u_t(x, 0) - f(x, 0)) \}.$$

Therefore, by taking $C = \max \{ \frac{1}{2} M_2^\frac{1}{n}, 2 \alpha_n^{-2} M_1^{-\frac{1}{n}} \}$, we have

$$C^{-1} \leq \frac{H}{R^2} \leq C. \quad \square$$

**Proposition 3.2.** For some positive constant $C$ depending on $n$, $c_0$, $C_0$, $\sup_{\mathbb{R}^{n+1}} f$ and $\inf_{\mathbb{R}^{n+1}} f$,

$$C^{-1} R \leq \text{dist} \left( T_H(Q_H (0)), \partial T_H(Q_H (0)) \right) \leq 2R. \quad (25)$$

Consequently,

$$B_{\frac{R}{2}}(0) \subset a_H(Q_H (0)) \subset B_{2R}(0). \quad (26)$$

**Proof.** Once estimate (25) has been established, estimate (26) can be deduced from (25) and the fact

$$\text{dist} \left( T_H(0), \partial T_H(Q_H (0)) \right) = \text{dist} \left( 0, \partial a_H(Q_H (0)) \right).$$

Since $T_H(Q_H (0)) \subset T_H(Q_H (0)) \subset B_R(0)$, it is obvious that

$$\text{dist} \left( T_H(Q_H (0)), \partial T_H(Q_H (0)) \right) \leq 2R.$$

So in order to obtain estimate (25), we only need to prove the first inequality in (25).
Let us consider the function

\[ w(y) = M^{-\frac{1}{n}}_2 \left( \frac{u \left( T_H^{-1}(Ry), 0 \right)}{2} - H \right), \quad y \in O_H(0) := \frac{1}{R} T_H(Q_H(0)), \]

where \( M_2 = \exp \{ \sup_{\mathbb{R}^n}(u_t(x, 0) - f(x, 0)) \} \). Then by (24), we have

\[ B_{\alpha n}(0) \subset O_H(0) \subset B_{\beta 1}(0), \quad w(y) = 0 \text{ on } \partial O_H(0), \]

and

\[ \det(D^2w) \leq 1 \text{ in } O_H(0). \]

It follows from Lemma 1 in [1] that

\[ w(y) \geq -C(n) \text{dist}(y, \partial O_H(0))^2, \quad y \in O_H(0). \]

For every \( y \in T_H(Q_H(0)) \), let \( \bar{y} = \frac{1}{R} \bar{y} \); we then have

\[ -M^{-\frac{1}{n}}_2 \frac{H}{2R^2} \geq M^{-\frac{n}{2}}_2 \left( \frac{H}{2} \right) - M^{-\frac{n}{2}}_2 \frac{H}{2R^2} \geq w(\bar{x}) \geq -C(n) \text{dist}(\bar{x}, \partial O_H(0))^2, \]

i.e.

\[ \text{dist}(\bar{y}, \partial T_H(Q_H(0))) \geq R \left( \frac{M^{-\frac{n}{2}}_2 H}{2R^2C(n)} \right)^\frac{1}{2}. \]

By Proposition 3.1, we obtain

\[ \text{dist}(\bar{y}, \partial T_H(Q_H(0))) \geq C^{-1} R, \]

where \( C = C(n, \sup_{\mathbb{R}^n}(u_t(x, 0) - f(x, 0)), \inf_{\mathbb{R}^n} u_t(x, 0) - f(x, 0)) \).

Proposition 3.3.

\[ \varepsilon_0 a_H^{-1}(B_R(0)) \times [-\varepsilon_1 H, 0] \subset Q_H \subset a_H^{-1}(B_{2R}(0)) \times [-\varepsilon_2 H, 0], \quad (27) \]

where the constant \( C \) is the same as in Proposition 3.2 and \( \varepsilon_0, \varepsilon_1, \varepsilon_2 \) are positive constants depending on \( n, c_0, C_0, \sup_{\mathbb{R}^{n+1}} f \) and \( \inf_{\mathbb{R}^{n+1}} f \).

Proof. Since \( u_t(x, t) \leq C_0 \) for \( t \leq 0 \), we have \( u(x, t) \geq u(x, 0) + C_0 t \). By (13) and (26), we then obtain \( u(x, t) \geq H \) for \( t < \frac{H}{C_0} \) or \( x \notin a_H^{-1}(B_{2R}(0)) \). So if we take \( \varepsilon_2 = \frac{1}{C_0} \), we have

\[ Q_H \subset a_H^{-1}(B_{2R}(0)) \times [-\varepsilon_2 H, 0]. \]
Due to (12), $Q_H(0)$ is a section of the convex function $u(x, 0)$ at $x = 0$. Particularly, from (26) and Lemma 2.1 of [9] we have that for any $\varepsilon_0 \in (0, 1)$,

$$
\varepsilon_0(\frac{1}{2C} a_H^{-1}(B_{2R}(0))) = \varepsilon_0 a_H^{-1}(\frac{B_x}{\varepsilon}(0)) \subset \varepsilon_0 Q_H(0) \subset Q_{(1 - \frac{1 - \varepsilon_0}{4C})H}(0).
$$

If $(x, t) \in \varepsilon_0 a_H^{-1}(\frac{B_x}{\varepsilon}(0)) \times [-1, 0]$, then

$$
u(x, t) = u(x, 0) - \int_0^t u_t(x, \tau) d\tau \leq (1 - \frac{1 - \varepsilon_0}{4C})H + c_0t \leq (1 - \frac{1 - \varepsilon_0}{4C})H < H.
$$

Therefore, if we take $\varepsilon_0$ and $\varepsilon_1$ sufficiently small, we can obtain $u(x, t) < H$,

$$
\varepsilon_0 a_H^{-1}(\frac{B_x}{\varepsilon}(0)) \times [-1, 0] \subset Q_H,
$$

which completes the proof of (27). \qed

Let

$$
\Gamma_H(x, t) := (\frac{a_H x}{R}, \frac{t}{R^2}) \quad \text{and} \quad Q_H^* := \Gamma_H(Q_H).
$$

By Proposition 3.3 and Proposition 3.1, we have

$$
B_{2\varepsilon}(0) \times [-1, 0] \subset Q_H^* \subset B_2(0) \times [-1, 0].
$$

Now we define the normalized function of $u$

$$
v(y, s) = \frac{1}{R^2} u(\Gamma_H^{-1}(y, s)) = \frac{1}{R^2} u(Ra_H^{-1} y, R^2 s), \quad (y, s) \in \overline{Q_H^*}.
$$

It is easy to verify that

$$
v_s - \log \left( \det(D^2 v) \right) = f(Ra_H^{-1} y, R^2 s), \quad c_0 \leq v_s \leq C_0 \quad \text{in} \quad Q_H^*,
$$

and by Proposition 3.1,

$$
v = \frac{H}{R^2} \in (C^{-1}, C) \quad \text{on} \quad \partial_H Q_H^*.
$$
3.2. Nonlinear perturbation

By [16], there exists a unique parabolically convex solution \( \bar{v} \in C^0(\overline{Q^*_H}) \cap C^\infty(Q^*_H) \) of

\[
\begin{aligned}
\bar{v}_s - \log \left( \det(D^2 \bar{v}) \right) &= a \quad \text{in } Q^*_H,
\bar{v} = \frac{H}{R^2} \quad \text{on } \partial_p Q^*_H,
\end{aligned}
\]

\( c_0 \leq \bar{v}_s \leq C_0 \) in \( Q^*_H \).

From the interior estimates, for every \( \delta > 0 \), there exists some positive constant \( C = C(\delta) \) such that for all \( (y, s) \in Q^*_H \) and \( \text{dist}_{p}((y, s), \partial_p Q^*_H) \geq \delta \), we have

\[
C^{-1} I \leq D^2 \bar{v}(y, s) \leq CI, \quad |D^3 \bar{v}(y, s)| \leq C, \quad |D\bar{v}_s(y, s)| \leq C, \quad |\bar{v}_{ss}(y, s)| \leq C.
\]

(28)

Lemma 3.4. For some positive constant \( C \) dependent on \( n \), \( \sup_{R^{n+1}} f, \inf_{R^{n+1}} f \) and \( \text{meas}\{f - a\} \), we have

\[
|v - \bar{v}| \leq CR^{\frac{n+2}{n+1}} \quad \text{in } Q^*_H.
\]

Proof. Let

\[
S^+ = \{(y, s) \in Q^*_H : (v - \bar{v})_s(y, s) < 0, D^2(v - \bar{v})(y, s) > 0\}.
\]

Since on \( S^+ \),

\[
\det \left( D^2 (v - \bar{v}) \right) \leq \det \left( D^2 v \right),
\]

we have

\[
-(v - \bar{v})_s + \log \left( \det D^2 (v - \bar{v}) \right) \leq -v_s + \log \left( \det D^2 v \right)
\]

\[
= -f(Ra_H^{-1}y, R^2s),
\]

and

\[
f(Ra_H^{-1}y, R^2s) = v_s - \log \left( \det D^2 v \right) < \bar{v}_s - \log \left( \det D^2 \bar{v} \right) = a.
\]

By the Alexandrov–Bakelman–Pucci estimate for parabolic equations (see [15]), we have
\[
\sup_{Q^*_{\H}} (v - \bar{v}) \leq \max \left\{ \frac{\partial P_{Q^*_{\H}}}{\partial p} (v - \bar{v}) + C(n) \left( \int_{S^+} -(v - \bar{v})_s \det \left( D^2 (v - \bar{v}) \right) dy ds \right) \right\}^{\frac{1}{\pi + 1}}
\]
\[
\leq C(n) \left( \int_{S^+} \exp\left( -(v - \bar{v})_s \right) \det \left( D^2 (v - \bar{v}) \right) dy ds \right)^{\frac{1}{\pi + 1}}
\]
\[
= C(n) \left( \int_{S^+} \exp\left( -(v - \bar{v})_s + \log \left( \det \left( D^2 (v - \bar{v}) \right) \right) \right) dy ds \right)^{\frac{1}{\pi + 1}},
\]

from which it follows that
\[
\sup_{Q^*_{\H}} (v - \bar{v}) \leq C(n) \left( \int_{S^+} \exp\left\{ -f \left( \mathcal{R}_{\H}^{-1} y, R^2 s \right) \right\} dy ds \right)^{\frac{1}{\pi + 1}}
\]
\[
= C(n) \left( \int_{\mathcal{H}^{-1}(S^+)} \exp\left\{ -f \left( x, t \right) \right\} \left\{ \frac{\det_{\H}}{R^{n+2}} \right\} dx dt \right)^{\frac{1}{\pi + 1}}
\]
\[
\leq C(n) R^{-\frac{n+2}{\pi + 1}} \left( \int_{\{ f < a \}} \exp\left\{ -f \left( x, t \right) \right\} dx dt \right)^{\frac{1}{\pi + 1}}
\]
\[
\leq CR^{-\frac{n+2}{\pi + 1}}.
\]

Similarly, we can obtain that
\[
\sup_{Q^*_{\H}} (\bar{v} - v) \leq CR^{-\frac{n+2}{\pi + 1}}.
\]

Lemma 3.4 is established. \(\square\)

3.3. Rough asymptotic behavior

Let \((\bar{y}, 0)\) be the unique minimum point of \(\bar{v}\) on \(Q^*_{\H}, \bar{\H} \in (\bar{v}(\bar{y}, 0), \H)\) and \(m > 0\). For simplification, we will denote that
\[
P_{\bar{\H}}(0, 0) := P_{\bar{\H}}(D^2 \bar{v}(\bar{y}, 0), \bar{v}_s(\bar{y}, 0), 0)
\]
\[
S_{\bar{\H}}(0, 0) := \partial_p P_{\bar{\H}}(0, 0)
\]
\[
P_{\bar{\H}}(\bar{y}, 0) := P_{\bar{\H}}(D^2 \bar{v}(\bar{y}, 0), \bar{v}_s(\bar{y}, 0), \bar{y})
\]
\[
S_{\bar{\H}}(\bar{y}, 0) := \partial_p P_{\bar{\H}}(\bar{y}, 0)
\]
\[
mP_{\bar{\H}}(0, 0) := \{(y, s) \in \mathbb{R}^{n+1}_- : \left( \frac{y}{m}, \frac{s}{m^2} \right) \in P_{\bar{\H}}(0, 0)\}.
\]
\[ m P_{H}(\bar{y}, 0) := \{(y, s) \in \mathbb{R}^{n+1}_{\geq 0} : \left(\frac{\gamma}{m}, \frac{s}{m^2}\right) \in P_{H}(\bar{y}, 0)\} \]
\[ m Q_{H} := \{(y, s) \in \mathbb{R}^{n+1}_{\geq 0} : \left(\frac{\gamma}{m}, \frac{s}{m^2}\right) \in Q_{H}\} . \]

**Lemma 3.5.** There exist \( \bar{k} \) and \( \bar{C} \), depending on \( n, c_0, C_0, \sup f, \inf f \) and \( \text{meas}\{f - a\} \), such that for \( \epsilon = \frac{1}{3} \), \( H = 2^{(1+\epsilon)k} \) and \( 2^{k-1} \leq H' \leq 2^k \), we have
\[ \left(\frac{H'}{R^2} - \bar{C}2^{-3\epsilon k}\right)\frac{1}{2} P_{1}(\bar{y}, 0) \subset \Gamma_{H}(Q_{H'}) \subset \left(\frac{H'}{R^2} + \bar{C}2^{-3\epsilon k}\right)\frac{1}{2} P_{1}(\bar{y}, 0), \quad \forall k \geq \bar{k}. \quad (29) \]

**Proof.**
\[ \Gamma_{H}(Q_{H'}) = \{v < \frac{H'}{R^2}\} := \{(y, s) \in Q_{H}^* : v(y, s) < \frac{H'}{R^2}\} . \]

By **Lemma 3.4**, the level surface of \( v \) can be well approximated by the level surface of \( \bar{v} \):
\[ \{\bar{v} < \frac{H'}{R^2} - \frac{C}{R \frac{n+2}{n+1}}\} \subset \{v < \frac{H'}{R^2}\} \subset \{\bar{v} < \frac{H'}{R^2} + \frac{C}{R \frac{n+2}{n+1}}\} , \]
and
\[ -\frac{C}{R \frac{n+2}{n+1}} \leq v(\bar{y}, 0) - \frac{C}{R \frac{n+2}{n+1}} \leq \bar{v}(\bar{y}, 0) \leq \bar{v}(0, 0) \leq v(0, 0) + \frac{C}{R \frac{n+2}{n+1}} = \frac{C}{R \frac{n+2}{n+1}} , \quad (30) \]

since \( v \geq 0 \) and \( v(0, 0) = 0 \). We also have that
\[ C^{-1} I \leq D^2\bar{v}(\bar{y}, 0) \leq CI \quad \bar{v}_{s}(\bar{y}, 0) \leq C^{-1} , \quad (31) \]
and
\[ |v(y, s) - \bar{v}(\bar{y}, 0) - \bar{v}_{s}(\bar{y}, 0)s - \frac{1}{2}(y - \bar{y})^T D^2\bar{v}(\bar{y}, 0)(y - \bar{y})| \leq C(|y - \bar{y}|^2 + |s|)^{\frac{3}{2}} , \quad (32) \]

for \( \text{dist}_{p}((y, s), (\bar{y}, 0)) < \frac{1}{C} \) by (28) and \( D\bar{v}(\bar{y}, 0) = 0 \).

It is clear by **Proposition 3.1** that
\[ C^{-1} 2^{-\epsilon k} \leq \frac{H'}{R^2} \leq C 2^{-\epsilon k} , \quad C^{-1} 2^{-(1+\epsilon)k} \leq R \leq C 2^{-(1+\epsilon)k} . \quad (33) \]

Next we will prove the two relations in (29) respectively. On one side, we shall take a positive constant \( C_1 \) such that
\[ \left(\frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2}}\right)\frac{1}{2} P_{1}(\bar{y}, 0) \subset \{\bar{v} < \frac{H'}{R^2} - \frac{C}{R \frac{n+2}{n+1}}\} . \quad (34) \]

For \( (y, s) \in \left(\frac{H'}{R^2} - C_1 2^{-\frac{3\epsilon k}{2}}\right)\frac{1}{2} P_{1}(\bar{y}, 0) \), we have
Thus, it follows from (30) and (32) that

\[
\bar{v}(y, s) \leq \bar{v}(\bar{y}, 0) + \bar{v}_s(\bar{y}, 0)s + \frac{1}{2}(y - \bar{y})^T D^2\bar{v}(\bar{y}, 0)(y - \bar{y}) + C(|y - \bar{y}|^2 + |s|)^\frac{3}{2}
\]

\[
\leq \frac{C}{R^{\frac{n+2}{n+1}}} + \frac{H'}{R^2} - C_1 2^{-\frac{3k}{2}} + C^2 (\frac{H'}{R^2} - C_1 2^{-\frac{3k}{2}})^\frac{3}{2}
\]

\[
\leq \frac{C}{R^{\frac{n+2}{n+1}}} + \frac{H'}{R^2} - C_1 2^{-\frac{3k}{2}} + C^4 2^{-\frac{3}{2}k}.
\]

We can take \( C_1 = 1 + C^4 \) such that

\[
\frac{2C}{R^{\frac{n+2}{n+1}}} \leq 2C^{\frac{2n+3}{n+1}} 2^{-\frac{(1+\epsilon)(n+2k)}{2(n+1)}} < 2^{-\frac{3k}{2}},
\]

if \( k \geq k_1, k_1 \) large enough. Therefore, we obtain

\[
\bar{v}(y, s) \leq \frac{C}{R^{\frac{n+2}{n+1}}} + \frac{H'}{R^2} + 2^{-\frac{3k}{2}} < \frac{H'}{R^2} - \frac{C}{R^{\frac{n+2}{n+1}}},
\]

and finish the proof of (34).

On the other side, we shall take a positive constant \( C_2 \) such that

\[
\{ v < \frac{H'}{R^2} + \frac{C}{R^{\frac{n+2}{n+1}}} \} \subset (\frac{H'}{R^2} + C_2 2^{-\frac{3k}{2}})^\frac{1}{2} P_1(\bar{y}, 0).
\]

By using the fact

\[
(\bar{y}, 0) \in \{ \bar{v} < \frac{H'}{R^2} + \frac{C}{R^{\frac{n+2}{n+1}}} \} \cap (\frac{H'}{R^2} + C_2 2^{-\frac{3k}{2}})^\frac{1}{2} P_1(\bar{y}, 0),
\]

we only need to prove

\[
(\frac{H'}{R^2} + C_2 2^{-\frac{3k}{2}})^\frac{1}{2} S_1(\bar{y}, 0) \subset \{ v < \frac{H'}{R^2} + \frac{C}{R^{\frac{n+2}{n+1}}} \} \subset C.
\]

For \((y, s) \in (\frac{H'}{R^2} + C_2 2^{-\frac{3k}{2}})^\frac{1}{2} S_1(\bar{y}, 0)\), then

\[
\bar{v}_s(\bar{y}, 0)s + \frac{1}{2}(y - \bar{y})^T D^2\bar{v}(\bar{y}, 0)(y - \bar{y}) = \frac{H'}{R^2} + C_2 2^{-\frac{3k}{2}},
\]

\[
|y - \bar{y}|^2 + |s| < C(\frac{H'}{R^2} + C_2 2^{-\frac{3k}{2}}).\]
Take $k_2$ large enough satisfying for $k \geq k_2$,
\[
|y - \bar{y}|^2 + |s| < C \left( \frac{H'}{R^2} + C_2 2^{-\frac{3k}{2}} \right) \leq \frac{1}{C^2}.
\]

Thus,
\[
\bar{v}(y, s) \geq \bar{v}(\bar{y}, 0) + \bar{v}_s(\bar{y}, 0)s + \frac{1}{2} (y - \bar{y})^T D^2 \bar{v}(\bar{y}, 0)(y - \bar{y}) - C(|y - \bar{y}|^2 + |s|)\frac{1}{2}
\]
\[
\geq - \frac{C}{R^{n+1}} + \frac{H'}{R^2} + C_2 2^{-\frac{3k}{2}} - C_2 \frac{5}{2} \left( \frac{H'}{R^2} + C_2 2^{-\frac{3k}{2}} \right) \frac{1}{2}
\]
\[
\geq - \frac{C}{R^{n+1}} + \frac{H'}{R^2} + C_2 2^{-\frac{3k}{2}} - C_2 \frac{5}{2} (\frac{H'}{R^2}) \frac{3}{2}
\]
\[
\geq - \frac{C}{R^{n+1}} + \frac{H'}{R^2} + C_2 2^{-\frac{3k}{2}} - C_4 2^{\frac{3}{2}} 2^{-\frac{3}{2}k}.
\]

We can take $C_2 = 1 + 2^{\frac{3}{2}} C^4$, then
\[
\frac{2C}{R^{n+1}} \leq 2C \frac{2n+3}{n+1} 2^{-\frac{(1+c)(n+2)k}{2(n+1)}} < 2^{-\frac{3k}{2}},
\]
if $k \geq k_2$, $k_2$ large enough. Therefore, we obtain
\[
\bar{v}(y, s) \geq - \frac{C}{R^{n+1}} + \frac{H'}{R^2} + 2^{-\frac{3k}{2}} > \frac{H'}{R^2} + \frac{C}{R^{n+1} 2^{\frac{3}{2}k}}.
\]

In conclusion, if we take $\bar{C} > \max\{C_1, C_2\}$ and $\bar{k} = \max\{k_1, k_2\}$, then (29) holds. \(\Box\)

**Proposition 3.6.** There exist $\bar{k}$ and $\bar{C}$, depending on $n$, $c_0$, $C_0$, $\sup f$, $\inf f$ and $\text{meas}(f - a)$, such that for $\epsilon = \frac{1}{4}$, $H = 2^{(1+\epsilon)k}$ and $2^{k-1} \leq H' \leq 2^k$, we have
\[
\left( \frac{H'}{R^2} - \bar{C} 2^{-\frac{3k}{2}} \right) \frac{1}{2} P_1(0, 0) \subset \Gamma_H(Q_{\bar{H}^r}) \subset \left( \frac{H'}{R^2} + \bar{C} 2^{-\frac{3k}{2}} \right) \frac{1}{2} P_1(0, 0), \quad \forall k \geq \bar{k}. \tag{35}
\]

**Proof.** In order to obtain (35), we first show that
\[
\partial Q_{\bar{H}^r + \bar{v}(\bar{y}, 0)}^n(\bar{v}) \subset N_{\delta_1} (S_{\bar{H}^r}(\bar{y}, 0)), \quad 0 < \bar{H} \leq \frac{H}{R^2} - \bar{v}(\bar{y}, 0), \quad \delta_1 \leq C \bar{H}^{\frac{1}{2}}, \tag{36}
\]
and neighborhood $N$ is measured by parabolic distance
\[
\text{dist}_p((y_1, s_1), (y_2, s_2)) := (|y_1 - y_2|^2 + |s_1 - s_2|)\frac{1}{2}.
\]

In fact, for $(y, s) \in \partial Q_{\bar{H}^r + \bar{v}(\bar{y}, 0)}^n(\bar{v})$, by the Mean Theorem, we have
\[ \tilde{H} = \tilde{v}(y, s) - \tilde{v}(\tilde{y}, 0) \]
\[ = \tilde{v}(y, s) - \tilde{v}(y, 0) + \tilde{v}(y, 0) - \tilde{v}(\tilde{y}, 0) \]
\[ = \tilde{v}_s(y, s') s + \frac{1}{2} (y - \tilde{y})^T D^2 \tilde{v}(y', 0)(y - \tilde{y}) \]
\[ \geq \frac{1}{2C} (|s| + |y - \tilde{y}|^2), \]

where \((y', s') \in Q^*_{\tilde{H} + \tilde{v}(\tilde{y}, 0)}\). Writing
\[ \tilde{H} = \tilde{v}(y, s) - \tilde{v}(\tilde{y}, 0) \]
\[ = \tilde{v}_s(\tilde{y}, 0)s + (\tilde{v}_s(y, s') - \tilde{v}_s(\tilde{y}, 0))s + \frac{1}{2} (y - \tilde{y})^T D^2 \tilde{v}(\tilde{y}, 0)(y - \tilde{y}) \]
\[ + \frac{1}{2} (y - \tilde{y})^T (D^2 \tilde{v}(y', 0) - D^2 \tilde{v}(\tilde{y}, 0))(y - \tilde{y}), \]

for \((y, s) \in \partial Q^*_{\tilde{H} + \tilde{v}(\tilde{y}, 0)}(\tilde{v})\), then
\[ |\tilde{H} - \tilde{v}_s(\tilde{y}, 0)s - \frac{1}{2} (y - \tilde{y})^T D^2 \tilde{v}(\tilde{y}, 0)(y - \tilde{y})| \]
\[ = |(\tilde{v}_s(y, s') - \tilde{v}_s(\tilde{y}, 0))s + \frac{1}{2} (y - \tilde{y})^T (D^2 \tilde{v}(y', 0) - D^2 \tilde{v}(\tilde{y}, 0))(y - \tilde{y})| \]
\[ \leq C |s| + C |y - \tilde{y}|^2 \]
\[ \leq C \tilde{H}. \]

For any \((y, s) \in \partial Q^*_{\tilde{H} + \tilde{v}(\tilde{y}, 0)}(\tilde{v})\) and any \((\tilde{\gamma}, \tilde{s}) \in S_{\tilde{H}}(\tilde{y}, 0)\), by the above inequality, we have
\[ |\bar{v}_s(\tilde{y}, 0)s + \frac{1}{2} (y - \tilde{y})^T D^2 \tilde{v}(\tilde{y}, 0)(y - \tilde{y}) - \bar{v}_s(\tilde{y}, 0)s - \frac{1}{2} (y - \tilde{y})^T D^2 \tilde{v}(\tilde{y}, 0)(y - \tilde{y})| \leq C \tilde{H}. \]

For \(s = \tilde{s}\), take \(\tilde{\gamma}, \tilde{y}, y\) on the same line \(l\) with \(\tilde{\gamma}\) and \(y\) on the same side of the line \(l\) with respect to \(\tilde{y}\); rotating the coordinates again so that \(l\) is parallel to some axis, we have
\[ ||\tilde{y} - y||^2 - |y - \tilde{y}|^2| \geq |y - \tilde{y}|^2. \]

Then
\[ \frac{1}{2C} ||\tilde{y} - y||^2 - |y - \tilde{y}|^2| \leq C \tilde{H}. \]

In fact, there exists an orthogonal matrix \(O\) such that \(D^2 \tilde{v}(\tilde{y}, 0) = O^T \text{diag}(\lambda_1, \ldots, \lambda_n) O\), and the length of a vector in Euclidean space is invariant in orthogonal transformation.

Therefore, we obtain
\[ |y - \tilde{y}|^2 \leq C \tilde{H}. \]
Similarly, for \( y = \tilde{y} \),

\[
|\tilde{v}_s(\tilde{y}, 0)\tilde{s} - \tilde{v}_s(\tilde{y}, 0)s| \leq C\tilde{H}.
\]

So we get

\[
|s - \tilde{s}| \leq C\tilde{H}.
\]

This completes the proof of (36).

Next we estimate the distance between \((0, 0)\) and \((\tilde{y}, 0)\). By Lemma 3.4, we have

\[
0 \leq \tilde{v}(0, 0) - \tilde{v}(\tilde{y}, 0) = (\tilde{v}(0, 0) - v(0, 0)) + (v(0, 0) - v(\tilde{y}, 0)) + (v(\tilde{y}, 0) - \tilde{v}(\tilde{y}, 0)) \leq 2CR^{-\frac{n+2}{n+1}},
\]

so \((0, 0) \in Q^* \cdot \tilde{v}, \) and by (36) (taking \( \tilde{H} = 2CR^{-\frac{n+2}{n+1}} \)), we have

\[
\partial(Q^* \cdot 2CR^{-\frac{n+2}{n+1}} + \tilde{v}(\tilde{y}, 0)) \subset N_{\delta_1} (S \cdot 2CR^{-\frac{n+2}{n+1}}(\tilde{y}, 0)), \quad \delta_1 \leq C(2CR^{-\frac{n+2}{n+1}})^{1/2},
\]

thus we obtain

\[
|\tilde{y}| = \text{dist}_p((0, 0), (\tilde{y}, 0)) \leq C(2CR^{-\frac{n+2}{n+1}})^{1/2}.
\]

So by (29), we have for \( k \geq \tilde{k} \)

\[
\left( \frac{H'}{R^2} - C_32^{-\frac{3k}{2}} - 2C^3R^{-\frac{n+2}{n+1}} \right)^{1/2} P_1(0, 0) \subset \left\{ v < \frac{H'}{R^2} \right\}
\subset \left( \frac{H'}{R^2} + C_32^{-\frac{3k}{2}} + 2C^3R^{-\frac{n+2}{n+1}} \right)^{1/2} P_1(0, 0).
\]

Since \( 2^{-\frac{3k}{2}} \gg R^{-\frac{n+2}{n+1}} \) and letting \( \tilde{C} = C^2 + C_3 \), then we can obtain (35). □

**Proposition 3.7.** There exist positive constants \( \hat{k}, \) \( \hat{C} \) depending on \( n, c_0, C_0, \sup f, \inf f \) and \( \text{meas}\{f - a\} \), some real invertible upper-triangular matrices \( \{T_k\}_{k \geq \hat{k}} \) and negative number \( \{\tau_k\}_{k \geq \hat{k}} \) such that

\[
\tau_k - \log \left( \det(T_k^T T_k) \right) = a, \quad \|T_kT_{k-1} - I\| \leq \hat{C}2^{-\frac{\epsilon_k}{2}}, \quad |\tau_k \tau_{k-1}^{-1} - 1| \leq \hat{C}2^{-\frac{\epsilon_k}{2}}, \quad (37)
\]

and

\[
(1 - \hat{C}2^{-\frac{\epsilon_k}{2}})\sqrt{H'} P_1(I, -1, 0) \subset \Sigma_k(Q_{H'}) \subset (1 + \hat{C}2^{-\frac{\epsilon_k}{2}})\sqrt{H'} P_1(I, -1, 0),
\]

\[
\forall H' \in [2^{k-1}, 2^k],
\]

(38)
where $\Sigma_k = (T_k, -\tau_k)$. Consequently, for some invertible upper-triangular matrix $T$ and $\tau < 0$,
\[
\tau - \log \left( \det(T^T T) \right) = a, \quad \|T_k - T\| \leq \tilde{C} 2^{-\frac{\epsilon_k}{2}}, \quad |\tau_k - \tau| \leq \tilde{C} 2^{-\frac{\epsilon_k}{2}}. \tag{39}
\]

**Proof.** Let $H = 2^{(1+\epsilon)k}$ and $2^{k-1} \leq H' \leq 2^k$. By Proposition 3.6 and the definition of $\Gamma_H$, there exist some positive constants $\tilde{C}$ and $\tilde{k}$ depending on $n, a, c_0, C_0$ and $f$ such that
\[
(H' - \tilde{C} 2^{-\frac{3\epsilon_k}{2}} R^2)^{\frac{1}{2}} P_1(0, 0) \subset (a_H, id)(Q_{H'}) \subset (H' + \tilde{C} 2^{-\frac{3\epsilon_k}{2}} R^2)^{\frac{1}{2}} P_1(0, 0), \quad \forall k \geq \tilde{k}.
\]
Since
\[
C^{-1} 2^{-\epsilon_k} \leq \frac{H'}{R^2} \leq C 2^{-\epsilon_k},
\]
we can obtain
\[
(1 - \tilde{C} C 2^{-\frac{\epsilon_k}{2}})^{\frac{1}{2}} \sqrt{H'} P_1(0, 0) \subset (a_H, id)(Q_{H'}) \subset (1 + \tilde{C} C 2^{-\frac{\epsilon_k}{2}})^{\frac{1}{2}} \sqrt{H'} P_1(0, 0).
\]
In conclusion, if we take $\tilde{C}$ and $\tilde{k}$ large enough, then
\[
(1 - \tilde{C} 2^{-\frac{\epsilon_k}{2}}) \sqrt{H'} P_1(0, 0) \subset (a_H, id)(Q_{H'}) \subset (1 + \tilde{C} 2^{-\frac{\epsilon_k}{2}}) \sqrt{H'} P_1(0, 0), \quad k \geq \tilde{k}. \tag{40}
\]

Let $Q$ be the real symmetric positive definite matrix satisfying $Q^2 = Q^T Q = D^2 \tilde{v}(\tilde{y}, 0)$ and $O$ be an orthogonal matrix such that
\[
T_k := O Q a_H \quad \text{is upper-triangular},
\]
and we also define $\tau_k = \tilde{v}_s(\tilde{y}, 0)$ and $\Sigma_k = (T_k, -\tau_k)$. Clearly,
\[
\tau_k - \log \left( \det(T_k^T T_k) \right) = \tilde{v}_s(\tilde{y}, 0) - \log \left( (\det a_H)^2 \det D^2 \tilde{v}(\tilde{y}, 0) \right) = a,
\]
and $P_1(I, -1, 0) = (O Q, -\tau_k) P_1(0, 0)$.

From (40), we have
\[
(1 - \tilde{C} 2^{-\frac{\epsilon_k}{2}}) \sqrt{H'} P_1(I, -1, 0) \subset \Sigma_k(Q_{H'}) \subset (1 + \tilde{C} 2^{-\frac{\epsilon_k}{2}}) \sqrt{H'} P_1(I, -1, 0), \quad k \geq \tilde{k},
\]
that is (38).

By taking $H = 2^{(1+\epsilon)k}$, $H' = 2^{k-1}$ and $H = 2^{(1+\epsilon)(k-1)}$, $H' = 2^{k-1}$, we can obtain
\[
(1 - \tilde{C} 2^{-\frac{\epsilon_k}{2}}) \sqrt{2^{k-1}} P_1(I, -1, 0) \subset \Sigma_k(Q_{2^{k-1}}) \subset (1 + \tilde{C} 2^{-\frac{\epsilon_k}{2}}) \sqrt{2^{k-1}} P_1(I, -1, 0), \tag{41}
\]
\[
(1 - \tilde{C} 2^{-\frac{\epsilon(k-1)}{2}}) \sqrt{2^{k-1}} P_1(I, -1, 0) \subset \Sigma_{k-1}(Q_{2^{k-1}}) \subset (1 + \tilde{C} 2^{-\frac{\epsilon(k-1)}{2}}) \sqrt{2^{k-1}} P_1(I, -1, 0), \tag{42}
\]
respectively. Then
\[(1 - \hat{C}2^{-\frac{c(k-1)}{2}})\sqrt{2^{k-1}}\Sigma_k \Sigma_{k-1}^{-1} P_1(I, -1, 0) \subset \Sigma_k(Q_{2k-1})
\subset (1 + \hat{C}2^{-\frac{c(k-1)}{2}})\sqrt{2^{k-1}}\Sigma_k \Sigma_{k-1}^{-1} P_1(I, -1, 0),
\]

(43)

by taking \(k\) sufficiently large. Similarly,
\[(1 - 3\hat{C}2^{-\frac{ck}{2}}) P_1(I, -1, 0) \subset \Sigma_k \Sigma_{k-1}^{-1} P_1(I, -1, 0).
\]

So by taking \(k\) sufficiently large, we have
\[(1 - 3\hat{C}2^{-\frac{ck}{2}}) P_1(I, -1, 0) \subset \Sigma_k \Sigma_{k-1}^{-1} P_1(I, -1, 0) \subset (1 + 3\hat{C}2^{-\frac{ck}{2}}) P_1(I, -1, 0).
\]

Since \(\Sigma_k \Sigma_{k-1}^{-1}\) is still upper-triangular, we apply Lemma 2.1 (with \(U = \Sigma_k \Sigma_{k-1}^{-1}\)) to obtain that
\[\|\Sigma_k \Sigma_{k-1}^{-1} - I\| \leq C(n)\sqrt{3\hat{C}2^{-\frac{ck}{2}}} \leq \hat{C}2^{-\frac{ck}{4}}.
\]

Estimate (37) has been established. The existence of \(T, \tau\) and (39) follow by elementary consideration. \(\square\)

We can deduce from (38) and (39) that on one side,
\[
\Sigma(Q_{H'}) - \Sigma_k(Q_{H'}) \subset \hat{C}2^{-\frac{ck}{2}}\sqrt{H'}P_1(I, -1, 0),
\]
\[
\Sigma(Q_{H'}) \subset (1 + 2\hat{C}2^{-\frac{ck}{2}})\sqrt{H'}P_1(I, -1, 0),
\]
and on the other side,
\[
\Sigma_k(Q_{H'}) - \Sigma(Q_{H'}) \subset \hat{C}2^{-\frac{ck}{2}}\sqrt{H'}P_1(I, -1, 0),
\]
\[
(1 - 2\hat{C}2^{-\frac{ck}{2}})\sqrt{H'}P_1(I, -1, 0) \subset \Sigma(Q_{H'}).
\]

In particular, if we take \(H' = 2^k, k \geq \hat{k}\), then
\[(1 - 2\hat{C}(H')^{-\frac{c}{2}})\sqrt{H'}P_1(I, -1, 0) \subset \{(y, s) \in \mathbb{R}^{n+1}_+ : u(T^{-1}y, -\frac{s}{\tau}) < H'\}
\subset (1 + 2\hat{C}(H')^{-\frac{c}{2}})\sqrt{H'}P_1(I, -1, 0).
\]

So we have
\[(1 - 2\hat{C}(u(T^{-1}y, -\frac{s}{\tau}))^{-\frac{c}{2}}u(T^{-1}y, -\frac{s}{\tau}) < -s + \frac{1}{2}|y|^2
\]
\[< (1 + 2\hat{C}(u(T^{-1}y, -\frac{s}{\tau}))^{-\frac{c}{2}}u(T^{-1}y, -\frac{s}{\tau}),
\]
\(-4 \hat{C}(u(T^{-1}y, -\frac{s}{\tau}))^{1-\xi} - 4 \hat{C}^2(u(T^{-1}y, -\frac{s}{\tau}))^{1-\xi} < u(T^{-1}y, -\frac{s}{\tau}) - (-s + \frac{1}{2}|y|^2) < 4 \hat{C}(u(T^{-1}y, -\frac{s}{\tau}))^{1-\xi} - 4 \hat{C}^2(u(T^{-1}y, -\frac{s}{\tau}))^{1-\xi}.\)

Thus
\[
\left|u(T^{-1}y, -\frac{s}{\tau}) - (-s + \frac{1}{2}|y|^2)\right| < \hat{C}(u(T^{-1}y, -\frac{s}{\tau}))^{1-\xi}.
\]

Consequently, by the fact \(C^{-1}u(T^{-1}y, -\frac{s}{\tau}) \leq \frac{1}{2}|y|^2 - s\), we obtain
\[
\left|u(T^{-1}y, -\frac{s}{\tau}) - (-s + \frac{1}{2}|y|^2)\right| \leq C(\frac{1}{2}|y|^2 - s)^{\frac{\xi}{4}}, \quad (y, s) \in \mathbb{R}^{n+1} \setminus P_{22k}(I, \tau, 0).
\]

Let us define \(w(y, s) = u(T^{-1}y, s)\); then
\[
w_s - \log \left(\det(D^2w)\right) = \log \left(\det(T^T T)\right) + a = \tau, \quad (y, s) \in \mathbb{R}^{n+1} \setminus P_{22k}(I, \tau, 0),
\]
where \(\tilde{\Sigma}(Q_H) := (T, id)Q_H\) and
\[
|w(y, s) - (\tau s + \frac{1}{2}|y|^2)| \leq C(\frac{1}{2}|y|^2 + \tau s)^{\frac{\xi}{4}}, \quad (y, s) \in \mathbb{R}^{n+1} \setminus P_{22k}(I, \tau, 0). \tag{44}
\]

We call the above inequality the asymptotic behavior of \(w\).

### 3.4. Explicit asymptotic behavior

In this section, we will obtain the explicit asymptotic behavior of \(w\).

**Proposition 3.8.** There exist \(\tilde{b} \in \mathbb{R}^n, \tilde{c} \in \mathbb{R}\) and some positive constant \(C\) depending on \(n, c_0, C_0, \sup f, \inf f\) and \(\text{meas}\{f - a\}\) such that
\[
|w(y, s) - \tau s - \frac{|y|^2}{2} - \tilde{b}^T y - \tilde{c}| \leq C e^{-\tau s}(1 + |y|^2)^{-\frac{n-2}{2}}, \quad (y, s) \in \mathbb{R}^{n+1} \setminus P_{22k}(I, \tau, 0).
\]

**Proof.** Let
\[
g(y, s) := w(y, s) - (\tau s + \frac{|y|^2}{2}), \quad (y, s) \in \mathbb{R}^{n+1} \setminus P_{22k}(I, \tau, 0),
\]
and by (44) and Lemma 2.3,
\[
|D^2 g(y, s)| + |g_s(y, s)| \leq C(\frac{1}{2}|y|^2 - s)^{-\frac{\xi}{2}}.
\]
It follows that

$$g_s - \tilde{a}_{ij} D_{ij} g = 0 \quad \text{in } \mathbb{R}^{n+1}_- \setminus P_{2\tau} (I, \tau, 0), \quad (45)$$

where \((\tilde{a}_{ij}(y, s)) = \int_0^1 (I + \theta D^2 g)^{-1} d\theta.

Let \(e \in \mathbb{R}^n\) be a unit vector; now we apply \(D_s, D_e\) and \(D_{ee}\) to the following equation

$$g_s - \log \left( \det (I + D^2 g) \right) = 0, \quad \text{in } \mathbb{R}^{n+1}_- \setminus P_{2\tau} (I, \tau, 0),$$

in view of the concavity of \(\log(\det)\), we have

$$\begin{align*}
(g_s)_s - B_{ij} D_{ij}(g_s) &= 0, \quad \text{in } \mathbb{R}^{n+1}_- \setminus P_{2\tau} (I, \tau, 0), \\
(D_{eg})_s - B_{ij} D_{ij}(D_{eg}) &= 0, \quad \text{in } \mathbb{R}^{n+1}_- \setminus P_{2\tau} (I, \tau, 0),
\end{align*} \quad (46) \quad (47)$$

and

$$\begin{align*}
(D_{ee} g)_s - B_{ij} (D_{ee} g)_{ij} &\leq 0, \quad \text{in } \mathbb{R}^{n+1}_- \setminus P_{2\tau} (I, \tau, 0),
\end{align*} \quad (48)$$

where \((B_{ij}) = (I + D^2 g)^{-1}\). We claim that

$$|B_{ij} - \delta_{ij}| \leq C \left( \frac{1}{2} |y|^2 - s \right)^{-\frac{\epsilon}{4}}.$$

In fact, let \(\lambda_1, \lambda_2, \cdots, \lambda_n\) denote the eigenvalues of \(D^2 g\); then

$$\begin{align*}
| \frac{1}{1 + \lambda_i} - 1 | &= \frac{|\lambda_i|}{|1 + \lambda_i|}, \\
&\leq \frac{|\lambda_i|}{1 - |\lambda_i|}, \\
&= \frac{1}{|\lambda_i|^{-1} - 1}, \\
&\leq \frac{1}{\left( \frac{\epsilon}{4} |y|^2 - s \right)^{\epsilon/4} \frac{C}{\epsilon} - 1}, \\
&\leq \frac{1}{\left( \frac{\epsilon}{4} |y|^2 - s \right)^{\epsilon/4} \frac{C}{\epsilon} - \frac{\epsilon}{4} |y|^2 - s \right)^{\epsilon/4}}, \\
&\leq C \left( \frac{1}{2} |y|^2 - s \right)^{-\frac{\epsilon}{4}},
\end{align*}$$

this completes the proof of the claim.

By Lemma 2.2, for such coefficients, there exists a positive solution \(G(y, s)\) of

$$G_s - B_{ij} D_{ij} G = e^{-\tau s} (1 + |y|^2)^{-\frac{n}{2} - 1} \left( -\tau (1 + |y|^2)^2 + n(n - 2) \right) \geq 0 \quad \text{in } \mathbb{R}^{n+1}_-. \quad (49)$$
By (46), (48), (49) and the maximum principle, we have

\[-g_s(y, s) \leq G(y, s) \leq Ce^{-ts} (1 + |y|^2)^{-\frac{\alpha+2}{2}}, \tag{50}\]

and

\[D_{ee} g(y, s) \leq G(y, s) \leq Ce^{-ts} (1 + |y|^2)^{-\frac{\alpha+2}{2}}. \tag{51}\]

Thus

\[|D^2 g(y, s)| \leq Ce^{-ts} (1 + |y|^2)^{-\frac{\alpha+2}{2}}.\]

Then for \(1 \leq m \leq n\), there exists \(\tilde{b}_m \in \mathbb{R}\) satisfying \(D_m g(y, s) - \tilde{b}_m \to 0\), as \(|y|^2 - s \to 0\) (see [18]) and \(D_m g - \tilde{b}_m\) satisfies

\[(D_m g - \tilde{b}_m)_s - B_{ij}(D_m g - \tilde{b}_m)_{ij} = 0, \quad (y, s) \in \mathbb{R}^{n+1} \setminus P_{2t^*}(I, \tau, 0),\]

from (47). Let \(\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)^T\); so we can also deduce from the maximum principle that

\[|Dg(y, s) - \tilde{b}| \leq Ce^{-ts} (1 + |y|^2)^{-\frac{\alpha+2}{2}},\]

by combining (49).

Similarly, there exists \(\tilde{c} \in \mathbb{R}\) such that

\[|g(y, s) - \tilde{b}^T y - \tilde{c}| \leq Ce^{-ts} (1 + |y|^2)^{-\frac{\alpha+2}{2}}, \quad \forall (y, s) \in \mathbb{R}^{n+1} \setminus P_{2t^*}(I, \tau, 0). \quad \square\]

3.5. Finishing proof of Theorem 1.1

**Proof.** Since \(w(y, s) = u(T^{-1} y, s)\), then letting \(x = T^{-1} y\) and \(t = s\), by Proposition 3.8, we have

\[|u(x, t) - \tau t - \frac{x^T T^T T x}{2} - \tilde{b}^T T x - \tilde{c}| \leq Ce^{-\tau t} (1 + |x|^2)^{n+2}, \quad \frac{1}{2} |x|^2 - t \text{ is large enough.}\]

By taking \(A = T^T T, b = T^T \tilde{b}, c = \tilde{c}\), we have

\[|u(x, t) - \tau t - \frac{x^T A x}{2} - b^T x - c| \leq Ce^{-\tau t} (1 + |x|^2)^{\frac{2n+2}{2}}, \quad \frac{1}{2} |x|^2 - t \text{ is large enough.} \tag{52}\]

If \(t^* \leq t \leq 0\), then by (52), we have

\[u(x, t) = \tau t + \frac{x^T A x}{2} + b^T x + c + O(|x|^{2-n}), \quad |x| \to \infty, t^* \leq t \leq 0.\]

If \(t \leq t^*\), by defining

\[u^*(x, t) = u(x, t + t^*) \quad \text{in } \mathbb{R}^{n+1},\]
we have that $c_0 \leq u^*_t \leq C_0$ in $\mathbb{R}^{n+1}_-$,

$$u^*_t - \log \left( \det(D^2 u^*) \right) = f(x, t + t^*) = a \quad \text{in } \mathbb{R}^{n+1}_-$$

and

$$|u^*(x, t) - \tau(t + t^*) - \frac{x^T A x}{2} - b^T x - c| \leq C e^{-\tau t} (1 + |x|^2)^{\frac{2-n}{2}}, \quad \frac{1}{2} |x|^2 - t \to +\infty.$$

Now we define

$$E^*(x, t) := u^*(x, t) - \tau(t + t^*) - \frac{x^T A x}{2} - b^T x - c,$$

then

$$E^*(x, t) \to 0, \quad \text{as } \frac{1}{2} |x|^2 - t \to +\infty.$$

Since

$$E^*_t - \widehat{a}_{ij} D_{ij} E^* = 0 \quad \text{in } \mathbb{R}^{n+1}_-,$$

where

$$(\widehat{a}_{ij}) = \int_0^1 (\theta D^2 u + (1 - \theta) A)^{-1} d\theta,$$

by the maximum principle, $E^* \equiv 0$ in $\mathbb{R}^{n+1}_-$, i.e.,

$$u^*(x, t) = \tau(t + t^*) + \frac{x^T A x}{2} + b^T x + c \quad \text{in } \mathbb{R}^{n+1}_-.$$

In conclusion, $u(x, t) = \tau t + \frac{x^T A x}{2} + b^T x + c - \tau t^*$ in $\mathbb{R}^{n+1}_-$. □

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