Exponential stability of an active constrained layer beam actuated by a voltage source without magnetic effects✩

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ABSTRACT

We study the boundary stabilization of an active constrained layer (ACL) beam consisting of a stiff layer, a viscoelastic layer and a piezoelectric layer. The piezoelectric layer is actuated by a voltage source without magnetic effects. The system is modeled as a Rayleigh beam coupled with two wave equations. By using an asymptotic technique, we present the asymptotic expressions for the eigenpairs of the system. We show that the generalized eigenfunctions form a Riesz basis in the state space, and hence the spectrum determined growth condition holds. Finally, the exponential stability of the closed-loop system is established.

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1. Introduction

Active constrained layer (ACL) damping treatments have been one of the main research topics in the vibration suppression of various flexible structures due to their reliability, robustness, low weight, adjustability, and high efficiency [1,2,16,19,22]. The ACL damping treatments integrate both active and passive dampings through constrained layer treatments [18]. The ACL beam is composite material, which usually represents a beam treated with an ACL damping treatment. A typical design of an ACL beam consists of a viscoelastic shear (passive damping) layer sandwiched between a piezoelectric constraining (active damping) layer and the vibrating structure (e.g. an Euler–Bernoulli beam) [17].

Much research of composite beam systems has been carried out in the past decade, and many methods, such as Riesz basis approach, multiplier method, Carleman estimates, Lyapunov’s energy method, and so on, have been used to analyze the systems. For example, the exponential stability and the exact controllability of a sandwich beam system with a boundary control are considered using the Riesz basis approach in [24]. The exact controllability of a Rao–Nakra sandwich beam with boundary controls is studied by the multiplier method in [14]. The analyticity of the solution and the exponential stability of a sandwich beam system are

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obtained via the Riesz basis approach in [23]. By the method of Carleman estimates, the exact controllability results of a multilayer plate system with clamped (or hinged) and free boundary conditions are obtained in [6,7], respectively. The boundary feedback stabilization of a multilayer Rao–Nakra sandwich beam is investigated via the Riesz basis approach and the compact perturbation method in [12]. Recently, the exact controllability of a multilayer Rao–Nakra sandwich beam with different boundary conditions: clamped, hinged, and clamped-hinged is considered using the multiplier method in [13]. In [15], the exponential stability of a laminated beam with interfacial slip and frictional damping is investigated by the energy method. An ACL beam actuated by charge (or current) is studied in [11], where the exponential stability of the closed-loop system with mechanical feedback controls is considered.

In this paper, we consider an ACL beam consisting of three layers: a viscoelastic layer sandwiched between a stiff (bottom) layer and a piezoelectric (top) layer. The piezoelectric layer is actuated by a voltage source without magnetic effects. The ACL beam is modeled by the following equations [10]:

\[
\begin{align*}
\rho_\varphi h_\varphi \varphi_{tt} - \alpha_\varphi h_\varphi \varphi_{xx} - G \phi &= 0, & 0 < x < L, \ t > 0, \\
\rho_\psi h_\psi \psi_{tt} - \alpha_\psi h_\psi \psi_{xx} + G \phi &= 0, & 0 < x < L, \ t > 0, \\
mw_{tt} - K_1 w_{xxtt} + K_2 w_{xxxx} - GH \phi_x &= 0, & 0 < x < L, \ t > 0, \\
\phi &= \frac{1}{h} (-\varphi + \psi + H w_x),
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
\varphi(0,t) &= \psi(0,t) = w(0,t) = w_x(0,t) = 0, \\
\alpha_\varphi h_\varphi \varphi_x(L,t) &= u_1(t), \\
\alpha_\psi h_\psi \psi_x(L,t) &= u_2(t), \\
K_2 w_{xx}(L,t) &= u_3(t), \\
K_1 w_{xtt}(L,t) - K_2 w_{xxx}(L,t) + GH \phi(L,t) &= 0,
\end{align*}
\]

and the initial data

\[
(\varphi, \varphi_t, \psi, \psi_t, w, w_t)(x,0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1),
\]

where \(\varphi\) and \(\psi\) are the longitudinal displacements of bottom and top layers respectively, \(w\) denotes the transverse displacement of each layer (the transverse displacements of three layers are considered to be equal), and \(u_i\) \((i = 1, 2, 3)\) are the boundary controls. Moreover, \(\rho_\varphi, \rho_\psi, h_\varphi, h_\psi, G, m, H\) and \(K_i\) \((i = 1, 2)\) are positive parameters. For details of (1.1) and its physical analysis, the reader can refer to [9,10].

For brevity in making, we make the following transformation

\[
\begin{align*}
u(x,t) &= \varphi \left( Lx, \sqrt{\frac{mL^4}{K_2}} \right), \ w(x,t) = \psi \left( Lx, \sqrt{\frac{mL^4}{K_2}} \right), \ h_1 = \frac{1}{h}, \\
y(x,t) &= w \left( Lx, \sqrt{\frac{mL^4}{K_2}} \right), \ \alpha_u = \frac{\alpha_\varphi mL^2}{\rho_\varphi K_2}, \ G_u = \frac{Gl}{\rho_\varphi h_\varphi K_2}, \ K = \frac{H}{L}, \\
\alpha_v &= \frac{\alpha_\varphi mL^2}{\rho_\varphi K_2}, \ G_v = \frac{Gl}{\rho_\varphi h_\varphi K_2}, \ I = \frac{K_1}{ML^2}, \ U_1(t) = \frac{L}{\alpha_v h_\varphi} u_1 \left( \sqrt{\frac{mL^4}{K_2}} \right), \\
G_1 &= \frac{GH}{K_2}, \ U_2(t) = \frac{L}{\alpha_v h_\varphi} u_2 \left( \sqrt{\frac{mL^4}{K_2}} \right), \ U_3(t) = \frac{L^2}{\alpha_v K_2} u_3 \left( \sqrt{\frac{mL^4}{K_2}} \right), \\
\varphi_0(Lx) &= \varphi_0(Lx), \ u_1(x) = \sqrt{\frac{mL^4}{K_2}} \varphi_1(Lx), \ \psi_0(x) = \psi_0(Lx), \\
\psi_1(x) &= \sqrt{\frac{mL^4}{K_2}} \psi_1(Lx), \ \psi_0(x) = \psi_0(Lx), \ y_1(x) &= \sqrt{\frac{mL^4}{K_2}} w_1(Lx).
\end{align*}
\]
Then (1.1) can be written as

$$
\begin{align*}
&\begin{cases}
    u_{tt} - \alpha_u u_{xx} - G_u z = 0, & 0 < x < 1, \ t > 0, \\
v_{tt} - \alpha_v v_{xx} + G_v z = 0, & 0 < x < 1, \ t > 0, \\
y_{tt} - Ky_{xxtt} + y_{xxxx} - G_1 z_x = 0, & 0 < x < 1, \ t > 0, \\
z = h_1(-u + v + Ky_x),
\end{cases}
\end{align*}
$$

with the boundary conditions

$$
\begin{align*}
&\begin{cases}
    u(0, t) = v(0, t) = y(0, t) = y_x(0, t) = 0, \\
    u_x(1, t) = U_1(t), \\
v_x(1, t) = U_2(t), \\
y_{xx}(1, t) = U_3(t), \\
Iy_{xxtt}(1, t) - y_{xxxx}(1, t) + G_1 z(1, t) = 0,
\end{cases}
\end{align*}
$$

and the initial data

$$(u, u_t, v, v_t, y, y_t)(x, 0) = (u_0, u_1, v_0, v_1, y_0, y_1).$$

We propose the following boundary feedback controls

$$U_1(t) = -l_1 u_t(1, t), \ U_2(t) = -l_2 v_t(1, t), \ U_3(t) = -l_3 y_{xxt}(1, t),$$

where $l_i \ (i = 1, 2, 3)$ are positive constant feedback gains. Then the closed-loop system is

$$
\begin{align*}
&\begin{cases}
    u_{tt} - \alpha_u u_{xx} - G_u z = 0, & 0 < x < 1, \ t > 0, \\
v_{tt} - \alpha_v v_{xx} + G_v z = 0, & 0 < x < 1, \ t > 0, \\
y_{tt} - Ky_{xxtt} + y_{xxxx} - G_1 z_x = 0, & 0 < x < 1, \ t > 0, \\
z = h_1(-u + v + Ky_x),
\end{cases}
\end{align*}
$$

with the boundary conditions

$$
\begin{align*}
&\begin{cases}
    u(0, t) = v(0, t) = y(0, t) = y_x(0, t) = 0, \\
    u_x(1, t) = -l_1 u_t(1, t), \\
v_x(1, t) = -l_2 v_t(1, t), \\
y_{xx}(1, t) = -l_3 y_{xxt}(1, t), \\
Iy_{xxtt}(1, t) - y_{xxxx}(1, t) + G_1 z(1, t) = 0,
\end{cases}
\end{align*}
$$

and the initial data

$$(u, u_t, v, v_t, y, y_t)(x, 0) = (u_0, u_1, v_0, v_1, y_0, y_1).$$

The energy function of the system (1.3) is given by

$$E(t) = \frac{1}{2} \int_0^1 \left( \frac{1}{G_u} u_t^2 + \frac{1}{G_v} v_t^2 + \frac{K}{G_1} y_{tt}^2 + \frac{\alpha_u}{G_u} u_x^2 + \frac{\alpha_v}{G_v} v_x^2 + \frac{IK}{G_1} y_{xxt}^2 + \frac{K}{G_1} y_{xx}^2 + \frac{1}{h_1} z_x^2 \right) dx. \quad (1.4)$$

It is easy to check that $\dot{E}(t) \leq 0$. So the energy of the system (1.3) is nonincreasing.
Our purpose is to prove that the closed-loop system (1.3) is exponentially stable in the state Hilbert space by Riesz basis approach. To this end, we first prove that the system is well-posed in the state space and the system operator has compact resolvent. Secondly, we show that there are three branches of asymptotic eigenvalues for the system. Finally, we prove that the generalized eigenfunctions of the system form a Riesz basis in the state space and hence the spectrum determined growth condition and the exponential stability hold for the system.

The paper is organized as follows. In section 2, we establish the well-posedness of the system. In section 3, we give asymptotic estimates of the eigenvalues for the system. Section 4 deals with the asymptotic expansion of the corresponding eigenfunctions. Finally, in section 5, we obtain the main result that the generalized eigenfunctions of the system form a Riesz basis in the state Hilbert space. Therefore, the spectrum determined growth condition and the exponential stability are concluded.

2. Well-posedness of the system

In this section, we transform system (1.3) into an abstract evolution equation and then discuss the well-posedness of the system.

Integrating the third equation of (1.3a) over $[x, 1]$ with respect to the spatial variable yields

\[
\begin{align*}
\begin{cases}
u_{tt} - \alpha_v v_{xx} + G_v z &= 0, \\y_{tt} - \alpha_y y_{xxx} + G_1 z &= 0, \\z = h_1(-u + v + Ky_x).
\end{cases}
&= \int_0^1 \left[ \int_0^{\sqrt{\frac{1}{T}} (1 - \tau)} d\tau \right] d\tau \\
&= \int_0^1 \frac{1}{\cosh \left( \sqrt{\frac{1}{T}} x \right)} dx \\
&+ \int_0^x \frac{1}{\cosh \left( \sqrt{\frac{1}{T}} (x - \tau) \right)} d\tau, \quad \forall g \in L^2(0, 1).
\end{align*}
\]

Lemma 2.1. $A$ has a continuous inverse in $L^2(0, 1)$ and is given by

\begin{align*}
A^{-1} g(x) &= -\int_0^1 \frac{g(\tau) \sinh \left[ \sqrt{\frac{1}{T}} (1 - \tau) \right]}{\cosh \left( \sqrt{\frac{1}{T}} x \right)} d\tau \\
&= \int_0^x \frac{1}{\cosh \left( \sqrt{\frac{1}{T}} (x - \tau) \right)} d\tau, \quad \forall g \in L^2(0, 1).
\end{align*}

For detailed proof of Lemma 2.1, we refer the reader to [5, Lemma 2.1]. Hence, system (1.3) can be written as

\[
\begin{align*}
u_{tt} - \alpha_v v_{xx} + G_v z &= 0, \quad 0 < x < 1, \ t > 0, \\
y_{tt} - \alpha_y y_{xxx} + G_1 z &= 0, \quad 0 < x < 1, \ t > 0, \\
z = h_1(-u + v + Ky_x), \quad 0 < x < 1, \ t > 0.
\end{align*}
\]
with the boundary conditions

\[
\begin{align*}
    u(0, t) &= v(0, t) = y(0, t) = y_x(0, t) = 0, \\
    u_x(1, t) &= -l_1u_t(1, t), \\
    v_x(1, t) &= -l_2v_t(1, t), \\
    y_{xx}(1, t) &= -l_3y_{tt}(1, t).
\end{align*}
\]

Let \( H^2_E(0, 1) = \{ f \in H^2(0, 1) : f(0) = f'(0) = 0 \} \). We consider system (1.3) in the state Hilbert space

\[
\mathcal{H} := H^1_E(0, 1) \times L^2(0, 1) \times H^1_E(0, 1) \times L^2(0, 1) \times H^2_E(0, 1) \times H^1_E(0, 1)
\]

with inner product induced norm

\[
\| (u_1, u_2, v_1, v_2, y_1, y_2) \|^2_{\mathcal{H}} = \int_0^1 \left( \frac{1}{G_u} |u_2|^2 + \frac{\alpha_u}{G_v} |u'_1|^2 + \frac{1}{G_v} |v_2|^2 + \frac{\alpha_v}{G_1} |v'_1|^2 + \frac{K}{G_1} |y_2|^2 + \frac{IK}{G_1} |y'_1|^2 \\
    + h_1 | -u_1 + v_1 + Ky'_1|^2 \right) dx, \quad \forall (u_1, u_2, v_1, v_2, y_1, y_2) \in \mathcal{H}.
\]

Define a new variable \( \mathcal{A}: \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H} \) by

\[
\mathcal{A} = \begin{pmatrix}
    u_1 \\
    u_2 \\
    v_1 \\
    v_2 \\
    y_1 \\
    y_2
\end{pmatrix} =
\begin{pmatrix}
    u_2 \\
    \alpha_u u'_1 + G_u h_1 (-u_1 + v_1 + Ky'_1) \\
    v_2 \\
    \alpha_v v'_1 - G_v h_1 (-u_1 + v_1 + Ky'_1) \\
    y_1 \\
    A^{-1} y''_1 - G_1 h_1 A^{-1} (-u_1 + v_1 + Ky'_1)
\end{pmatrix}
\]

with

\[
\mathcal{D}(\mathcal{A}) = \left\{ (u_1, u_2, v_1, v_2, y_1, y_2)^T \in \left( (H^2(0, 1) \times H^1_E(0, 1))^2 \times H^3(0, 1) \times H^2_E(0, 1) \right) \cap \mathcal{H} : \\
    u'_1(1) = -l_1u_2(1), v'_1(1) = -l_2v_2(1), y''_1(1) = -l_3y'_2(1) \right\}.
\]

Denote \( Y(t) := (u(\cdot, t), u_1(\cdot, t), v(\cdot, t), v_1(\cdot, t), y(\cdot, t), y_1(\cdot, t))^T \), then system (1.3) can be formulated into an evolution equation in \( \mathcal{H} \)

\[
\begin{align*}
    \frac{dY(t)}{dt} &= \mathcal{A}Y(t), & t &> 0, \\
    Y(0) &= (u_0, u_1, v_0, v_1, y_0, y_1)^T.
\end{align*}
\]

**Lemma 2.2.** Let \( \mathcal{A} \) be defined by (2.6) and (2.7). Then \( \mathcal{A}^{-1} \) exists and is compact on \( \mathcal{H} \). Hence, the spectrum \( \sigma(\mathcal{A}) \) consists entirely of isolated eigenvalues only.

**Proof.** For any \( Q = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H} \), we look for \( Y = (u_1, u_2, v_1, v_2, y_1, y_2)^T \in \mathcal{D}(\mathcal{A}) \) such that \( \mathcal{A}Y = Q \), which yields
\[
\begin{aligned}
\begin{cases}
\alpha_u u'' + G_u h_1 (-u_1 + v_1 + Ky'_1) = f_2(x), \\
\alpha_v v'' - G_v h_1 (-u_1 + v_1 + Ky'_1) = f_4(x), \\
A^{-1}y''_1 - G_1 h_1 A^{-1} (-u_1 + v_1 + Ky'_1) = f_6(x), \\
u_2(x) = f_1(x), \quad v_2(x) = f_3(x), \quad y_2(x) = f_5(x), \\
u_1(0) = v_1(0) = y_1(0) = 0, \\
u'_1(1) = -l_1 u_2(1), \quad v'_1(1) = -l_2 v_2(1), \quad y'_1(1) = -l_3 y_2(1).
\end{cases}
\end{aligned}
\]  

(2.9)

Let \( z_1(x) := -u_1(x) + v_1(x) + Ky'_1(x) \). From (2.9), we have

\[
\begin{aligned}
\begin{cases}
z''_1(x) = \alpha z_1(x) + q_1(x), \quad \alpha := \frac{G_u h_1}{\alpha_u} + \frac{G_v h_1}{\alpha_v} + G_1 K h_1, \\
z_1(0) = 0, \quad z'_1(1) = \beta := l_1 f_1(1) - l_2 f_3(1) - l_3 K f'_1(1)
\end{cases}
\end{aligned}
\]

(2.10)

with

\[
q_1(x) := -\frac{1}{\alpha_u} f_2(x) + \frac{1}{\alpha_v} f_4(x) + K A f_6(x).
\]

We solve equation (2.10) and obtain

\[
z_1(x) = \left( \beta - \frac{1}{\alpha} \int_0^x \cosh \left( \sqrt{\alpha} (1 - \tau) \right) q_1(\tau) d\tau \right) \frac{\sinh \left( \sqrt{\alpha} x \right)}{\sqrt{\alpha} \cosh \sqrt{\alpha}} + \frac{1}{\sqrt{\alpha}} \int_0^x \sinh \left( \sqrt{\alpha} (x - \tau) \right) q_1(\tau) d\tau.
\]

(2.11)

By the first equation of (2.9), (2.11) and the boundaries \( u_1(0) = 0, \ u'_1(1) = -l_1 u_2(1) \), we have

\[
\begin{aligned}
u_1(x) &= \int_0^x (x - \tau) q_2(\tau) d\tau - \left( l_1 f_1(1) + \int_0^1 q_2(\tau) d\tau \right) x, \quad q_2(x) := -\frac{G_u h_1}{\alpha_u} z_1(x) + \frac{1}{\alpha_u} f_2(x). \\
\end{aligned}
\]

(2.12)

Similarly, \( v_1(x) \) and \( y_1(x) \) can be solved as follows:

\[
\begin{aligned}
v_1(x) &= \int_0^x (x - \tau) q_3(\tau) d\tau - \left( l_2 f_3(1) + \int_0^1 q_3(\tau) d\tau \right) x, \quad q_3(x) := \frac{G_v h_1}{\alpha_v} z_1(x) + \frac{1}{\alpha_v} f_4(x) \\
\end{aligned}
\]

(2.13)

and

\[
\begin{aligned}
y_1(x) &= \frac{1}{2} \int_0^x (x - \tau)^2 q_4(\tau) d\tau - \left( \frac{l_3 f'_1(1)}{2} + \int_0^1 q_4(\tau) d\tau \right) x^2, \quad q_4(x) := G_1 h_1 z_1(x) + A f_6(x). \\
\end{aligned}
\]

(2.14)

Therefore \( A^{-1} \) exists and is bounded. In light of the Sobolev embedding theorem \([8]\), \( A^{-1} \) is compact. The proof is complete. \( \square \)

**Lemma 2.3.** Let \( A \) be defined by (2.6) and (2.7). Then \( A \) generates a \( C_0 \)-semigroup \( e^{At} \) of contractions on \( \mathcal{H} \).

**Proof.** We first prove that \( A \) is dissipative in \( \mathcal{H} \). For any \( Y = (u_1, u_2, v_1, v_2, y_1, y_2)^T \in \mathcal{D}(A) \), we denote

\[
\tilde{\phi} := h_1 \left( -u_1 + v_1 + Ky'_1 \right), \quad \tilde{\psi} := A^{-1} y''_1 - G_1 h_1 A^{-1} \left( -u_1 + v_1 + Ky'_1 \right).
\]
Since

\[
\langle AY, Y \rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} u_2, \alpha_u u''_1 + G_u \tilde{\phi}, v_2, \alpha_v v''_1 - G_v \tilde{\phi}, y_2, \tilde{\psi} \end{pmatrix}^T, (u_1, u_2, v_1, v_2, y_1, y_2)^T \right\rangle_{\mathcal{H}}
\]

\[
= \frac{1}{G_u} \int_0^1 (\alpha_u u''_1 + G_u \tilde{\phi}) \overline{u_2} dx + \frac{\alpha_u}{G_u} \int_0^1 u'_2 \overline{u'_1} dx + \frac{1}{G_v} \int_0^1 (\alpha_v v''_1 - G_v \tilde{\phi}) \overline{v_2} dx
\]

\[
+ \frac{\alpha_v}{G_v} \int_0^1 v'_2 \overline{v'_1} dx + \frac{K}{G_1} \int_0^1 \overline{\tilde{\psi}} \overline{y_2} dx + \frac{IK}{G_1} \int_0^1 \overline{\tilde{\psi}} \overline{y_2} dx + \frac{1}{G_1} \int_0^1 y''_2 \overline{y'_1} dx
\]

\[
+ \int_0^1 (-u_2 + v_2 + Ky'_2) \tilde{\phi} dx
\]

\[
= -\frac{\alpha_u}{l_1 G_u} |u'_1(1)|^2 - \frac{\alpha_v}{l_2 G_v} |v'_1(1)|^2 - \frac{\alpha_u}{G_u} \int_0^1 \left( u'_1 \overline{u'_2} - u'_2 \overline{u'_1} \right) dx
\]

\[
- \frac{K}{l_3 G_1} |y''_1(1)|^2 - \frac{\alpha_v}{G_v} \int_0^1 \left( v'_1 \overline{v'_2} - v'_2 \overline{v'_1} \right) dx
\]

\[
+ \int_0^1 \left[ \tilde{\phi} \left( \overline{\tilde{u}_2} - \overline{\tilde{v}_2} - Ky'_2 \right) \right] dx - (u_2 - v_2 - Ky'_2) \tilde{\phi} dx,
\]

it follows that

\[
\text{Re}(\langle AY, Y \rangle_{\mathcal{H}}) = -\frac{\alpha_u}{l_1 G_u} |u'_1(1)|^2 - \frac{\alpha_v}{l_2 G_v} |v'_1(1)|^2 - \frac{K}{l_3 G_1} |y''_1(1)|^2 \leq 0.
\]

Hence \( A \) is dissipative in \( \mathcal{H} \). Therefore, \( A \) generates a \( C_0 \)-semigroup \( e^{At} \) of contractions on \( \mathcal{H} \) by using the Lumer–Phillips theorem. \( \square \)

Let \( \lambda \in \sigma(A) \) and \( Y_\lambda := (u_1, u_2, v_1, v_2, y_1, y_2)^T \in \mathcal{D}(A) \) be an eigenfunction corresponding to \( \lambda \). From \( AY_\lambda = \lambda Y_\lambda \), we obtain \( u_2 = \lambda u_1, \ v_2 = \lambda v_1, \ y_2 = \lambda y_1 \) and

\[
\begin{align*}
\alpha_u u''_1 + G_u h_1 (-u_1 + v_1 + Ky'_1) &= \lambda^2 u_1, \\
\alpha_v v''_1 - G_v h_1 (-u_1 + v_1 + Ky'_1) &= \lambda^2 v_1, \\
A^{-1} y''_1 - G_1 h_1 A^{-1} (-u_1 + v_1 + Ky'_1) &= \lambda^2 y_1,
\end{align*}
\]

\[
u_1(0) = 0, \ u'_1(1) = -\lambda l_1 u_1(1),
\]

\[
v_1(0) = 0, \ v'_1(1) = -\lambda l_2 v_1(1),
\]

\[
y_1(0) = y'_1(0) = 0, \ y''_1(1) = -\lambda l_3 y'_1(1).
\]

Lemma 2.4. If \( \lambda \in \sigma(A) \), then \( \text{Re} \lambda < 0 \).
Proof. From (2.15), we have

\[
\begin{cases}
\alpha_u u''_1 + G_u h_1 (-u_1 + v_1 + Ky'_1) = \lambda^2 u_1,
\alpha_v v''_1 - G_v h_1 (-u_1 + v_1 + Ky'_1) = \lambda^2 v_1, \\
y''_1 - G_1 h_1 (-u_1' + v_1' + Ky''_1) = \lambda^2 Iy''_1 - \lambda^2 y_1,
\end{cases}
\]

(2.16)

\[
u(0) = 0, \quad u_1'(1) = -\lambda u_1 u_1(1),
\]

\[
v(0) = 0, \quad v_1'(1) = -\lambda v_1 v_1(1),
\]

\[
y(0) = y_1'(0) = 0, \quad y_1'(1) = -\lambda y_1 y_1(1),
\]

\[
y_1''(1) - G_1 h_1 [-u_1(1) + v_1(1) + Ky_1'(1)] - \lambda^2 Iy_1''(1) = 0.
\]

For simplicity, set \(k_1 = \frac{1}{G_u h_1}, \quad k_2 = \frac{1}{G_v h_1}\) and \(k_3 = \frac{K}{G_1 h_1}\). Multiplying \(k_1 \overline{u_1}\), the conjugate of \(k_1 u_1\), on both side of the first equation in (2.16) and integrating from 0 to 1 with respect to \(x\), we obtain

\[
\lambda_1 \alpha_u k_1 |u_1(1)|^2 + \lambda^2 k_1 \int_0^1 |u_1|^2 dx + \alpha_u k_1 \int_0^1 |u_1'|^2 dx - \frac{1}{0} (-u_1 + v_1 + Ky_1') \overline{u_1} dx = 0.
\]

(2.17)

Similarly, multiplying \(k_2 \overline{v_1}\), the conjugate of \(k_2 v_1\), on both side of the second equation in (2.16) and integrating from 0 to 1 with respect to \(x\), we obtain

\[
\lambda_2 \alpha_v k_2 |v_1(1)|^2 + \lambda^2 k_2 \int_0^1 |v_1|^2 dx + \alpha_v k_2 \int_0^1 |v_1'|^2 dx + \frac{1}{0} (-u_1 + v_1 + Ky_1') \overline{v_1} dx = 0.
\]

(2.18)

In addition, multiplying \(k_3 \overline{y_1}\), the conjugate of \(k_3 y_1\), on both side of the third equation in (2.16) and integrating from 0 to 1 with respect to \(x\), we obtain

\[
\lambda k_3 |y_1'(1)|^2 + \lambda^2 \left( k_3 I \int_0^1 |y_1'|^2 dx + k_3 \int_0^1 |y_1|^2 dx \right) + k_3 \int_0^1 |y_1''|^2 dx + \frac{1}{0} (-u_1 + v_1 + Ky_1') \overline{y_1'} dx = 0.
\]

(2.19)

From (2.17), (2.18) and (2.19), we have

\[
\lambda \left[ l_1 \alpha_u k_1 |u_1(1)|^2 + l_2 \alpha_v k_2 |v_1(1)|^2 + l_3 k_3 |y_1'(1)|^2 \right]
\]

\[
+ \lambda^2 \left( k_1 \int_0^1 |u_1|^2 dx + k_2 \int_0^1 |v_1|^2 dx + I k_3 \int_0^1 |y_1'|^2 dx + k_3 \int_0^1 |y_1|^2 dx \right)
\]

\[
+ \alpha_u k_1 \int_0^1 |u_1'|^2 dx + \alpha_v k_2 \int_0^1 |v_1'|^2 dx + k_3 \int_0^1 |y_1''|^2 dx + \frac{1}{0} (-u_1 + v_1 + Ky_1') \overline{y_1'} dx = 0.
\]

(2.20)

Let \(\lambda = \text{Re}\lambda + i\text{Im}\lambda\), where \(\text{Re}\lambda\) and \(\text{Im}\lambda\) are real. We have

\[
\text{Re}\lambda \left[ l_1 \alpha_u k_1 |u_1(1)|^2 + l_2 \alpha_v k_2 |v_1(1)|^2 + l_3 k_3 |y_1'(1)|^2 \right]
\]

\[
+ \left[ (\text{Re}\lambda)^2 - (\text{Im}\lambda)^2 \right] \left( k_1 \int_0^1 |u_1|^2 dx + k_2 \int_0^1 |v_1|^2 dx + I k_3 \int_0^1 |y_1'|^2 dx + k_3 \int_0^1 |y_1|^2 dx \right)
\]
\begin{equation}
+ \alpha_u k_1 \int_0^1 |u_1'|^2 dx + \alpha_v k_2 \int_0^1 |v_1'|^2 dx + k_3 \int_0^1 |y_1'|^2 dx + \int_0^1 | - u_1 + v_1 + Ky_1'|^2 dx = 0, \tag{2.21}
\end{equation}

and

\begin{equation}
\text{Im} \lambda \left[ l_1 \alpha_u k_1 |u_1(1)|^2 + l_2 \alpha_v k_2 |v_1(1)|^2 + l_3 k_3 |y_1'(1)|^2 \right] \\
+ 2 \text{Re} \lambda \text{Im} \lambda \left( k_1 \int_0^1 |u_1|^2 dx + k_2 \int_0^1 |v_1|^2 dx + I k_3 \int_0^1 |y_1'|^2 dx + k_3 \int_0^1 |y_1|^2 dx \right) = 0. \tag{2.22}
\end{equation}

If \( \text{Im} \lambda = 0 \), then \( \text{Re} \lambda < 0 \) follows from (2.21). If \( \text{Im} \lambda \neq 0 \), then \( \text{Re} \lambda < 0 \) by (2.22). The proof is complete. \( \square \)

We next formulate the eigenvalue problem for the operator \( \mathcal{A} \). For simplicity of notation, we define

\begin{equation}
r_1 := \sqrt{\frac{1}{\alpha_u}}, \quad r_2 := \sqrt{\frac{1}{\alpha_v}}, \quad c_1 := \frac{G_u h_1}{\alpha_u}, \quad c_2 := \frac{G_v h_1}{\alpha_v}, \quad c_3 := G_1 h_1. \tag{2.23}
\end{equation}

Then (2.15) can be written as

\[
\begin{cases}
r_1^2 \lambda^2 u_1 - c_1 Ky_1' - c_1 v_1 + c_1 u_1 - u_1'' = 0, \\
r_2^2 \lambda^2 v_1 + c_2 Ky_1' + c_2 v_1 - c_2 u_1 - v_1'' = 0, \\
\lambda^2 I y_1'' - \lambda^2 y_1 + c_3 Ky_1'' + c_3 v_1' - c_3 u_1' - y_1''' = 0, \\
u_1(0) = 0, \quad u_1'(1) = -\lambda l_1 u_1(1), \\
v_1(0) = 0, \quad v_1'(1) = -\lambda l_2 v_1(1), \\
y_1(0) = y_1'(0) = 0, \quad y_1''(1) = -\lambda l_3 y_1'(1), \\
y_1'''(1) = -c_3 u_1(1) + c_3 v_1(1) + (c_3 K + \lambda^2 I) y_1'(1).
\end{cases} \tag{2.24}
\]

We use the matrix operator pencil method [20] to solve the equations (2.24). Set

\begin{equation}
r_3 := \sqrt{T}, \quad \tilde{u}_1 := u_1, \quad \tilde{u}_2 := u_1', \quad \tilde{v}_1 := v_1, \quad \tilde{v}_2 := v_1', \quad \tilde{y}_1 := y_1, \quad \tilde{y}_2 := y_1', \quad \tilde{y}_3 := y_1'', \quad \tilde{y}_4 := y_1''' \tag{2.25}
\end{equation}

and

\begin{equation}
\Phi := (\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)^T. \tag{2.26}
\end{equation}

Then (2.24) becomes

\[
\begin{cases}
T^D(\lambda) \Phi := \Phi'(x) + M(\lambda) \Phi(x) = 0, \\
T^R(\lambda) \Phi := W^0(\lambda) \Phi(0) + W^1(\lambda) \Phi(1) = 0,
\end{cases} \tag{2.27}
\]

where

\begin{equation}
W^0(\lambda) := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}, \tag{2.28}
\end{equation}

\( O_{4 \times 8} \)
\[
W^1(\lambda) := \begin{pmatrix}
\lambda l_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda l_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda l_3 & 1 & 0 & 0 & 0 & 0 \\
c_3 & 0 & -c_3 & 0 & 0 & -c_3K - r_3^2\lambda^2 & 0 & 1 \\
\end{pmatrix},
\tag{2.29}
\]
and
\[
M(\lambda) := D_0 - \lambda^2 D_1
\tag{2.30}
\]
with \(D_0, D_1\) defined by
\[
D_0 := \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_1 & 0 & c_1 & 0 & 0 & c_1K & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
c_2 & 0 & -c_2 & 0 & 0 & -c_2K & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & c_3 & 0 & -c_3 & 0 & 0 & -c_3K & 0 \\
\end{pmatrix},
\tag{2.31}
\]
\[
D_1 := \begin{pmatrix}
r_1^2 D_{11} & O_{2\times2} & O_{2\times2} & O_{2\times2} \\
O_{2\times2} & r_2^2 D_{11} & O_{2\times2} & O_{2\times2} \\
O_{2\times2} & O_{2\times2} & r_2^2 D_{11} & O_{2\times2} \\
O_{2\times2} & O_{2\times2} & O_{2\times2} & -D_{11} & r_3^2 D_{11} \\
\end{pmatrix}, \quad D_{11} := \begin{pmatrix}
0 & 0 \\
1 & 0 \\
\end{pmatrix}.
\tag{2.32}
\]

**Theorem 2.1.** The characteristic equation (2.15) is equivalent to the first order linear system (2.27). Also \(\lambda \in \sigma(A)\) if and only if (2.27) has a nontrivial solution.

3. Asymptotic behavior of eigenvalues

In this section, we shall give the asymptotic expressions for the eigenvalues of \(A\). We first solve system (2.27) and obtain a fundamental matrix solution by modifying a standard technique of Birkhoff–Langer [3] and Tretter [20] (or [21]) for dealing with the matrix operator pencils. Then we derive the asymptotic expressions of the eigenvalues through the characteristic determinant \(\Delta(\lambda)\) of (2.27). A key step is providing an invertible matrix transformation in calculation.

Firstly, we diagonalize the leading term \(\lambda^2 D_1\) in (2.30). For any \(0 \neq \lambda \in \mathbb{C}\), define an invertible matrix in \(\lambda\) by
\[
P(\lambda) := \begin{pmatrix}
P_1(\lambda) \\
P_2(\lambda) \\
P_3(\lambda) \\
\end{pmatrix}, \quad P_1(\lambda) := \begin{pmatrix}
r_1^2 \lambda & r_1 \lambda \\
r_1^2 \lambda^2 & -r_1^2 \lambda^2 \\
\end{pmatrix}, \quad P_2(\lambda) := \begin{pmatrix}
r_2 \lambda & r_2 \lambda \\
r_2 \lambda^2 & -r_2 \lambda^2 \\
\end{pmatrix}, \quad P_3(\lambda) := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{r_3} & 1 & -1 \\
0 & 0 & r_3 \lambda & r_3 \lambda \\
0 & 0 & r_2 \lambda^2 & -r_2 \lambda^2 \\
\end{pmatrix}.
\tag{3.1}
\]
For any $\lambda \neq 0$, by a simple computation, we obtain

\[
P^{-1}(\lambda) := \begin{pmatrix}
P_1^{-1}(\lambda) & P_2^{-1}(\lambda) \\
P_2^{-1}(\lambda) & P_3^{-1}(\lambda)
\end{pmatrix}, \quad P_1^{-1}(\lambda) := \begin{pmatrix}
\frac{1}{2r_1\lambda} & \frac{1}{2r_1\lambda^2} \\
\frac{1}{2r_2\lambda} & \frac{1}{2r_2\lambda^2}
\end{pmatrix},
\]

with

\[
P_2^{-1}(\lambda) := \begin{pmatrix}
\frac{1}{2r_2\lambda} & \frac{1}{2r_2\lambda^2} \\
\frac{1}{2r_1\lambda} & \frac{1}{2r_1\lambda^2}
\end{pmatrix}, \quad P_3^{-1}(\lambda) := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & r_2 & 0 & -\frac{1}{r_2} \lambda \\
0 & 0 & \frac{1}{2r_3\lambda} & \frac{1}{2r_3\lambda^2} \\
0 & 0 & \frac{1}{2r_3\lambda^2} & -\frac{1}{2r_3\lambda^2}
\end{pmatrix}.
\]

For simplicity, let

\[
\begin{cases}
b_1 := \frac{c_1}{r_1}, & b_2 := \frac{c_1 r_2}{r_1 r_3}, & b_3 := \frac{c_1 K}{r_1 r_3}, & b_4 := \frac{c_2 r_1}{r_2}, & b_5 := \frac{c_2}{r_2}, \\\nb_6 := \frac{c_2 K}{r_2 r_3}, & b_7 := \frac{c_3 r_1^2}{r_2 r_3}, & b_8 := \frac{c_3 r_2^2}{r_2 r_3}, & b_9 := \frac{c_3 K}{r_3}.
\end{cases}
\]

In addition, define

\[
\Psi(x) := P^{-1}(\lambda)\Psi(x), \quad \hat{T}^D(\lambda) := P^{-1}(\lambda)T^D(\lambda)P(\lambda).
\]

From (2.27), we have

\[
\hat{T}^D(\lambda)\Psi = \Psi'(x) - \hat{M}(\lambda)\Psi(x) = 0,
\]

where

\[
\hat{M}(\lambda) = -P^{-1}(\lambda)M(\lambda)P(\lambda) := \lambda\hat{M}_1 + \hat{M}_0 + \lambda^{-1}\hat{M}_{-1} + \lambda^{-2}\hat{M}_{-2},
\]

with

\[
\hat{M}_1 := \text{diag}(r_1, -r_1, r_2, -r_2, 0, 0, r_3, -r_3), \quad \hat{M}_0 := \begin{pmatrix} O_{5 \times 5} & \hat{M}_{5 \times 3} \\ \hat{M}_{3 \times 5} & O_{3 \times 3} \end{pmatrix}
\]

by

\[
\hat{M}_{5 \times 3} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{r_3} & 1 & -1 \end{pmatrix}, \quad \hat{M}_{3 \times 5} := \begin{pmatrix} c_3 r_1^2 & -c_3 r_1^2 & -c_3 r_2^2 & c_3 r_2^2 & -\frac{1}{r_2} \lambda \\ -\frac{b_7}{2} & \frac{b_7}{2} & -\frac{b_7}{2} & \frac{b_7}{2} & -\frac{1}{2r_3} \lambda \\ -\frac{b_5}{2} & \frac{b_5}{2} & -\frac{b_5}{2} & \frac{b_5}{2} & -\frac{1}{2r_3} \lambda \\ \frac{1}{r_3} & 1 & -1 \end{pmatrix},
\]

and

\[
\hat{M}_{-1} := \begin{pmatrix} \hat{M}_{1 \times 4} & O_{4 \times 4} \\ O_{4 \times 4} & \hat{M}_{1 \times 4} \end{pmatrix}, \quad \hat{M}_{-2} := \begin{pmatrix} O_{4 \times 5} & \hat{M}_{4 \times 3} \\ \hat{M}_{4 \times 3} & O_{4 \times 5} \end{pmatrix}
\]

by
\[ \hat{M}_{4 \times 4}^1 := \begin{pmatrix} \frac{b_1}{4} & \frac{b_2}{4} & -\frac{b_3}{4} & -\frac{b_4}{4} \\ -\frac{b_1}{4} & -\frac{b_2}{4} & \frac{b_3}{4} & \frac{b_4}{4} \\ \frac{b_5}{4} & \frac{b_6}{4} & -\frac{b_7}{4} & -\frac{b_8}{4} \\ -\frac{b_5}{4} & -\frac{b_6}{4} & \frac{b_7}{4} & \frac{b_8}{4} \end{pmatrix}, \]

\[ \hat{M}_{4 \times 4}^2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -c_3 r_3 K & -c_3 r_3 K & 0 \\ 0 & 0 & \frac{b_9}{2} & -\frac{b_9}{2} \\ 0 & 0 & -\frac{b_9}{2} & \frac{b_9}{2} \end{pmatrix}, \]

\[ \hat{M}_{3 \times 3} := \begin{pmatrix} -\frac{b_3^2}{2} & -\frac{b_3 r_3^2}{2} & -\frac{b_3 r_3^2}{2} \\ \frac{b_3^2}{2} & \frac{b_3 r_3^2}{2} & -\frac{b_3 r_3^2}{2} \\ \frac{b_3 r_3^2}{2} & \frac{b_3 r_3^2}{2} & \frac{b_3 r_3^2}{2} \end{pmatrix}. \]

Secondly, we give an asymptotic expression for the fundamental matrix solution of system (3.4).

**Theorem 3.1.** Let \( 0 \neq \lambda \in \mathbb{C} \), and \( \hat{M}(\lambda) \) given by (3.5) and assume that \( r_1 \neq r_2 \neq r_3 \). For \( x \in [0,1] \), set

\[ E(x, \lambda) := \text{diag} \left( e^{r_1 \lambda x}, e^{-r_1 \lambda x}, e^{r_2 \lambda x}, e^{-r_2 \lambda x}, e^{r_3 \lambda x}, e^{-r_3 \lambda x} \right). \] (3.8)

Then there exists a fundamental matrix solution \( \hat{\Psi}(x, \lambda) \) for system (3.4), which satisfies

\[ \hat{\Psi}'(x, \lambda) = \hat{M}(\lambda) \hat{\Psi}(x, \lambda) \] (3.9)

such that, for large enough \( |\lambda| \),

\[ \hat{\Psi}(x, \lambda) = \left( \hat{\Psi}_0(x) + \frac{\hat{\Theta}(x, \lambda)}{\lambda} \right) E(x, \lambda), \] (3.10)

where

\[ \hat{\Psi}_0(x) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cosh \left( \frac{1}{r_3} x \right) & \frac{1}{r_3} \sinh \left( \frac{1}{r_3} x \right) & 0 & 0 \\ 0 & 0 & 0 & 0 & r_3 \sinh \left( \frac{1}{r_3} x \right) & \cosh \left( \frac{1}{r_3} x \right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \] (3.11)

and

\[ \hat{\Theta}(x, \lambda) := \hat{\Psi}_1(x) + \lambda^{-1} \hat{\Psi}_2(x) + \cdots \] (3.12)

with all entries uniformly bounded in \([0,1]\).

**Proof.** It is easy to check that

\[ E'(x, \lambda) = \lambda \hat{M}_1 E(x, \lambda), \] (3.13)

where \( \hat{M}_1 \) is given in (3.6).
Now we look for a fundamental matrix solution of (3.9) in the form of

$$\hat{\Psi}(x, \lambda) = \left( \hat{\Psi}_0(x) + \lambda^{-1} \hat{\Psi}_1(x) + \cdots + \lambda^{-n} \hat{\Psi}_n(x) + \cdots \right) E(x, \lambda).$$

Then the left-hand side of (3.9) is

$$\hat{\Psi}'(x, \lambda) = \left( \hat{\Psi}'_0(x) + \lambda^{-1} \hat{\Psi}'_1(x) + \cdots + \lambda^{-n} \hat{\Psi}'_n(x) + \cdots \right) E(x, \lambda),$$

and the right-hand side of (3.9) is

$$\hat{M}(\lambda) \hat{\Psi}(x, \lambda) = \left( \lambda \hat{M}_1 + \hat{M}_0 + \lambda^{-1} \hat{M}_{-1} + \lambda^{-2} \hat{M}_{-2} \right) \left( \hat{\Psi}_0(x) + \lambda^{-1} \hat{\Psi}_1(x) + \cdots + \lambda^{-n} \hat{\Psi}_n(x) + \cdots \right) E(x, \lambda).$$

According to the coefficients of $\lambda^1, \lambda^0, \lambda^{-1}, \ldots, \lambda^{-n}, \ldots$, we have

$$\hat{\Psi}_0(x) \hat{M}_1 - \hat{M}_1 \hat{\Psi}_0(x) = 0,$$

$$\hat{\Psi}'_0(x) - \hat{M}_0 \hat{\Psi}_0(x) + \hat{\Psi}_1(x) \hat{M}_1 - \hat{M}_1 \hat{\Psi}_1(x) = 0,$$

$$\hat{\Psi}'_1(x) - \hat{M}_0 \hat{\Psi}_1(x) - \hat{M}_{-1} \hat{\Psi}_0(x) + \hat{\Psi}_2(x) \hat{M}_1 - \hat{M}_1 \hat{\Psi}_2(x) = 0,$$

$$\hat{\Psi}'_2(x) - \hat{M}_0 \hat{\Psi}_2(x) - \hat{M}_{-1} \hat{\Psi}_1(x) - \hat{M}_{-2} \hat{\Psi}_0(x) + \hat{\Psi}_3(x) \hat{M}_1 - \hat{M}_1 \hat{\Psi}_3(x) = 0,$$

$$\vdots$$

$$\hat{\Psi}'_n(x) - \hat{M}_0 \hat{\Psi}_n(x) - \hat{M}_{-1} \hat{\Psi}_{n-1}(x) - \hat{M}_{-2} \hat{\Psi}_{n-2}(x) + \hat{\Psi}_{n+1}(x) \hat{M}_1 - \hat{M}_1 \hat{\Psi}_{n+1}(x) = 0,$$

$$\vdots$$

Using the arguments in [20, p. 135] or [3], we deduce that there is an asymptotic fundamental matrix solution $\hat{\Psi}(x, \lambda)$ for system (3.9). It remains to show that the leading order term $\hat{\Psi}_0(x)$ is given by (3.11). Indeed, $\hat{\Psi}_0(x)$ can be determined by the matrix equations

$$\hat{\Psi}_0(x) \hat{M}_1 - \hat{M}_1 \hat{\Psi}_0(x) = 0, \quad (3.14)$$

and

$$\hat{\Psi}'_0(x) - \hat{M}_0 \hat{\Psi}_0(x) + \hat{\Psi}_1(x) \hat{M}_1 - \hat{M}_1 \hat{\Psi}_1(x) = 0, \quad (3.15)$$

where $\hat{M}_1$ and $\hat{M}_0$ are given in (3.6). If $\hat{\Psi}_0(x)$ is known, then one can deduce $\hat{\Psi}_1(x)$ of $\hat{\Theta}(x, \lambda)$ in (3.12) from (3.15) and

$$\hat{\Psi}'_1(x) - \hat{M}_0 \hat{\Psi}_1(x) - \hat{M}_{-1} \hat{\Psi}_0(x) + \hat{\Psi}_2(x) \hat{M}_1 - \hat{M}_1 \hat{\Psi}_2(x) = 0$$

with $\hat{M}_{-1}$ being given in (3.7). Similarly, we obtain all the terms $\hat{\Psi}_1(x), \hat{\Psi}_2(x), \ldots, \hat{\Psi}_n(x), \ldots$ of $\hat{\Theta}(x, \lambda)$ in (3.12). So the proof will be accomplished if we would find $\hat{\Psi}_0(x)$ in (3.10).

Let $c_{ij}(x)$ be the $(i, j)$-entry of the matrix $\hat{\Psi}_0(x)$ with $i, j = 1, 2, \ldots, 8$. Since $\hat{M}_1$ is diagonal, it follows from (3.14) and $r_1 \neq r_2 \neq r_3$ that the entries $c_{ij}(x)$ of $\hat{\Psi}_0(x)$ satisfy $c_{ij}(x) = 0 \ (i \neq j)$ except $c_{50}(x)$ and
where \( c_{65}(x) \). Moreover, the entries \( c_{ii}(x) \) \((i = 1, 2, \ldots, 8)\), \( c_{56}(x) \) and \( c_{65}(x) \) can be determined by substituting them into (3.15) to obtain

\[
\begin{align*}
\begin{cases}
\quad c'_{ii}(x) = 0, \text{ for } i = 1, 2, 3, 4, 7, 8, \\
\quad c_{55}(x) = \frac{1}{r^2_5} c_{65}(x), \quad c'_{56}(x) = \frac{1}{r^2_5} c_{66}(x), \\
\quad c_{65}(x) = c_{55}(x), \quad c'_{66}(x) = c_{56}(x).
\end{cases}
\end{align*}
\tag{3.16}
\]

Then (3.11) follows from \( \hat{\Psi}_0(0) = I \). The proof is complete. \( \square \)

The following corollary is the relationship of the solution between system (2.27) and system (3.4).

**Corollary 3.1.** Let \( 0 \neq \lambda \in \mathbb{C} \). Suppose that \( r_1 \neq r_2 \neq r_3 \) and \( \hat{\Psi}(x, \lambda) \) given by (3.10) is a fundamental matrix solution of system (3.4). Then

\[
\hat{\Phi}(x, \lambda) := P(\lambda)\hat{\Psi}(x, \lambda)
\tag{3.17}
\]

is a fundamental matrix solution for the first order linear system (2.27).

Thirdly, we estimate the asymptotic eigenvalues of the system. Note that the eigenvalues of the first order linear system (2.27) are given by the zeros of the characteristic determinant

\[
\Delta(\lambda) := \det \left(T^R(\lambda)\hat{\Phi}\right), \quad \lambda \in \mathbb{C},
\tag{3.18}
\]

where \( T^R(\lambda) \) is given in (2.27) and \( \hat{\Phi} \) is a fundamental matrix solution of \( T^D(\lambda)\Phi = 0 \) [20].

In fact, from (2.27), (3.10) and (3.17), we have

\[
T^R(\lambda)\Phi = W^0(\lambda)P(\lambda)\hat{\Psi}(0, \lambda) + W^1(\lambda)P(\lambda)\hat{\Psi}(1, \lambda).
\tag{3.19}
\]

Using (2.28) and (3.1), a simple computation gives

\[
W^0(\lambda)P(\lambda) = \begin{pmatrix}
    r_1 \lambda & r_1 \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & r_2 \lambda & r_2 \lambda & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{r^2_3} & 1 & -1
\end{pmatrix},
\]

Similarly,

\[
W^1(\lambda)P(\lambda) = \begin{pmatrix}
    r_4 r_4 \lambda^2 & r_5 r_5 \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & O_{4 \times 8} \\
    0 & 0 & r_6 r_6 \lambda^2 & r_7 r_7 \lambda^2 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & r_8 \lambda & r_9 \lambda & r_{10} \lambda \\
    c_3 r_1 \lambda & c_3 r_1 \lambda & -c_3 r_2 \lambda & -c_3 r_2 \lambda & 0 & -r_{11} - \lambda^2 & -c_3 K & c_4 K
\end{pmatrix},
\]

where

\[
\begin{align*}
\begin{cases}
\quad r_4 := l_1 + r_1, \quad r_5 := l_1 - r_1, \quad r_6 := l_2 + r_2, \quad r_7 := l_2 - r_2, \\
\quad r_8 := \frac{l_3}{r^2_3}, \quad r_9 := r_3 + l_3, \quad r_{10} := r_3 - l_3, \quad r_{11} := \frac{c_3 K}{r^2_3}.
\end{cases}
\tag{3.20}
\end{align*}
\]
For notational simplicity, set \([a]_1 := a + \mathcal{O}(\lambda^{-1})\) with \(\mathcal{O}(\lambda^{-m})\) satisfying
\[
\lim_{|\lambda| \to \infty} |\lambda^m \mathcal{O}(\lambda^{-m})| < \infty, \quad m \in \mathbb{Z}.
\]

From Theorem 3.1, an asymptotic fundamental matrix solution \(\tilde{\Psi}(x, \lambda)\) of system (3.4) can be given by
\[
\tilde{\Psi}(x, \lambda) := \left(\tilde{\Psi}_0(x) + \mathcal{O}(\lambda^{-1})\right) E(x, \lambda).
\] (3.21)

Since \(\tilde{\Psi}_0(0) = I\) and \(E(0, \lambda) = I\), a direct computation yields
\[
W^0(\lambda) P(\lambda) \tilde{\Psi}(0, \lambda) = \begin{pmatrix} M_{11} & M_{12} \\ O_{4 \times 4} & O_{4 \times 4} \end{pmatrix},
\]
where
\[
M_{11} := \begin{pmatrix} \lambda [r_1]_1 & \lambda [r_1]_1 & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) & \lambda [r_2]_1 & \lambda [r_2]_1 \\ \mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) \\ \mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) \end{pmatrix},
\]
and
\[
M_{12} := \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \\ [1]_1 & \mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) \\ \mathcal{O}(\lambda^{-1}) & \left[\frac{1}{\lambda E}\right]_1 & [1]_1 & [-1]_1 \end{pmatrix}.
\]

Similarly,
\[
W^1(\lambda) P(\lambda) \tilde{\Psi}(1, \lambda) = \begin{pmatrix} O_{4 \times 4} & O_{4 \times 4} \\ M_{21} & M_{22} \end{pmatrix},
\]
where
\[
M_{21} := \begin{pmatrix} \lambda^2 E_1 [r_1 r_4]_1 & \lambda^2 E_2 [r_1 r_5]_1 & \lambda E_3 \mathcal{O}(1) & \lambda E_4 \mathcal{O}(1) \\ \lambda E_1 \mathcal{O}(1) & \lambda E_2 \mathcal{O}(1) & \lambda^2 E_3 [r_2 r_6]_1 & \lambda^2 E_4 [r_2 r_7]_1 \\ E_1 \mathcal{O}(1) & E_2 \mathcal{O}(1) & E_3 \mathcal{O}(1) & E_4 \mathcal{O}(1) \\ \lambda E_1 [m_1]_1 & \lambda E_2 [m_1]_1 & \lambda E_3 [m_2]_1 & \lambda E_4 [m_2]_1 \end{pmatrix},
\]
and
\[
M_{22} := \begin{pmatrix} \lambda \mathcal{O}(1) & \lambda \mathcal{O}(1) & \lambda E_5 \mathcal{O}(1) & \lambda E_6 \mathcal{O}(1) \\ \lambda \mathcal{O}(1) & \lambda \mathcal{O}(1) & \lambda E_5 \mathcal{O}(1) & \lambda E_6 \mathcal{O}(1) \\ \lambda \left[r_8 r_3 \tilde{l}_2\right]_1 & \lambda \left[r_8 \tilde{l}_1\right]_1 & \lambda E_5 [r_9]_1 & \lambda E_6 [r_{10}]_1 \\ -\lambda^2 \left[r_3 \tilde{l}_2\right]_1 & -\lambda^2 \left[l_1\right]_1 & -\lambda E_5 \mathcal{O}(1) & -\lambda E_6 \mathcal{O}(1) \end{pmatrix},
\]
with
\[
\begin{align*}
E_1 &= e^{r_1 \lambda}, \quad E_2 = e^{-r_1 \lambda}, \quad E_3 := e^{r_2 \lambda}, \\
E_4 &= e^{-r_2 \lambda}, \quad E_5 := e^{r_3 \lambda}, \quad E_6 := e^{-r_3 \lambda},
\end{align*}
\] (3.22)

and \( m_1 = c_3 r_1 - \mathcal{O}(1), \quad m_2 = -c_3 r_2 - \mathcal{O}(1), \quad \tilde{t}_1 = \cosh \frac{1}{r_3}, \quad \tilde{t}_2 = \sinh \frac{1}{r_3}. \)

Hence, \( T^R(\lambda) \hat{\Phi} \) has the following asymptotic expression:

\[
W^0(\lambda)P(\lambda)\tilde{\Psi}(0, \lambda) + W^1(\lambda)P(\lambda)\tilde{\Psi}(1, \lambda) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.
\]

Therefore, the asymptotic expansion of the characteristic determinant can be expressed as

\[
\Delta(\lambda) = \det \left( W^0(\lambda)P(\lambda)\tilde{\Psi}(0, \lambda) + W^1(\lambda)P(\lambda)\tilde{\Psi}(1, \lambda) \right)
\]

\[= -\det \left( \begin{array}{cc} \lambda[r_1]_{1} & \lambda[r_1]_{1} \\ \lambda^2 E_1[r_1 r_4]_{1} & \lambda^2 E_2[r_1 r_5]_{1} \end{array} \right) \times \det \left( \begin{array}{ccc} \lambda[r_2]_{1} & \lambda[r_2]_{1} \\ \lambda^2 E_3[r_2 r_6]_{1} & \lambda^2 E_4[r_2 r_7]_{1} \end{array} \right) \times \det \left( \begin{array}{ccc} \lambda^{-1} & \lambda^{-1} & \lambda^{-1} \\ \lambda^{-1} & [\frac{1}{r_3}]_{1} & [\frac{1}{r_3}]_{1} \\ \lambda E_5[r_9]_{1} & \lambda E_6[r_{10}]_{1} \end{array} \right) \times \lambda^9 r_1^2 r_2^2 3_1 \Delta_1(\lambda) \Delta_2(\lambda) \Delta_3(\lambda),
\]

where

\[
\begin{align*}
\Delta_1(\lambda) &= r_5 E_2 - r_4 E_1 + \mathcal{O}(\lambda^{-1}), \\
\Delta_2(\lambda) &= r_7 E_4 - r_6 E_3 + \mathcal{O}(\lambda^{-1}), \\
\Delta_3(\lambda) &= r_9 E_6 + r_8 E_5 + \mathcal{O}(\lambda^{-1}).
\end{align*}
\] (3.23)

**Theorem 3.2.** If \( r_1 \neq r_2 \neq r_3 \) and if \( \Delta(\lambda) \) be the characteristic determinant of system (2.27), then the following asymptotic expansion for \( \Delta(\lambda) \) holds:

\[
\Delta(\lambda) = \lambda^9 r_1^2 r_2^2 3_1 \Delta_1(\lambda) \Delta_2(\lambda) \Delta_3(\lambda)
\] (3.24)

with \( \Delta_i(\lambda) \) being given in (3.23). If \( l_i \neq r_i \) (\( i = 1, 2, 3 \)), then there are three branches of asymptotic eigenvalues given by (as \( |n| \to \infty \) and \( n \in \mathbb{Z} \))

\[
\lambda_{jn} = \mu_j + r_j^{-1} n \pi i + \mathcal{O}(n^{-1}) \quad \text{for} \quad j = 1, 2, 3,
\] (3.25)

where

\[
\mu_j := \begin{cases} \frac{1}{2r_j} \ln \frac{l_j - r_j}{l_j + r_j}, & l_j > r_j \\ \frac{1}{2r_j} \left( \ln \frac{r_j - l_j}{l_j + r_j} + \pi i \right), & l_j < r_j \end{cases} \quad \text{for} \quad j = 1, 2, 3.
\] (3.26)

Moreover, we have, as \( |n| \to \infty \),

\[
\Re \lambda_{jn} \to \frac{1}{2r_j} \ln \left| \frac{l_j - r_j}{l_j + r_j} \right| < 0 \quad \text{for} \quad j = 1, 2, 3.
\] (3.27)

Furthermore, the zeros of \( \Delta(\lambda) \) are simple when their moduli are sufficiently large.
Proof. By $\Delta(\lambda) = 0$ and (3.24), we obtain that
\[ \Delta_1(\lambda)\Delta_2(\lambda)\Delta_3(\lambda) = 0, \] (3.28)
and
\[ \Delta_i(\lambda) = 0 \quad \text{for } i = 1, 2, 3. \]

Let $\Delta_1(\lambda) = 0$, we have
\[ r_5E_2 - r_4E_1 + O(\lambda^{-1}) = 0, \] (3.29)
that is (from (3.20) and (3.22))
\[ (l_1 - r_1)e^{-r_1\lambda} - (l_1 + r_1)e^{r_1\lambda} + O(\lambda^{-1}) = 0. \] (3.30)
The solutions of the equation
\[ (l_1 - r_1)e^{-r_1\lambda} - (l_1 + r_1)e^{r_1\lambda} = 0 \]
are given by
\[ \tilde{\lambda}_{1n} = \mu_1 + r_1^{-1}n\pi i, \quad n \in \mathbb{Z}. \]
By the Rouché’s theorem, the solutions of (3.30) can be expressed as
\[ \lambda_{1n} = \tilde{\lambda}_{1n} + O(n^{-1}) = \mu_1 + r_1^{-1}n\pi i + O(n^{-1}), \quad n \in \mathbb{Z} \text{ and } |n| \to \infty. \] (3.31)

Similarly, let $\Delta_2(\lambda) = 0$. Then the equation
\[ (l_2 - r_2)e^{-r_2\lambda} - (l_2 + r_2)e^{r_2\lambda} + O(\lambda^{-1}) = 0 \] (3.32)
has the solutions
\[ \lambda_{2n} = \mu_2 + r_2^{-1}n\pi i + O(n^{-1}), \quad n \in \mathbb{Z} \text{ and } |n| \to \infty. \] (3.33)

Also, let $\Delta_3(\lambda) = 0$. The equation
\[ (r_3 - l_3)e^{-r_3\lambda} + (r_3 + l_3)e^{r_3\lambda} + O(\lambda^{-1}) = 0 \] (3.34)
has the solutions
\[ \lambda_{3n} = \mu_3 + r_3^{-1}n\pi i + O(n^{-1}), \quad n \in \mathbb{Z} \text{ and } |n| \to \infty. \] (3.35)

Finally, by $r_1 \neq r_2 \neq r_3$, it is easy to see that the last assertion is concluded. \[ \square \]

Theorem 3.3. Suppose $r_1 \neq r_2 \neq r_3$ and $l_i \neq r_i$ ($i = 1, 2, 3$). Let $A$ be defined by (2.6) and (2.7). Then all eigenvalues of $A$ have the asymptotic expressions given by (3.25). Moreover, all eigenvalues of the system with sufficiently large moduli are simple.
4. Asymptotic behavior of eigenfunctions

In this section, we shall give the asymptotic expressions for the eigenfunctions of $A$.

**Theorem 4.1.** Suppose $r_1 \neq r_2 \neq r_3$ and $l_j \neq r_j$ ($i = 1, 2, 3$). Let $\sigma(A) := \{\lambda_{1n}, \lambda_{2n}, \lambda_{3n}, n \in \mathbb{Z}\}$ be the eigenvalues of $A$ with $\lambda_{jn}$ ($j = 1, 2, 3$) being given in (3.25). Then the corresponding eigenfunctions

$$\left\{ (\tilde{u}_{jn}, \lambda_{jn}\tilde{u}_{jn}, \tilde{v}_{jn}, \lambda_{jn}\tilde{v}_{jn}, \lambda_{jn}\tilde{y}_{jn})^T, \quad j = 1, 2, 3, \quad n \in \mathbb{Z} \right\}$$

have the following asymptotic expressions for $|n| \to \infty$, $n \in \mathbb{Z}$:

$$\begin{align*}
\tilde{u}_{1n}'(x) &= \cosh \left( (r_1\mu_1 + n\pi i)x + \mathcal{O}(n^{-1}) \right) + \mathcal{O}(n^{-1}), \\
\tilde{u}_{jn}'(x) &= \mathcal{O}(n^{-1}) \text{ for } j = 2, 3, \\
\lambda_{1n}\tilde{u}_{jn}(x) &= r_1^{-1} \sinh \left( (r_1\mu_1 + n\pi i)x + \mathcal{O}(n^{-1}) \right) + \mathcal{O}(n^{-1}), \\
\lambda_{jn}\tilde{u}_{jn}(x) &= \mathcal{O}(n^{-1}) \text{ for } j = 2, 3, \\
\tilde{v}_{2n}'(x) &= \cosh \left( (r_2\mu_2 + n\pi i)x + \mathcal{O}(n^{-1}) \right) + \mathcal{O}(n^{-1}), \\
\tilde{v}_{jn}'(x) &= \mathcal{O}(n^{-1}) \text{ for } j = 1, 3, \\
\lambda_{2n}\tilde{v}_{jn}(x) &= r_2^{-1} \sinh \left( (r_2\mu_2 + n\pi i)x + \mathcal{O}(n^{-1}) \right) + \mathcal{O}(n^{-1}), \\
\lambda_{jn}\tilde{v}_{jn}(x) &= \mathcal{O}(n^{-1}) \text{ for } j = 1, 3, \\
\tilde{y}_{3n}'(x) &= \cosh \left( (r_3\mu_3 + n\pi i)x + \mathcal{O}(n^{-1}) \right) + \mathcal{O}(n^{-1}), \\
\tilde{y}_{jn}'(x) &= \mathcal{O}(n^{-1}) \text{ for } j = 1, 2, \\
\lambda_{3n}\tilde{y}_{jn}(x) &= r_3^{-1} \sinh \left( (r_3\mu_3 + n\pi i)x + \mathcal{O}(n^{-1}) \right) + \mathcal{O}(n^{-1}), \\
\lambda_{jn}\tilde{y}_{jn}(x) &= \mathcal{O}(n^{-1}) \text{ for } j = 1, 2,
\end{align*}$$

where $r_1$, $r_2$ and $r_3$ are given in (2.23) and (2.25) respectively. Moreover,

$$\left\{ (\tilde{v}_{jn}, \lambda_{jn}\tilde{u}_{jn}, \tilde{v}_{jn}, \lambda_{jn}\tilde{v}_{jn}, \lambda_{jn}\tilde{y}_{jn})^T, \quad j = 1, 2, 3, \quad n \in \mathbb{Z} \right\}$$

are approximately normalized in $\mathcal{H}$ in the sense that there exist positive constants $d_1$ and $d_2$ independent of $n$ such that ($j = 1, 2, 3$)

$$d_1 \leq \|\tilde{v}_{jn}\|_{L^2} \cdot \|\lambda_{jn}\tilde{u}_{jn}\|_{L^2} \cdot \|\tilde{v}_{jn}\|_{L^2} \cdot \|\lambda_{jn}\tilde{v}_{jn}\|_{L^2} \cdot \|\tilde{y}_{jn}\|_{L^2} \cdot \|\lambda_{jn}\tilde{y}_{jn}\|_{L^2} \leq d_2$$

for all integers $n$.

**Proof.** Note that the $j$th component of $\Phi(x)$ in (2.26) with respect to the eigenvalue $\lambda$ can be obtained by taking the determinant of the matrices which are replaced one of the rows of $T^R(\lambda)\tilde{\Phi}$ in (3.19) by $e_j^T\tilde{\Phi}$ so that their determinants are not zero, where $e_j$ is the $j$th column of the identity matrix.

From (3.17), an asymptotic fundamental matrix solution $\tilde{\Phi}_1(x, \lambda)$ of system (2.27) can be given by

$$\tilde{\Phi}_1(x, \lambda) := P(\lambda)\tilde{\Psi}(x, \lambda),$$

where $\tilde{\Psi}(x, \lambda)$ is given in (3.21). A direct computation gives

$$\tilde{\Phi}_1(x, \lambda) = \begin{pmatrix} \hat{\Phi}_{11}(x, \lambda) & \hat{\Phi}_{12}(x, \lambda) \\ \hat{\Phi}_{21}(x, \lambda) & \hat{\Phi}_{22}(x, \lambda) \end{pmatrix},$$
where

\[
\tilde{\Phi}_{11}(x, \lambda) := \begin{pmatrix}
\lambda [r_1] e^{r_1 \lambda x} & \lambda [r_1] e^{-r_1 \lambda x} & \mathcal{O}(1) e^{r_2 \lambda x} & \mathcal{O}(1) e^{-r_2 \lambda x} \\
\lambda^2 \left[ r_1^2 \right]_1 e^{r_1 \lambda x} & -\lambda^2 \left[ r_1^2 \right]_1 e^{-r_1 \lambda x} & 0 & 0 \\
\mathcal{O}(1) e^{r_1 \lambda x} & -\mathcal{O}(1) e^{-r_1 \lambda x} & \lambda [r_2] e^{r_2 \lambda x} & -\lambda [r_2] e^{-r_2 \lambda x} \\
0 & 0 & \mathcal{O}(1) e^{r_2 \lambda x} & \mathcal{O}(1) e^{-r_2 \lambda x}
\end{pmatrix},
\]

\[
\tilde{\Phi}_{12}(x, \lambda) := \begin{pmatrix}
\mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) e^{r_3 \lambda x} & \mathcal{O}(1) e^{-r_3 \lambda x} \\
0 & 0 & 0 & 0 \\
\mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) e^{r_3 \lambda x} & \mathcal{O}(1) e^{-r_3 \lambda x} \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\tilde{\Phi}_{21}(x, \lambda) := \begin{pmatrix}
\mathcal{O}(\lambda^{-1}) e^{r_1 \lambda x} & \mathcal{O}(\lambda^{-1}) e^{-r_1 \lambda x} & \mathcal{O}(\lambda^{-1}) e^{r_2 \lambda x} & \mathcal{O}(\lambda^{-1}) e^{-r_2 \lambda x} \\
\mathcal{O}(\lambda^{-1}) e^{r_1 \lambda x} & \mathcal{O}(\lambda^{-1}) e^{-r_1 \lambda x} & \mathcal{O}(\lambda^{-1}) e^{r_2 \lambda x} & \mathcal{O}(\lambda^{-1}) e^{-r_2 \lambda x} \\
\mathcal{O}(\lambda^{-1}) e^{r_1 \lambda x} & \mathcal{O}(\lambda^{-1}) e^{-r_1 \lambda x} & \mathcal{O}(\lambda^{-1}) e^{r_2 \lambda x} & \mathcal{O}(\lambda^{-1}) e^{-r_2 \lambda x} \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

and

\[
\tilde{\Phi}_{22}(x, \lambda) := \begin{pmatrix}
\left[ \tilde{I}_3(x) \right]_1 & \left[ \tilde{I}_4(x) \right]_1 & \mathcal{O}(\lambda^{-1}) e^{r_3 \lambda x} & \mathcal{O}(\lambda^{-1}) e^{-r_3 \lambda x} \\
\left[ \tilde{I}_3(x) \right]_1 & \left[ \tilde{I}_4(x) \right]_1 & \mathcal{O}(\lambda^{-1}) e^{r_3 \lambda x} & \mathcal{O}(\lambda^{-1}) e^{-r_3 \lambda x} \\
\mathcal{O}(1) & \mathcal{O}(1) & \lambda [r_3] e^{r_3 \lambda x} & \lambda [r_3] e^{-r_3 \lambda x} \\
0 & 0 & \lambda^2 \left[ r_3^2 \right]_1 e^{r_3 \lambda x} & -\lambda^2 \left[ r_3^2 \right]_1 e^{-r_3 \lambda x}
\end{pmatrix},
\]

with \( \tilde{I}_3(x) := \cosh \left( \frac{1}{r_3} x \right) \) and \( \tilde{I}_4(x) := \sinh \left( \frac{1}{r_3} x \right) \).

Thus, the first component of \( \Phi(x) \) is given by

\[
\tilde{u}_1(x, \lambda) = -\lambda^{-8} r_1^{-2} r_2^{-2} \tilde{r}_1^{-1} \det \left( \begin{array}{cc}
\lambda [r_1]_1 & \lambda [r_1]_1 \\
\lambda [r_1]_1 & \lambda [r_1]_1 e^{-r_1 \lambda x}
\end{array} \right) \times \det \left( \begin{array}{cc}
\lambda [r_2]_1 & \lambda [r_2]_1 \\
\lambda^2 E_3 [r_2 r_6]_1 & \lambda^2 E_4 [r_2 r_7]_1
\end{array} \right) \times \det \left( \begin{array}{cc}
\mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) \\
\mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) \\
\lambda \left[ r_8 \tilde{r}_2 \tilde{r}_3 \right]_1 & \lambda \left[ r_8 \tilde{r}_1 \tilde{r}_3 \right]_1 \\
\lambda E_5 [r_9]_1 & \lambda E_6 [r_{10}]_1 \end{array} \right) \times \det \left( \begin{array}{cc}
\mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) \\
\mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) \\
\lambda E_5 [r_9]_1 & \lambda E_5 [r_{10}]_1 \end{array} \right) \times \mathcal{O}(\lambda^{-1})
\]

\[
\tilde{u}_1(x, \lambda) = \begin{cases}
\frac{r_{12}(\lambda) \left( e^{-r_1 \lambda x} - e^{r_1 \lambda x} + \mathcal{O}(\lambda^{-1}) \right)}{\mathcal{O}(\lambda^{-1})} & \text{if } \lambda = \lambda_{1n}, \\
\frac{r_{12}(\lambda) \left( e^{-r_1 \lambda x} - e^{r_1 \lambda x} + \mathcal{O}(\lambda^{-1}) \right)}{\mathcal{O}(\lambda^{-1})} & \text{if } \lambda = \lambda_{2n} \text{ or } \lambda_{3n},
\end{cases}
\]

From (3.23), (3.25) and (3.28), it follows that
where

$$r_{12}(\lambda) := (r_7 E_4 - r_6 E_3 + O(\lambda^{-1})) (r_{10} E_6 + r_9 E_5 + O(\lambda^{-1})) .$$

(4.4)

In the same manner we can see that the third component of $\Phi(x)$ is given by

$$\tilde{v}_1(x, \lambda) = -\lambda^{-8} r_1^{-2} r_2^{-2} \tilde{t}_1^{-1} \det \begin{pmatrix}
\lambda [r_1]_1 & \lambda [r_1]_1 \\
\lambda^2 E_1 [r_1 r_4]_1 & \lambda^2 E_2 [r_1 r_5]_1
\end{pmatrix}
\times \det \begin{pmatrix}
\lambda [r_2]_1 & \lambda [r_2]_1 & \lambda [r_2]_1 e^{-r_2 \lambda x} \\
\lambda [r_2]_1 e^{r_2 \lambda x} & \lambda [r_2]_1 & \lambda [r_2]_1 e^{-r_2 \lambda x}
\end{pmatrix}
\times \det \begin{pmatrix}
[1]_1 & O(\lambda^{-1}) & O(\lambda^{-1}) & O(\lambda^{-1}) \\
O(\lambda^{-1}) & [\frac{1}{r_7}]_1 & [1]_1 & -[1]_1 \\
\lambda [r_8 \tilde{t}_2 r_3]_1 & \lambda [r_8 \tilde{t}_1]_1 & \lambda E_5 [r_9]_1 & \lambda E_6 [r_{10}]_1 \\
-\lambda^2 \tilde{t}_2 r_3 & -\lambda^2 \tilde{t}_1 & -\lambda E_5 \Omega(1) & -\lambda E_6 \Omega(1)
\end{pmatrix}
= (r_5 E_2 - r_4 E_1 + O(\lambda^{-1})) (e^{-r_2 \lambda x} - e^{r_2 \lambda x} + O(\lambda^{-1}))
\times (r_{10} E_6 + r_9 E_5 + O(\lambda^{-1})) .$$

By (3.23), (3.25) and (3.28), we have

$$\tilde{v}_1(x, \lambda) = \begin{cases}
r_{13}(\lambda) (e^{-r_2 \lambda x} - e^{r_2 \lambda x} + O(\lambda^{-1})) & \text{if } \lambda = \lambda_{2n}, \\
O(\lambda^{-1}) & \text{if } \lambda = \lambda_{1n} \text{ or } \lambda_{3n},
\end{cases}$$

(4.5)

where

$$r_{13}(\lambda) := (r_5 E_2 - r_4 E_1 + O(\lambda^{-1})) (r_{10} E_6 + r_9 E_5 + O(\lambda^{-1})) .$$

(4.6)

Furthermore, the sixth component of $\Phi(x)$ can be given by

$$\tilde{y}_2(x, \lambda) = -\lambda^{-8} r_1^{-2} r_2^{-2} \tilde{t}_1^{-1} \det \begin{pmatrix}
\lambda [r_1]_1 & \lambda [r_1]_1 \\
\lambda^2 E_1 [r_1 r_4]_1 & \lambda^2 E_2 [r_1 r_5]_1
\end{pmatrix}
\times \det \begin{pmatrix}
\lambda [r_2]_1 & \lambda [r_2]_1 \\
\lambda [r_2]_1 e^{r_2 \lambda x} & \lambda [r_2]_1 e^{-r_2 \lambda x}
\end{pmatrix}
\times \det \begin{pmatrix}
[1]_1 & O(\lambda^{-1}) & O(\lambda^{-1}) & O(\lambda^{-1}) \\
O(\lambda^{-1}) & [\frac{1}{r_7}]_1 & [1]_1 & -[1]_1 \\
[\frac{1}{r_3}]_1 & [\frac{1}{r_3}]_1 & [1]_1 e^{r_3 \lambda x} & -[1]_1 e^{-r_3 \lambda x} \\
-\lambda^2 \tilde{t}_2 r_3 & -\lambda^2 \tilde{t}_1 & -\lambda E_5 \Omega(1) & -\lambda E_6 \Omega(1)
\end{pmatrix}
= - (r_5 E_2 - r_4 E_1 + O(\lambda^{-1})) (r_7 E_4 - r_6 E_3 + O(\lambda^{-1}))
\times (e^{-r_3 \lambda x} - e^{r_3 \lambda x} + O(\lambda^{-1})) .$$

From (3.23), (3.25) and (3.28), we see that

$$\tilde{y}_2(x, \lambda) = \begin{cases}
r_{14}(\lambda) (e^{-r_3 \lambda x} - e^{r_3 \lambda x} + O(\lambda^{-1})) & \text{if } \lambda = \lambda_{3n}, \\
O(\lambda^{-1}) & \text{if } \lambda = \lambda_{1n} \text{ or } \lambda_{2n},
\end{cases}$$

(4.7)
where
\[ r_{14}(\lambda) := - \left( r_2 E_2 - r_4 E_1 + O(\lambda^{-1}) \right) (r_7 E_4 - r_6 E_3 + O(\lambda^{-1})) . \]

Together with \((3.25), (4.1)\) can be deduced from \((4.3)-(4.8)\) by setting
\[
\tilde{u}_n(x) = - \frac{\tilde{u}_1(x, \lambda)}{2r_1 \lambda r_{12}(\lambda)}, \quad \tilde{v}_n(x) = - \frac{\tilde{v}_1(x, \lambda)}{2r_2 \lambda r_{13}(\lambda)}, \quad \tilde{y}_n(x) = - \frac{\tilde{y}_2(x, \lambda)}{2r_3 \lambda r_{14}(\lambda)} .
\]

Finally, it follows from \((3.25)\) that
\[
\begin{align*}
\left\| e^{-r_j \lambda n x} \right\|_{L^2} &= \frac{1 - e^{-2r_j \mu_j}}{2r_j \mu_j} + O(n^{-1}) \quad \text{for } j = 1, 2, 3, \\
\left\| e^{r_j \lambda n x} \right\|_{L^2} &= \frac{e^{2r_j \mu_j} - 1}{2r_j \mu_j} + O(n^{-1}) \quad \text{for } j = 1, 2, 3,
\end{align*}
\]
where \(\mu_j \ (j = 1, 2, 3)\) are given in \((3.26)\). These together with \((4.1)\) yield \((4.2)\).

\section{The Riesz basis property and exponential stability of the system}

In this section, we prove that the generalized eigenfunctions of \(A\) form a Riesz basis for \(\mathcal{H}\). Furthermore, we establish the exponential stability of the system \((1.3)\) by Riesz basis approach \([4]\).

Let \(Y_j := (u_j, \xi_j, v_j, \zeta_j, y_j, \eta_j)^T \in \mathcal{H} \ (j = 1, 2)\), define a new inner product in \(\mathcal{H}\) by
\[
\langle Y_1, Y_2 \rangle_{\mathcal{H}} := \langle u'_1, u'_2 \rangle_{L^2} + \langle \xi_1, \xi_2 \rangle_{L^2} + \langle v'_1, v'_2 \rangle_{L^2} + \langle \zeta_1, \zeta_2 \rangle_{L^2} + \langle y'_1, y'_2 \rangle_{L^2} + \langle \eta'_1, \eta'_2 \rangle_{L^2},
\]
and write its induced norm by \(\| \cdot \|_{\mathcal{H}}\) which is equivalent to \((2.5)\). It is easy to check that \(\mathcal{H}\) is a Hilbert space under \((5.1)\). For convenience, we introduce another Hilbert space
\[
\mathcal{L} := (L^2(0, 1))^6
\]
with an inner product (for any \(X_j := (u_j, \xi_j, v_j, \zeta_j, y_j, \eta_j)^T \in \mathcal{L}, \ j = 1, 2\))
\[
\langle X_1, X_2 \rangle_{\mathcal{L}} := \langle u_1, u_2 \rangle_{L^2} + \langle \xi_1, \xi_2 \rangle_{L^2} + \langle v_1, v_2 \rangle_{L^2} + \langle \zeta_1, \zeta_2 \rangle_{L^2} + \langle y_1, y_2 \rangle_{L^2} + \langle \eta_1, \eta_2 \rangle_{L^2},
\]
and define the subspaces of \(\mathcal{H}\) and \(\mathcal{L}\), respectively, by
\[
\begin{align*}
\mathcal{H}_1 &:= \{ Y \in \mathcal{H} \mid Y = (u, \xi, 0, 0, 0, 0)^T \}, \\
\mathcal{H}_2 &:= \{ Y \in \mathcal{H} \mid Y = (0, 0, v, \zeta, 0, 0)^T \}, \\
\mathcal{H}_3 &:= \{ Y \in \mathcal{H} \mid Y = (0, 0, 0, 0, y, \eta)^T \},
\end{align*}
\]
and
\[
\begin{align*}
\mathcal{L}_1 &:= \{ X \in \mathcal{L} \mid X = (u, \xi, 0, 0, 0, 0)^T \}, \\
\mathcal{L}_2 &:= \{ X \in \mathcal{L} \mid X = (0, 0, v, \zeta, 0, 0)^T \}, \\
\mathcal{L}_3 &:= \{ X \in \mathcal{L} \mid X = (0, 0, 0, 0, y, \eta)^T \}.
\end{align*}
\]

Obviously, we have
\[
\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \quad \text{and} \quad \mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3,
\]
where the sign \(\oplus\) denotes the direct sum in the sense of orthogonality with respect to the inner products \(\langle \cdot, \cdot \rangle_{\mathcal{H}}\) and \(\langle \cdot, \cdot \rangle_{\mathcal{L}}\) in \(\mathcal{H}\) and \(\mathcal{L}\), respectively.
We recall a lemma [25, Lemma 5.2] which is needed in establishing the Riesz basis property of the operator \( \mathcal{A} \).

**Lemma 5.1.** Let \( \{ \phi_n(x), n \in \mathbb{N} \} \) and \( \{ 1, \psi_n(x), n \in \mathbb{N} \} \) be two subsets in \( L^2(0, 1) \) defined by, respectively,

\[
\phi_n(x) := \sin n\pi x \quad \text{and} \quad \psi_n(x) := \cos n\pi x \quad \forall \, x \in (0, 1), \, n \in \mathbb{N}.
\]

Then \( \{ \phi_n(x), n \in \mathbb{N} \} \) and \( \{ 1, \psi_n(x), n \in \mathbb{N} \} \) are two orthogonal bases in \( L^2(0, 1) \). Moreover, for any scalars \( \alpha_1, \beta_1 \neq 0 \in \mathbb{C} \) the vector family

\[
\{ \Psi_n := (\cosh [(\alpha_1 + i\pi)x], \beta_1 \sinh [(\alpha_1 + i\pi)x])^T, \, n \in \mathbb{Z} \}
\]

forms a Riesz basis on the Hilbert space \( L^2(0, 1) \times L^2(0, 1) \).

**Theorem 5.1.** Suppose \( r_1 \neq r_2 \neq r_3 \) and \( l_i \neq r_i \) (\( i = 1, 2, 3 \)). Let

\[
\Psi_{jn} := (\tilde{u}_{jn}', \lambda_j \tilde{u}_{jn}, \tilde{v}_{jn}', \lambda_j \tilde{v}_{jn}, \tilde{y}_{jn}', \lambda_j \tilde{y}_{jn})^T, \quad j = 1, 2, 3, \, n \in \mathbb{Z}, \quad (5.4)
\]

where the entries are given as (4.1) corresponding to the eigenvalues \( \lambda_{jn} \). Then \( \{ \Psi_{1n}, \Psi_{2n}, \Psi_{3n}, n \in \mathbb{Z} \} \) forms a Riesz basis in Hilbert space \( \mathcal{L} \) provided that \( \{ \Psi_{jn}, \, j = 1, 2, 3, \, n \in \mathbb{Z} \} \) is \( \omega \)-linearly independent in \( \mathcal{L} \).

**Proof.** Define three vector families by

\[
\Phi_{1n} := (\cosh [(r_1\mu_1 + i\pi)x], r_1^{-1} \sinh [(r_1\mu_1 + i\pi)x], 0, 0, 0, 0)^T, \quad (5.5)
\]

\[
\Phi_{2n} := (0, 0, \cosh [(r_2\mu_2 + i\pi)x], r_2^{-1} \sinh [(r_2\mu_2 + i\pi)x], 0, 0)^T, \quad (5.5)
\]

\[
\Phi_{3n} := (0, 0, 0, 0, \cosh [(r_3\mu_3 + i\pi)x], r_3^{-1} \sinh [(r_3\mu_3 + i\pi)x])^T. \quad (5.5)
\]

From Lemma 5.1, we see that the families \( \{ \Phi_{jn}, n \in \mathbb{Z} \} \) (\( j = 1, 2, 3 \)) are the Riesz bases for \( \mathcal{L}_j \), respectively. From the asymptotic expressions of eigenvalues (3.25) and eigenfunctions (4.1), we conclude that

\[
\begin{align*}
\| \tilde{u}_{1n}' - \cosh [(r_1\mu_1 + i\pi)x] \|_{L^2} &= \mathcal{O}(n^{-1}), \\
\| r_1 \lambda_{1n} \tilde{u}_{1n} - \sinh [(r_1\mu_1 + i\pi)x] \|_{L^2} &= \mathcal{O}(n^{-1}), \\
\| \tilde{v}_{2n}' - \cosh [(r_2\mu_2 + i\pi)x] \|_{L^2} &= \mathcal{O}(n^{-1}), \\
\| r_2 \lambda_{2n} \tilde{v}_{2n} - \sinh [(r_2\mu_2 + i\pi)x] \|_{L^2} &= \mathcal{O}(n^{-1}), \\
\| \tilde{y}_{3n}' - \cosh [(r_3\mu_3 + i\pi)x] \|_{L^2} &= \mathcal{O}(n^{-1}), \\
\| r_3 \lambda_{3n} \tilde{y}_{3n} - \sinh [(r_3\mu_3 + i\pi)x] \|_{L^2} &= \mathcal{O}(n^{-1}).
\end{align*}
\]

Hence, we obtain

\[
\begin{align*}
\| \Psi_{1n} - \Phi_{1n} \|_{\mathcal{L}_1} &= \mathcal{O}(n^{-1}), \\
\| \Psi_{2n} - \Phi_{2n} \|_{\mathcal{L}_2} &= \mathcal{O}(n^{-1}), \\
\| \Psi_{3n} - \Phi_{3n} \|_{\mathcal{L}_3} &= \mathcal{O}(n^{-1}).
\end{align*}
\]

Therefore, by Bari’s theorem [26], \( \{ \Psi_{1n}, \Psi_{2n}, \Psi_{3n}, n \in \mathbb{Z} \} \) forms a Riesz basis for \( \mathcal{L} \) provided that \( \{ \Psi_{jn}, \, j = 1, 2, 3, \, n \in \mathbb{Z} \} \) is \( \omega \)-linearly independent in \( \mathcal{L} \). \( \Box \)

We now come to the main results of the paper.
Theorem 5.2. Suppose $r_1 \neq r_2 \neq r_3$ and $l_i \neq r_i$ $(i = 1, 2, 3)$. Let $A$ be defined by (2.6) and (2.7). Then the generalized eigenfunctions of $A$ form a Riesz basis for $\mathcal{H}$.

Proof. Let $Y_{jn} := (\tilde{u}_{jn}, \lambda_j \tilde{v}_{jn}, \tilde{v}_{jn}, \lambda_j \tilde{y}_{jn}, \lambda_j \tilde{y}_{jn})^T$ $(j = 1, 2, 3$ and $n \in \mathbb{Z})$ be eigenfunctions corresponding to the eigenvalues $\lambda_{jn}$ $(j = 1, 2, 3)$ in which the entries are given in (4.1). Then $\{Y_{jn}, j = 1, 2, 3, n \in \mathbb{Z}\}$ is $\omega$-linearly independent in $\mathcal{H}$. The proof will be completed via an isomorphic mapping between two Hilbert spaces $\mathcal{H}$ and $\mathcal{L}$ that maps $Y_{jn}$ to $\Psi_{jn}$, where $\{\Psi_{jn}, j = 1, 2, 3, n \in \mathbb{Z}\}$ is given by (5.4).

To this end, for any $V := (s_1, s_2, s_3, s_4, s_5, s_6)^T \in \mathcal{H}$, we define a linear bounded operator $T : \mathcal{H} \to \mathcal{L}$ by
\[
TV = (s'_1, s_2, s'_3, s_4, s'_5, s'_6)^T := \hat{V}.
\]
Since $\langle V, Y_{jn} \rangle_H = \langle \hat{V}, \Psi_{jn} \rangle_L$, it is easily seen that $T$ is isomorphic and satisfies
\[
\|TV\|_L = \|\hat{V}\|_L = \|V\|_H. \tag{5.7}
\]
In particular, for $j = 1, 2, 3$ and $n \in \mathbb{Z}$,
\[
TY_{jn} = \Psi_{jn} \quad \text{and} \quad \|Y_{jn}\|_H = \|\Psi_{jn}\|_L.
\]
Moreover, $\{Y_{jn}, j = 1, 2, 3, n \in \mathbb{Z}\}$ is $\omega$-linearly independent in $\mathcal{H}$, so is $\{\Psi_{jn}, j = 1, 2, 3, n \in \mathbb{Z}\}$ in $\mathcal{L}$. Therefore, by Theorem 5.1, $\{\Psi_{jn}, j = 1, 2, 3, n \in \mathbb{Z}\}$ forms a Riesz basis in $\mathcal{L}$. Hence, $\{Y_{jn}, j = 1, 2, 3, n \in \mathbb{Z}\}$ forms a Riesz basis for $\mathcal{H}$. \Box

Theorem 5.3. Suppose $r_1 \neq r_2 \neq r_3$ and $l_i \neq r_i$ $(i = 1, 2, 3)$. Let $A$ be defined by (2.6) and (2.7), and $T(t)$ a $C_0$-semigroup generated by $A$ in $\mathcal{H}$. Then $T(t)$ is exponentially stable.

Proof. From Theorems 3.3 and 5.2, we see that the spectrum determined growth condition $\omega(A) = s(A)$ for $T(t)$ holds, where $\omega(A)$ is the growth order of $T(t)$ and $s(A)$ is the spectral bound of $A$. From Lemma 2.4, $\text{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$. This, together with (3.25) and the spectrum determined growth condition, shows that $T(t)$ is an exponentially stable semigroup in $\mathcal{H}$. \Box

References


