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The active disturbance rejection control of the rotating disk–beam system with boundary input disturbances

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ABSTRACT
This paper deals with the stabilisation of the rotating disk–beam system, where the control ends of beam and disk are suffered from disturbances, respectively. The active disturbance rejection control (ADRC) approach is adopted in investigation. The disturbances are first estimated by the extended state observers, and the observer-based feedback control laws are then designed to cancel the disturbances. When the angular velocity of the disk is less than the square root of the first nature frequency of the beam, it is shown that the feedback control laws are robust to the external disturbances, that is, the vibration of the beam can be suppressed while the disk rotating with a desired angular velocity in presence of the disturbances. Finally, the simulation results are provided to illustrate the effectiveness of ADRC approach.

1. Introduction
Many flexible structures in mechanical systems can be modelled by the Euler–Bernoulli beam equation interconnected with ordinary differential equation (ODE), such as flexible robots arms (Chen, Chentouf, & Wang, 2015), aerospace structures (Baillieul & Levi, 1987) and flexible marine risers Do & Pan (2008), etc. The stabilisation problem of the rotating flexible system has been the object of a considerable mathematical endeavour (Chen, Chentouf, & Wang, 2014; Chentouf & Wang, 2006, 2015; Laousy, Xu, & Sallet, 1996; Xu & Baillieul, 1993). Indeed, there are two categories of works: in the first one, only a torque control is exerted (Baillieul & Levi, 1987; Xu & Baillieul, 1993). In turn, in the second one, at least one boundary control (usually force or moment) is presented in the feedback law in addition of the torque control (Chentouf & Wang, 2006; Laousy et al., 1996). In the practical application, the disturbances (e.g., temperature, sound waves and vibration) and uncertainties (e.g., unknown system parameters) have strong adverse effects on performances of the flexible structures (He & Ge, 2015). To our knowledge, the stabilisation for the rotating disk–beam system with the disturbance suffered from the boundary control end is still open. The rotating disk–beam system is nonlinear coupling systems and the state of the disk and beam is interdependence. Hence, in the presence of the disturbances or uncertainties, the stability analysis of the rotating disk–beam systems is not easy.

Many control approaches have been developed to deal with the disturbances or uncertainties. These include the sliding mode control for systems with the external disturbances (Wang, Liu, Ren, & Chen, 2015), modelling uncertainties (Qin, Zhong, & Sun, 2013), the adaptive control for systems with unknown parameters (Ge, Zhang, & He, 2011; He, Ge, How, Choo, & Hong, 2011), and the Lyapunov approach for disturbance (Guo & Kang, 2014; He, Sun, & Ge, 2015), to name just a few. Most of these approaches are the worst-case concern strategy in dealing with disturbance. These control approaches designed are usually over-conservative, thus tend to result in using very large control energy (Feng & Guo, 2014), which is often unnecessary for any particular system. The active disturbance rejection control (ADRC), as an unconventional design approach, is first introduced by Han (2009).

In the past, the ADRC approach has been developed to obtain satisfactory performance of the closed-loop system, and its applications can be found in lots of literatures (Guo & Jin, 2013a, 2013b; Guo & Zhou, 2015; Xia, Dai, Fu, Li, & Wang, 2014). The ADRC strategy has been well known for its performance that estimate or compensate the system uncertainty or external disturbances. Thus, in the frame of ADRC, general nonlinear uncertainties or external disturbances are permitted (Xia, Shi, Liu, Rees, & Han, 2007). For example, the stabilisation problem of one-dimensional wave equation and Euler–Bernoulli beam equation with nonlinear disturbances have been
dealt in Feng and Guo (2014) and Guo and Jin (2013a) by the ADRC strategy, respectively. The ADRC has been also applied to DC–DC power converts in power electronics (Sun & Gao, 2005), high pointing accuracy and rotation speed (Li, Yang, & Yang, 2009), motion control (Gao, Hu, & Jiang, 2001), flight system (Huang, Xu, Han, & Lam, 2001) and other industrial systems (Xia, Chen, Pu, & Dai, 2014). For more detail of the ADRC, we can refer to the references in Guo and Zhao (2011).

The objective of this paper is to design the feedback control, by means of ADRC approach, to achieve stabilisation for the rotating disk–beam system with boundary input disturbances. The disturbances are estimated through extended state observers, and the observer-based feedback control laws are then designed for the rotating disk–beam system. The stabilisation and well-posedness of the closed-loop system are proved by dividing the system into two parts: one is for system (1) with controls, and another is for the observer system. For angular velocity of the disk less than a well-defined value, we show that the feedback controllers are robust to the external disturbances, that is, regardless of the presence of the disturbances on the control end, the beam vibrations can be suppressed to zero while the disk will rotate with the desired angular velocity as time goes to infinity. Moreover, we relaxed the desired angular velocity to the square root of the first frequency of the beam

\[
\sqrt{\mu_1 EI/\rho} \simeq 3.516 \sqrt{EI/\rho} \quad (EI \text{ and } \rho \text{ are, respectively, the flexural rigidity and the mass per unit length of the beam}).
\]

This improve the desired value \(3\sqrt{EI/\rho}\) in Laousy et al. (1996).

The rest of this paper is organised as follows. In Section 2, the model of the rotating disk–beam system is given, along with some results. In Section 3, a feedback control laws are designed for the disk–beam system via the ADRC method, where the ADRC is used to estimate the disturbances. Section 4 is devoted to stability analysis of the closed-loop system, respectively. In Section 5, simulation results are provided to demonstrate the effectiveness of the ADRC. The conclusion is presented in Section 6. When angular velocity of the disk is less than a critical value, the stability of the linear beam subsystem is proved in the Appendix.

### 2. Problem statement and preliminaries

This paper investigates the stabilisation of the rotating disk–beam system subject to the boundary disturbances. Assume that a beam (B) clamped at the center of a disk (D) and free at the other end. The disk rotates freely about its axis with a non-uniform angular velocity and the motion of the beam is confined to a plane perpendicular to the disk (see Figure 1). Indeed, we can imagine Figure 1 as a satellite, the beam being its antenna. The purpose of this paper is to show that under the effects of external disturbance, for any desired angular velocity of the disk less than a well-defined value, the beam vibrations can be forced to decay to zero while the disk will rotate with the desired angular velocity. The rotating disk–beam system with boundary moment control and disturbance as follows (see Figure 1):

\[
\begin{align*}
\rho u_{tt} + EI u_{xxxx} = \rho \omega^2(t) u, & \quad x \in (0, 1), t > 0, \\
u(0, t) = u_x(0, t) = 0, & \quad t \geq 0, \\
u_{xx}(1, t) = U_1(t) + d_1(t), & \quad t \geq 0, \\
u_{xxx}(1, t) = 0, & \quad t \geq 0, \\
\frac{d}{dt} \left\{ I_d + \rho \int_0^1 u^2 dx \, \omega(t) \right\} = U_2(t) + d_2(t), & \quad t > 0,
\end{align*}
\]

For simplicity and without loss of generality, assume that \(EI = \rho = 1\) (in fact a simple change of variables will lead to unit physical parameters). Let \(u(x, t)\) denote the transverse displacement of the beam at time \(t\) and position \(x\), \(\omega(t)\) be the angular velocity of the disk at time \(t\), \(I_d\) denote the disk's moment of inertia, \(U_1(t)\) be the control input through bend moment, and \(U_2(t)\) be the control to be applied on the disk. The functions \(d_1(t)\) and \(d_2(t)\) are the external disturbances acting on the beam and the disk, respectively, and we have

**Assumption:** \(|d_i(t)|, |\dot{d}_i(t)| \leq M, i = 1, 2,\) for \(M > 0\) and all \(t \geq 0\).

When \(d_1(t)\) and \(d_2(t)\) are absent, the collocated feedback controls are designed to stabilise system (1) in Laousy.
et al. (1996):

\[
\begin{align*}
U_1(t) &= -\alpha u_{xx}(1, t), \
U_2(t) &= -y(\omega(t) - \hat{\omega}), \
u_{xx}(1, t) &= \beta u_1(1, t),
\end{align*}
\]

where \(\hat{\omega} \in \mathbb{R}\) is the desired angular velocity of the disk. It is shown that the system is globally exponentially stable with the assumption \(\hat{\omega} < \sqrt{9EI/\rho}\).

Let \(H^2(0, 1) = \{u \in H^4(0, 1) : u(0) = u_x(0) = 0\}, \ n \in \mathbb{N}\). We consider system (1) in state space

\[X = \mathcal{H} \times \mathbb{R}, \quad \text{where } \mathcal{H} = H^2(0, 1) \times L^2(0, 1)\]

with the norm induced by the inner product: for \(X_i = (f_i, g_i, \omega_i) \in X, i = 1, 2,\)

\[(X_1, X_2)_X = \int_0^1 \left[ f''_1(x) f''_2(x) - \hat{\omega}^2 f_1(x) f_2(x) + g_1(x) g_2(x) \right] dx + \omega_1 \omega_2\]

which is equivalent to the usual one of \(H^2(0, 1) \times L^2(0, 1) \times \mathbb{R}\) provided that \(\hat{\omega} < \sqrt{\mu_1}\). Here, \(\mu_1\) is the first nature frequency of the beam which can be given by the first eigenvalue of the self-adjoint positive operator \(\mathcal{F}\) in \(L^2(0, 1)\):

\[
\mathcal{F} z = z'', \quad D(\mathcal{F}) = \{z \in H^4(0, 1) : z''(1) = z''(1) = 0 \}. \]

Moreover, \(\mu_1\) satisfies (see Luo & Guo, 1997): 1 + \(\cos(4\mu_1)\) \(\cosh(4\mu_1)\) = 0 and the numerical simulation tells \(\sqrt{\mu_1} \simeq 3.516\). In addition, for \(|\hat{\omega}| < \sqrt{\mu_1},\)

\[1 + \cos(\sqrt{|\hat{\omega}|}) \cosh(\sqrt{|\hat{\omega}|}) > 0. \quad (2)\]

We define the unbounded linear operators \(\hat{A}\) and \(B\) as follows:

\[
\begin{align*}
\hat{A}(y, z) &= (z, -y_{xxxx} + \hat{\omega}^2 y), \\
\mathcal{V}(y, z) &\in D(\hat{A})(\subset \mathcal{H}) \rightarrow \mathcal{H}, \quad D(\hat{A}) = \{ (y, z) \in \mathcal{H} \cap (H^4(0, 1) \\
\times H^2(0, 1)) : y_{xx}(1) = 0, y_{xxxx}(1) = 0 \}, \quad \mathcal{B} = (0, \delta(x - 1)),
\end{align*}
\]

and define the nonlinear function \(J\) in \(X = \mathcal{H} \times \mathbb{R}\): for \(\phi, \omega) \in X\) with \(\phi = (y, z) \in \mathcal{H},\)

\[
J(\phi, \omega, t) = \begin{pmatrix}
0, & (\omega^2(t) - \hat{\omega}^2)y, \\
-2\omega(t) \int_0^1 yzdx + U_2(t) + d_2(t)
\end{pmatrix}
\]

Then system (1) can be written as

\[
\begin{align*}
d\begin{pmatrix}
\phi(t) \\
\omega(t)
\end{pmatrix}
&= \begin{pmatrix}
\hat{A} & 0 \\
0 & \mathcal{B}
\end{pmatrix}
\begin{pmatrix}
\phi(t) \\
\omega(t)
\end{pmatrix}
+ \begin{pmatrix}
0 \\
0
\end{pmatrix}
	\times \left( U_1(t) + d_1(t) \right) + \mathcal{J}(\phi(t), \omega(t), t),
\end{align*}
\]

(5)

By Xu and Baillieul (1993), we know that \(\hat{A}\) generates a \(C_0\)-semigroup on \(\mathcal{H}\). However, it is easy to prove that \(\mathcal{B}\) is not admissible to \(e^{\hat{A}t}\). This is different from those in Feng and Guo (2014) and Guo and Jin (2013a), where the systems are always well-posed. To overcome this difficulty, we design the following control:

\[
U_1(t) = -\alpha u_{xx}(1, t) + V_1(t), \quad (6)
\]

where \(V_1(t)\) is a new control variable. Under control (6), system (1) becomes

\[
\begin{align*}
\begin{cases}
\rho u_{tt} + \rho u_{xxxx} = \rho \hat{\omega}^2 u, & x \in (0, 1), t > 0, \\
u(0, t) = u_x(0, t) = 0, & t \geq 0, \\
u_x(1, t) = -\alpha u_x(1, t), \quad V_1(t) + d_1(t), & t \geq 0, \\
u_{xxxx}(1, t) = 0, \quad t \geq 0, \\
\frac{d}{dt} \begin{pmatrix}
I_d + \rho \int_0^1 u^2 dx \\
\omega(t)
\end{pmatrix} = \begin{pmatrix}
U_1(t) + d_2(t), & t > 0
\end{pmatrix}
\end{cases}
\end{align*}
\]

(7)

Define the unbounded linear operators \(\hat{A}_\delta\) as follows:

\[
\begin{align*}
\hat{A}_\delta(y, z) &= (z, -y_{xxxx} + \hat{\omega}^2 y), \\
\mathcal{V}(y, z) &\in D(\hat{A}_\delta)(\subset \mathcal{H}) \rightarrow \mathcal{H}, \\
D(\hat{A}_\delta) = \{ (y, z) \in \mathcal{H} \cap (H^4(0, 1) \times H^2(0, 1)) : y_{xx}(1) = -\alpha z_x(1), y_{xxxx}(1) = 0 \}.
\end{align*}
\]

Then, system (7) can be reformulated into the following evolution equation in \(X:\)

\[
\frac{d}{dt} \begin{pmatrix}
\phi(t) \\
\omega(t)
\end{pmatrix} = \hat{A}_\delta \begin{pmatrix}
\phi(t) \\
\omega(t)
\end{pmatrix} + \begin{pmatrix}
0 \\
0
\end{pmatrix}
	\times \left( V_1(t) + d_1(t) \right) + \mathcal{J}(\phi(t), \omega(t), t),
\]

(9)

where \( \hat{A}_\delta = \text{diag}(\hat{A}_\delta, 0)\). Furthermore, a direct computation shows that \(\hat{A}_{\delta}^*\), the adjoint operator of \(\hat{A}_\delta\), has the form:

\[
\begin{align*}
\hat{A}_{\delta}^*(\psi, \psi') &= (-\psi, \varphi_{xxxx} - \hat{\omega}^2 \varphi), \\
\mathcal{V}(\psi, \psi') &\in D(\hat{A}_{\delta}^*), \quad D(\hat{A}_{\delta}^*) = \{ (\varphi, \psi') \in \mathcal{H} \cap (H^4(0, 1) \times H^2(0, 1)) : \varphi_{xx}(1) = 0, \varphi_{xxxx}(1) = \alpha \varphi_x(1) \}.
\end{align*}
\]

(10)
2.1 Well-posedness of the system (9)

The well-posedness of (9) is discussed in this subsection. To this end, we need the following results.

**Lemma 2.1:** Assume that \( \hat{\omega} < \sqrt{\mu_1} \) and let \( \mathcal{A}_0 \) be defined by (8). Then, \( \mathcal{A}_0^{-1} \) exists and is compact on \( \mathcal{H} \). Therefore, \( \sigma(\mathcal{A}_0) \), the spectrum of \( \mathcal{A}_0 \), consists of isolated eigenvalues of finite algebraic multiplicity only. Moreover, \( \mathcal{A}_0 \) generates a \( C_0 \)-semigroup \( e^{\mathcal{A}_0 t} \) of contractions on \( \mathcal{H} \).

**Proof:** For given \((f, g) \in \mathcal{H}\), solve \( \mathcal{A}_0(u, v) = (f, g) \) to get \( v = f \) and \( u \) satisfies

\[
\begin{aligned}
  -u_{xxxx} + \hat{\omega}^2 u &= g, \\
  u(0) = u_x(0) = u_{xxx}(1) = 0, &\quad u_{xx}(1) = -\alpha f_x(1).
\end{aligned}
\]

Let \( m(x) = u(x) + \alpha/2 f_x(1) x^2 \). Then, \( m(x) \) satisfies

\[
\begin{aligned}
  m_{xxxx} - \hat{\omega}^2 m(x) &= -g - \hat{\omega}^2 \alpha/2 f_x(1) x^2, \\
  m(0) = m_x(0) = m_{xx}(1) = m_{xxx}(1) = 0. &\quad (11)
\end{aligned}
\]

Thus, (12) has a unique solution if and only if the homogeneous equation of (12)

\[
\begin{aligned}
  m_{xxxx} - \hat{\omega}^2 m(x) &= 0, \\
  m(0) = m_x(0) = m_{xx}(1) = m_{xxx}(1) = 0 &\quad (13)
\end{aligned}
\]

only admits a zero solution. Noting that (13) has the fundamental solution,

\[
m(x) = c_1 e^{\sqrt{\hat{\omega}} x} + c_2 e^{-\sqrt{\hat{\omega}} x} + c_3 e^{i \sqrt{|\hat{\omega}|} x} + c_4 e^{-i \sqrt{|\hat{\omega}|} x},
\]

we substitute this into the boundary conditions of (13) to get

\[
\begin{aligned}
  c_1 + c_2 + c_3 + c_4 &= 0, &\quad c_1 - c_2 + ic_3 - ic_4 &= 0, \\
  c_1 e^{\sqrt{\hat{\omega}}} + c_2 e^{-\sqrt{\hat{\omega}}} - c_3 e^{i \sqrt{|\hat{\omega}|}} - c_4 e^{-i \sqrt{|\hat{\omega}|}} &= 0. &\quad (14)
\end{aligned}
\]

Hence, we have the determinant of the coefficient matrix of (14):

\[
\begin{vmatrix}
  1 & 1 & 1 & 1 \\
  1 & -1 & i & -i \\
  e^{\sqrt{\hat{\omega}}} & e^{-\sqrt{\hat{\omega}}} & -e^{i \sqrt{|\hat{\omega}|}} & -e^{-i \sqrt{|\hat{\omega}|}} \\
  e^{i \sqrt{|\hat{\omega}|}} & -e^{-i \sqrt{|\hat{\omega}|}} & ie^{i \sqrt{|\hat{\omega}|}} & -ie^{-i \sqrt{|\hat{\omega}|}}
\end{vmatrix}
\]

\[
= 8i [1 + \cosh \sqrt{|\hat{\omega}|} \cos \sqrt{|\hat{\omega}|}] \neq 0,
\]

where we have used \( \hat{\omega} < \sqrt{\mu_1} \) and (2). This tells us that (13) has only zero solution and (12) has the unique solution \( m(x) \). Thus, there is a unique \( u(x) = m(x) - \frac{\alpha}{2} f_x(1) x^2 \) to (11). Consequently, \( \mathcal{A}_0^{-1} \) exists and is compact on \( \mathcal{H} \) by the Sobolev embedding theorem. Therefore, \( \sigma(\mathcal{A}_0) \) consists of isolated eigenvalues of finite algebraic multiplicity only.

Now we show that \( \mathcal{A}_0 \) is dissipative in \( \mathcal{H} \). Let \( X = (f, g) \in D(\mathcal{A}_0) \). Then we have

\[
\text{Re}(\mathcal{A}_0 X, X)_{\mathcal{H}} = -\alpha |g_x(1)|^2 \leq 0.
\]

Thereby, \( \mathcal{A}_0 \) is dissipative and \( \mathcal{A}_0 \) generates a \( C_0 \)-semigroup \( e^{\mathcal{A}_0 t} \) of contractions on \( \mathcal{H} \) by the Lumer–Phillips theorem (see Pazy, 1983). The proof is complete. \( \square \)

**Lemma 2.2:** Assume that \( \hat{\omega} < \sqrt{\mu_1} \) and let \( \mathcal{A}_0 \) and \( \mathcal{B} \) be defined by (8). Then \( \mathcal{B} \) is admissible to the semigroup \( e^{\mathcal{A}_0 t} \).

**Proof:** It is noted that \( \mathcal{B} \) is admissible for \( e^{\mathcal{A}_0 t} \), or equivalently, \( \mathcal{B}^* \) is admissible for \( e^{\mathcal{A}_0^* t} \). So, we only need to show that \( \mathcal{B}^* \) is admissible for \( e^{\mathcal{A}_0^* t} \), where \( \mathcal{A}_0^* \) is defined by (10).

Consider the adjoint system

\[
\begin{aligned}
  \frac{d}{dt} \begin{pmatrix} u^* \\ u_t^* \end{pmatrix} &= \mathcal{A}_0^* \begin{pmatrix} u^* \\ u_t^* \end{pmatrix}, &\quad \mathcal{B}^* \begin{pmatrix} u^* \\ u_t^* \end{pmatrix} &= -u_{xx}^*(1),
\end{aligned}
\]

which yields

\[
\begin{aligned}
  u_{xx}^*(1) + u_{xxx}^* - \hat{\omega}^2 u^* &= 0, &\quad x \in (0, 1), &\quad t > 0, \\
  u^*(0, t) = u_{xx}^*(0, t) &= u_{xx}^*(1, t) = 0, &\quad t \geq 0, \\
  u_{xx}^*(1, t) &= -\alpha u_t^*(1, t), &\quad t \geq 0, \\
  y_0(t) &= u_t^*(1, t), &\quad t \geq 0.
\end{aligned}
\]

The energy function of (15) is given by

\[
\begin{aligned}
  E^*(t) &= \frac{1}{2} \int_0^1 \left[ u_{xx}^*(x, t) + u_t^*(x, t) - \hat{\omega}^2 u^*(x, t) \right] dx, \\
  \hat{\omega} &< \sqrt{\mu_1}.
\end{aligned}
\]

Differentiating \( E^*(t) \) with respect to time, we get

\[
\dot{E}^*(t) = -\alpha u_{xx}^*(1, t) \leq 0,
\]

which means that \( E^*(t) \) is non-increasing in \( t \). Integrating from 0 to \( T \) with respect to \( t \) in the above equation, we have

\[
\int_0^T u_{xx}^*(1, t) dt = -\frac{1}{\alpha} \int_0^T \dot{E}^*(t) dt
\]

\[
= \frac{1}{\alpha} \left( E^*(0) - E^*(T) \right) \leq \frac{1}{\alpha} E^*(0).
\]
On the other hand, for any given \((\phi, \varphi) \in \mathcal{H}\), solve \((A_{\hat{\omega}}^* - \hat{\omega}) (f, g) = (\phi, \varphi)\), we have
\[
\begin{align*}
\begin{cases}
    g = -\hat{\omega} f - \phi, & f_{xxx} = \varphi - \hat{\omega} \phi, \\
    f(0) = f_x(0) = 0, & f_{xx}(1) = \alpha g_x(1),
\end{cases}
\end{align*}
\]
which has the solution
\[
\begin{align*}
    f(x) &= \frac{1}{2} x^2 f_{xx}(1) + \frac{x}{2} \int_0^1 s^2 (\varphi(s) - \hat{\omega} \phi(s)) \, ds \\
    &\quad + \frac{1}{6} \int_0^1 (s - x)^3 (\varphi(s) - \hat{\omega} \phi(s)) \, ds \\
    &\quad - \frac{1}{6} \int_0^1 s^3 (\varphi(s) - \hat{\omega} \phi(s)) \, ds, \\
    g(x) &= -\hat{\omega} f(x) - \phi(x), \\
    f_{xx}(1) &= \alpha g_x(1) = -\frac{\alpha \hat{\omega}}{2(1 + \alpha \hat{\omega})} \int_0^1 s^2 (\psi - \hat{\omega} \phi) \, ds \\
    &\quad - \frac{\alpha}{1 + \alpha \hat{\omega}} \phi_x(1).
\end{align*}
\]
So, \(\hat{\omega} \in \sigma(A_{\hat{\omega}}^*)\) and
\[
B^* (A_{\hat{\omega}}^* - \hat{\omega})^{-1} (\phi, \varphi) = \frac{\hat{\omega}}{2(1 + \alpha \hat{\omega})} \int_0^1 s^2 (\psi - \hat{\omega} \phi) \, ds \\
+ \frac{1}{1 + \alpha \hat{\omega}} \phi_x(1).
\]
Hence, \(B^* (A_{\hat{\omega}}^* - \hat{\omega})^{-1}\) is bounded and thus \(B^*\) is admissible for \(e^{A_{\hat{\omega}} t}\). Therefore, \(B\) is admissible for \(e^{A t}\) (see Weiss, 1989).

We are now in a position to show that (9) is well-posed and deduce the corresponding variation of constant formula for mild solutions.

**Theorem 2.1:** Assume that \(\hat{\omega} < \sqrt{\mu_1}\) and both \(d_i(t)\) and \(\hat{d}_i(t)\), \(i=1,2\), are bounded measurable. Then for any initial value \((u(\cdot, 0), \phi(\cdot, 0), \omega(\cdot, 0)) \in \mathcal{H} \times \mathbb{R}, V_1(t), U_2(t) \in L^2_{loc}(0, \infty)\) and \(d_i \in L^2_{loc}(0, \infty), \omega \in C(0, \infty; \mathcal{H})\) which is given by

\[
\phi(t) = e^{A t} \phi_0 + \int_0^t e^{A(t-s)} B (V_1(t) + d_i(t)) \, ds \\
+ \int_0^t e^{A(t-s)} (\omega^2(s) - \hat{\omega}^2) P \phi(s) \, ds,
\]
and

\[
\omega(t) = \omega_0 - \int_0^t 2 \omega(s) \int_0^1 uu_t \, dx - U_2(t) - d_2(t) \int_0^1 u^2 \, dx \, ds,
\]
where \(\phi(t) = (u, u_t)\) and \(P(u, v) = (0, u)\) is the compact operator on \(\mathcal{H}\).

**Proof:** By Lemma 2.1, \(A_{\hat{\omega}}\) generates a \(C_0\)-semigroup \(S(t) = e^{A_{\hat{\omega}} t}\) on \(\mathcal{H}\), by (9), so is for \(\hat{A}\). A trivial verification shows that (9) is equivalent to the following equation:

\[
(\phi(t), \omega(t)) = e^{\hat{A} t} (\phi_0, \omega_0) \\
+ \int_0^t e^{\hat{A}(t-s)} \hat{B} (V_1(t) + d_i(t)) \, ds \\
+ \int_0^t e^{\hat{A}(t-s)} J(\phi(s), \omega(s), s) \, ds,
\]
(18)
where \(\hat{B} = (B, 0)\) and \(e^{\hat{A} t} = \text{diag}(e^{A t^2}, I)\). One can use Fréchet derivative to verify that the operator \(J\) is continuously differentiable in \(X\) (Xu & Baillieul, 1993). Since \(B\) is admissible to the semigroup \(e^{A_{\hat{\omega}} t}\), \(\hat{B} = (B, 0)\) is admissible to the semigroup \(e^{A t}\). It then follows from the Corollary 1.3 of Pazy (1983) that for any initial value \((\phi_0, \omega_0) \in \mathcal{H} \times \mathbb{R}\) and \(T > 0\), the Equation (18) has a unique solution \((\phi(t), \omega(t)) \in C(0, T; \mathcal{H}) \times \mathbb{R}\). In fact, the integral Equation (18) is the unique global mild solution of (9). The above Equation (18) can be decomposed into (16) and (17).

### 3. Feedback via active disturbance rejection control

In this section, we use the ADRC approach to attenuate the disturbances and then design the feedback controls for (7).

#### 3.1 Extended state observer of the beam

This subsection is devoted to estimating \(d_i(t)\). By Theorem 2.1, the solution of the beam of (9) satisfies

\[
\frac{d}{dt} \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} \phi \\ \varphi \end{pmatrix} \in \mathcal{H} \\
= \begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} \phi \\ \varphi \end{pmatrix} + (V_1(t) + d_i(t)) B^* \begin{pmatrix} \phi \\ \varphi \end{pmatrix} + \begin{pmatrix} 0 \\ \omega^2(t) - \hat{\omega}^2 \end{pmatrix} u \begin{pmatrix} \phi \\ \varphi \end{pmatrix},
\]
(19)
where \([\phi, \varphi]^T \in D(A_{\hat{\omega}}^*)\) is test function. Taking specially \([\phi, \varphi]^T = [\alpha x^2, x^2]^T\). By (10), we have

\[
\tilde{y}(t) = -2[V_1(t) + d_i(t)] + y_0(t) + \omega^2(t) y(t),
\]
(20)
where

\[
y(t) = \int_0^1 x^2 u(x, t) \, dx. \quad y_0(t) = -2\alpha u_x(1, t) - 2 u_x(1, t).
\]
(21)
We can design an extended state observer to estimate \( d_1 \) as follows (see Guo & Zhao, 2011):

\[
\begin{align*}
\dot{\hat{y}_1}(t) &= \hat{y}_2(t) - 6\varepsilon_1^{-1}\hat{y}_1(t) - y(t), \\
\dot{\hat{y}_2}(t) &= -2\hat{d}_1(t) + \omega^2(t)\hat{y}_1(t) \\
-11\varepsilon_1^{-2} + \omega^2(t)\hat{y}_1(t) - y(t) - 2V_1(t) + y_0(t), \\
\dot{\hat{d}_1}(t) &= 3\varepsilon_1^{-3}\hat{y}_1(t) - y(t),
\end{align*}
\]

where \( \varepsilon_1 \) is the tuning small parameter. The errors are

\[
\begin{align*}
\hat{y}_1(t) &= \hat{y}_2(t) - y(t), \\
\hat{y}_2(t) &= \hat{y}_2(t) - \hat{y}_1(t), \\
\hat{d}_1(t) &= -\hat{d}_1(t) + d_1(t),
\end{align*}
\]

which satisfy

\[
\begin{align*}
\begin{cases}
\dot{\hat{y}_1}(t) = \hat{y}_2(t) - 6\varepsilon_1^{-1}\hat{y}_1(t), \\
\dot{\hat{y}_2}(t) = -2\hat{d}_1(t) + 11\varepsilon_1^{-2}\hat{y}_1(t), \\
\dot{\hat{d}_1}(t) = -3\varepsilon_1^{-3}\hat{y}_1(t) + \hat{d}_1(t).
\end{cases}
\end{align*}
\]

**Lemma 3.1:** Suppose that the disturbance \( d_1(t) \) and its derivative \( \dot{d}_1(t) \) are uniformly bounded on \([0, \infty)\). Then, for any given \( \alpha > 0 \), it follows that

\[
|\hat{y}_1(t)| + |\hat{y}_2(t)| + |\hat{d}_1(t)| \to 0,
\]

as \( \varepsilon_1 \to 0 \) uniformly in \([\alpha, \infty)\).

**Proof:** A direct computation gives

\[
\varepsilon_1^3 \hat{y}_1(t) + 6\varepsilon_1^2\hat{y}_1(t) + 11\varepsilon_1\hat{y}_1(t) + 6\hat{y}_1(t) = 2\varepsilon_1^3\hat{d}_1(t).
\]

Let \( t = \varepsilon_1 s \) and \( v(s) = \hat{y}_1(\varepsilon_1 s) \). It then follows from the above equation that

\[
\frac{d^3}{ds^3}v(s) + 6\frac{d^2}{ds^2}v(s) + 11\frac{d}{ds}v(s) + 6v(s) = 2\varepsilon_1^3\hat{d}_1(\varepsilon_1 s),
\]

which can be rewritten as follows:

\[
\frac{df(s)}{ds} = A_0 f(s) + D(s),
\]

where

\[
A_0 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-6 & -11 & -6
\end{pmatrix},
\]

\[
f(s) = \begin{pmatrix}
v(s) \\
\dot{v}(s) \\
\ddot{v}(s)
\end{pmatrix},
\]

\[
D(s) = \begin{pmatrix}
0 \\
0 \\
2\varepsilon_1^3\hat{d}_1(\varepsilon_1 s)
\end{pmatrix}.
\]

The solution of \((25)\) is

\[
f(s) = e^{A_0(s\tau)} f(0) + \int_0^s e^{A_0(\tau\tau)} D(\tau) d\tau.
\]

Since \( A_0 \) is Hurwitz, there exist constants \( \omega, L > 0 \), such that \( \| e^{A_0} \|_F \leq L e^{-\omega s} \), where \( \| \cdot \|_F \) is the Frobenius norm. From \((26)\) and \((27)\), we get

\[
|f(s)| \leq |e^{A_0} f(0)| + \left| \int_0^s e^{A_0(s\tau)} D(\tau) d\tau \right| \\
\leq |f(0)|L e^{-\omega s} + \int_0^s |D(\tau)| e^{-\omega s} d\tau \leq |f(0)|L e^{-\omega s} + 2\varepsilon_1\omega^{-1}L \sup_{t \in [0, \infty)} |\hat{d}_1(t)|.
\]

Hence,

\[
\left| \begin{pmatrix}
\hat{y}_1(t), \\
\varepsilon_1^{-1}\hat{y}_1(t), \\
\varepsilon_1^{-2}\hat{y}_1(t)
\end{pmatrix} \right| \leq \varepsilon_1^{-2}L |f(0)| e^{-\varepsilon_1^{-1}t} + 2\varepsilon_1\omega^{-1}L \sup_{t \in [0, \infty)} |\hat{d}_1(t)|.
\]

Due to bounded of \(|\hat{d}_1(t)|\), for any \( \alpha > 0 \), we have

\[
\left| \hat{y}_1(t) \right| + \left| \varepsilon_1^{-1}\hat{y}_1(t) \right| + \left| \varepsilon_1^{-2}\hat{y}_1(t) \right| \to 0,
\]

as \( \varepsilon_1 \to 0 \) uniformly in \([\alpha, \infty)\).

On the other hand, it follows from \((23)\) that

\[
\begin{align*}
\hat{y}_2(t) &= \hat{y}_1(t) + 6\varepsilon_1^{-1}\hat{y}_1(t), \\
\hat{d}_1(t) &= \frac{1}{2} \left( \hat{y}_1(t) + 6\varepsilon_1^{-1}\hat{y}_1(t) + 11\varepsilon_1^{-2}\hat{y}_1(t) \right).
\end{align*}
\]

This together with \((29)\) yield \((24)\). The proof is complete.

### 3.2 Extended state observer of the disk

This subsection mainly estimate the disturbance \( d_2(t) \). Define

\[
\Gamma(t) = \left( I_4 + \int_0^t u^2 dx \right) \omega(t).
\]

By the last condition of \((7)\), we get \( \hat{\Gamma}(t) = U_2(t) + \hat{d}_2(t) \), where \( \Gamma(t) \) is the state of ODE and \( U_2 \) is the control. Now we are able to design an extended state observer to estimate both \( \Gamma(t) \) and \( d_2(t) \):

\[
\begin{align*}
\hat{\Gamma}_{\varepsilon_2}(t) &= U_2(t) + \hat{d}_2(t) - \varepsilon_2^{-1}(\hat{\Gamma}_{\varepsilon_2} - \Gamma), \\
\hat{d}_2(t) &= -\varepsilon_2^{-2}(\hat{\Gamma}_{\varepsilon_2} - \Gamma).
\end{align*}
\]
where $\varepsilon_2$ is the tuning small parameter. The errors are
\[
\hat{\Gamma}_{e_2}(t) = \hat{\Gamma}_{e_2}(t) - \Gamma(t), \quad \hat{d}_2(t) = -\hat{d}_2(t) + d_2(t).
\]  
(32)

**Lemma 3.2:** Suppose that the disturbance $d_2(t)$ and its derivative $d_2(t)$ are uniformly bounded on $[0, \infty)$. Then, for any given $\alpha_0 > 0$, it follows that
\[
|\hat{\Gamma}_{e_2}(t)| + |\hat{d}_2(t)| \to 0, \quad \text{as} \quad \varepsilon_2 \to 0 \quad \text{uniformly in} \quad [\alpha_0, \infty).
\]  
(33)

**Proof:** A simple calculation gives that the errors (32) satisfy
\[
\begin{align*}
\hat{\Gamma}_{e_2}(t) &= -\hat{d}_2(t) - \varepsilon_2^{-1} \hat{\Gamma}_{e_2}(t), \\
\hat{d}_2(t) &= \varepsilon_2^{-2} \hat{\Gamma}_{e_2}(t) + d_2(t),
\end{align*}
\]  
(34)

which can be rewritten as
\[
\begin{align*}
\frac{d}{dt} \left( \frac{\hat{\Gamma}_{e_2}(t)}{\hat{d}_2(t)} \right) &= A_1 \left( \frac{\hat{\Gamma}_{e_2}(t)}{\hat{d}_2(t)} \right) + D_1 \hat{d}_2(t), \\
A_1 &= \left( -\frac{1}{\varepsilon_2^2}, 1 \right), \quad D_1 = \left( 0, 1 \right).
\end{align*}
\]  
(35)

The solution of the above equation is
\[
\left( \hat{\Gamma}_{e_2}(t) \quad \hat{d}_2(t) \right) = e^{A_1 t} \left( \hat{\Gamma}_{e_2}(0) \quad \hat{d}_2(0) \right) + \int_0^t e^{A_1 (t-s)} D_1 \hat{d}_2(s) \, ds,
\]  
(36)

From (35) and (36), we obtain (33). The proof is complete. \hfill \Box

### 3.3 Feedback control design

Based on the estimates of $\hat{d}_i(t), i = 1, 2$, we design the feedback controllers for system (7) as follows:
\[
\begin{align*}
V_1(t) &= -\text{sat} \left( \hat{d}_1(t) \right), \\
U_2(t) &= -\gamma (\omega(t) - \hat{\omega}) - \text{sat} \left( \hat{d}_2(t) \right), \quad \alpha > 0, \gamma > 0,
\end{align*}
\]  
(37)

where the saturation function $\text{sat} (\cdot)$ defined by
\[
\text{sat}(x) := \begin{cases}
M, & x \geq M + 1, \\
-M, & x \leq -M - 1, \\
x, & x \in (-M - 1, M + 1).
\end{cases}
\]  
(38)

Here, $M$ is the upper bound of $d_i(t)$ assumed in Introduction. Under the feedback (37), the closed-loop system of (7) becomes
\[
\left\{ \begin{array}{l}
\dot{\Gamma}_{e_1}(t) + \dot{\Gamma}_{e_2}(t) = \frac{\lambda_1 \lambda_2 e^{\lambda_1 t} - \lambda_2 \lambda_1 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \\
\dot{\Gamma}_{e_2}(t) - \dot{\Gamma}_{e_1}(t) = \frac{\lambda_1 \lambda_2 e^{\lambda_2 t} - \lambda_2 \lambda_1 e^{\lambda_1 t}}{\lambda_1 - \lambda_2},
\end{array} \right.
\]  
(39a)

and
\[
\begin{align*}
\dot{\hat{y}}_1(t) &= \frac{\dot{\gamma}_2(t) - 6 \varepsilon_1^{-2} (\hat{\gamma}_1(t) - \gamma(t))}{\lambda_2 - \lambda_1}, \\
\dot{\gamma}_2(t) &= \omega^2(t) \hat{y}_1(t) - \left( \frac{1}{\lambda_2 - \lambda_1} \right) \frac{\lambda_2 \lambda_1 e^{\lambda_2 t} - \lambda_2 \lambda_1 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} + 2 \alpha u_{sat}(1, t) + 2 \text{sat} \left( \hat{d}_1(t) \right) - 2 \hat{d}_1(t) + y_0(t),
\end{align*}
\]  
(39b)

and
\[
\left\{ \begin{array}{l}
\frac{d}{dt} \Gamma(t) = -\gamma (\omega(t) - \hat{\omega}) - \text{sat} \left( \hat{d}_2(t) \right) + d_2(t), \\
\dot{\Gamma}_{e_1}(t) = -\gamma (\omega(t) - \hat{\omega}) - \varepsilon_2^{-1} (\Gamma_{e_2} - \Gamma) - \text{sat} \left( \hat{d}_2(t) \right) + \hat{d}_2(t), \\
\dot{\hat{d}}_2(t) = -\varepsilon_2^{-2} (\hat{\Gamma}_{e_2} - \Gamma),
\end{array} \right.
\]  
(39c)

where $\gamma(t)$ and $\Gamma(t)$ are given by (21) and (30), respectively. Using the error dynamics defined by (22), (32), it is
found that (39) can be rewritten as

\[
\begin{aligned}
&u_{tt} + u_{xxxx} - \omega^2(t)u = 0, \\
u(0, t) = u_x(0, t) = u_{xxx}(1, t) = 0, \\
u_x(1, t) = -\alpha u_{xt}(1, t) + \gamma(t), \\
& + \text{sat} \left( \tilde{d}_1(t) - d_1(t) \right) + d_2(t) \triangleq -\alpha u_{xt}(1, t) + h(t),
\end{aligned}
\]  

(40a)

(40b)

and

\[
\begin{aligned}
&\frac{d\Gamma}{dt} (t) = -\gamma \left( \omega(t) - \tilde{\omega} \right) + \text{sat} \left( \tilde{d}_2(t) - d_2(t) \right) + d_2(t) \triangleq -\gamma \left( \omega(t) - \tilde{\omega} \right) + h(t), \\
&\tilde{\Gamma}_{x2}(t) = -\tilde{d}_2(t) - e_2^{-1} \Gamma_{x2}(t), \\
&\tilde{d}_2(t) = e_2^{-2} \tilde{\Gamma}_{x2}(t) + \tilde{d}_2(t).
\end{aligned}
\]  

(40c)

From (23) and (34), it is seen that \((\tilde{y}_1, \tilde{y}_2, \tilde{d}_1)\) and \((\tilde{\Gamma}, \tilde{d}_2)\) are independent of the ‘u-part’ and ‘\'part’. \((\tilde{y}_1, \tilde{y}_2, \tilde{d}_1)\) and \((\tilde{\Gamma}, \tilde{d}_2)\) can be arbitrarily small as \(t \to \infty, \varepsilon_1, \varepsilon_2 \to 0\) by Lemmas 3.1 and 3.2, respectively. Hence, we only need to consider the ‘u-part’ and ‘\'part’ of (40),

\[
\begin{aligned}
&u_{tt} + u_{xxxx} - \omega^2(t)u = 0, \\
u(0, t) = u_x(0, t) = u_{xxx}(1, t) = 0, \\
u_x(1, t) = -\alpha u_{xt}(1, t) + h(t), \\
& + \text{sat} \left( d_1(t) - d_1(t) \right) + d_2(t) \triangleq -\alpha u_{xt}(1, t) + h(t),
\end{aligned}
\]  

(41)

4. Stability of the system (41)

In this section, the closed-loop stability of system (41) under the proposed controllers (6) and (37) is established.

Theorem 4.1: Assume that \(\tilde{\omega} < \sqrt{\omega_1}\). The semigroup \(e^{A_{\tilde{\omega}}}\), generated by the operator \(A_{\tilde{\omega}}\), is exponentially stable in \(\mathcal{H}\), i.e., there exist positive constants \(L_0\) and \(\beta\) such that \(\|e^{A_{\tilde{\omega}}t}\| \leq L_0 e^{-\beta t}\).

Proof: Since it is a tedious work, we put the proof in the Appendix.

Theorem 4.2: Suppose that \(\tilde{\omega} < \sqrt{\omega_1}\) and both \(d_2(t)\) and \(\tilde{d}_2(t)\), \(i = 1, 2\), are uniformly bounded. For any initial value \((u(\cdot, 0), u_x(\cdot, 0), \omega_0) \in \mathcal{H} \times \mathbb{R}\), then the solution \((u(\cdot, t), \omega_0(\cdot, t, \tilde{\omega})\) of system (41) asymptotically tends to equilibrium point \((0, 0, \tilde{\omega})\) in \(\mathcal{X}\), as \(t \to \infty, \varepsilon_1, \varepsilon_2 \to 0\).

Proof: The proof is divided into two steps. In the first step, we prove

\[
\omega(t) \to \tilde{\omega}, \quad \text{as } t \to \infty, \varepsilon_1 \to 0, \varepsilon_2 \to 0.
\]

Consider the Lyapunov function \(V : \mathcal{X} \to \mathbb{R}^+\):

\[
V = \frac{1}{2} \left( I_d (\omega(t) - \tilde{\omega})^2 + (\omega(t) - \tilde{\omega})^2 \|u\|^2_{L^2(0,1)} + \|u_t\|^2_{H^1(0,1)} + \|u\|^2_{L^2(0,1)} - \tilde{\omega}^2 \|u\|^2_{L^2(0,1)} \right),
\]

where

\[
\|u\|^2_{L^2(0,1)} = \int_0^1 u^2 dx, \quad \|u\|^2_{H^1(0,1)} = \int_0^1 u_x^2 dx.
\]

Differentiating \(V(\phi(t), \omega(t))\) along the solution of (41) gives

\[
\dot{V} = (\omega(t) - \tilde{\omega}) \dot{\omega}(t) (I_d + \|u\|^2_{L^2(0,1)}) + (\omega(t) - \tilde{\omega})^2 (u, u_t)_{L^2(0,1)} + (u, u_t)_{H^1(0,1)} + (u, u_t)_{L^2(0,1)} - \tilde{\omega}^2 (u, u_t)_{L^2(0,1)}
\]

\[
\leq -\gamma (\omega(t) - \tilde{\omega})^2 + \gamma (\omega(t) - \tilde{\omega}) \dot{h}_2(t) - \alpha u_{xt}(1, t) + h(t)
\]

\[
\leq -\gamma (\omega(t) - \tilde{\omega})^2 + \gamma (\omega(t) - \tilde{\omega})^2 + \frac{1}{2} h_2(t) - \alpha u_{xt}(1, t) + h(t)
\]

\[
\triangleq -\tilde{K}_1 + \tilde{K}_2,
\]

where \(\tilde{K}_1 = \gamma (\omega(t) - \tilde{\omega})^2 + \frac{\gamma}{2} u_{xt}(1, t)\) and \(\tilde{K}_2 = \frac{1}{2\gamma} h_2(t) + \frac{1}{2\alpha} h_1^2(t)\). There are two cases:

(1) If \(\tilde{K}_1 \geq \tilde{K}_2\), we get \(\dot{V} \leq 0\). Hence, each solution is bounded and \(\int_0^\infty \gamma (\omega(t) - \tilde{\omega})^2 dt\) converges. This implies that \(\lim_{t \to \infty} \omega(t, \varepsilon_1, \varepsilon_2) \to \tilde{\omega}\) according to the Barbalat’s lemma.

(2) If \(\tilde{K}_1 < \tilde{K}_2\), by Lemmas 3.1 and 3.2, we get

\[
\lim_{t \to \infty} \omega(t, \varepsilon_1, \varepsilon_2) \leq \lim_{t \to \infty} \dot{\omega}(t) \to \tilde{\omega}
\]

\[
\leq \lim_{t \to \infty} \tilde{K}_2 = \lim_{t \to \infty} \tilde{K}_2 + \frac{1}{2\gamma} h_2(t) + \frac{1}{2\alpha} h_1^2(t) = 0.
\]
In Summary, for any $\nu > 0$, there exist $T_0$ and $\varepsilon > 0$ such that

$$|\omega^2(t) - \hat{\omega}^2| < \nu, \quad \text{for all } t \geq T_0, 0 < \varepsilon_1, \varepsilon_2 < \varepsilon.$$  \hspace{1cm} (42)

In the second step, we show that $\phi(t) = (u(\cdot, t), u_t(\cdot, t))$ is asymptotically stable in $\mathcal{H}$, that is,

$$(u(\cdot, t), u_t(\cdot, t)) \to (0, 0), \quad \text{when } t \to \infty, \varepsilon_1, \varepsilon_2 \to 0.$$  

By Lemma 3.1 and (40a), for any given $\varepsilon_0 > 0$, there exist $t_0 \geq T_0 > 0$ and $\varepsilon_0^* > 0$ such that

$$|h_1(t)| = |\text{sat}(\tilde{d}_1(t) - d_1(t)) + d_1(t)| < \varepsilon_0 \quad \text{for all } t > t_0 \quad \text{and } 0 < \varepsilon_1, \varepsilon_2 < \varepsilon_0^*.$$  

We rewrite the solution of (16) as

$$\phi(t) = e^{A_{\hat{\omega}}t} \phi_0 + \int_0^t e^{A_{\hat{\omega}}(t-s)} (\omega^2(s) - \hat{\omega}^2) P\phi(s)ds$$

$$+ \int_0^t e^{A_{\hat{\omega}}(t-s)} Bh_1(s)ds$$

$$= e^{A_{\hat{\omega}}t} \phi_0 + \int_0^t e^{A_{\hat{\omega}}(t-s)} (\omega^2(s) - \hat{\omega}^2) P\phi(s)ds$$

$$+ e^{A_{\hat{\omega}}(t-t_0)} \int_{t_0}^t e^{A_{\hat{\omega}}(t-s)} Bh_1(s)ds$$

$$+ \int_{t_0}^t e^{A_{\hat{\omega}}(t-s)} Bh_1(s)ds.$$  \hspace{1cm} (43)

The admissibility of $B$ implies that

$$\left\| \int_0^{t_0} e^{A_{\hat{\omega}}(t_0-s)} Bh_1(s)ds \right\|_{L^2}^2$$

$$\leq C_{t_0} \| h_1 \|_{L^2(0,t_0)} C_0 \| \text{sat} \left( \tilde{d}_1(t) - d_1(t) \right) + d_1(t) \|_{L^2(0,t_0)}^2$$

$$\leq t_0 C_6 (2M + 1)^2, \quad \forall \tilde{d}_1 \in L^\infty(0, \infty).$$  \hspace{1cm} (44)
where the constant $C_{b0}$ is independent of $\tilde{d}_1$. Because $e^{A_{b1}t}$ is exponential stable (see Theorem 4.1) and $B$ is admissible to $e^{A_{b1}t}$ with $L_0^2$ control, $B$ is admissible to $e^{A_{b1}t}$ with $L_0^\infty$ control. From Proposition 2.5 in Weiss (1989), it follows that

\[
\|\phi(t)\|_{H} \leq L_0 e^{-\beta t} \|\phi(0)\|_{H} \\
+ L_0 \int_{t_0}^{t} e^{-\beta (t-s)} (\omega^2(s) - \tilde{\omega}^2) P\phi(s) ds \\
+ L_0 e^{-\beta (t-t_0)} t_0 C_{b0} (2M + 1)^2 + C_{\varepsilon_{b0}}. \quad (46)
\]

Set $m(t) = L_0 e^{-\beta t} \|\phi(0)\|_{H} + L_0 e^{-\beta (t-t_0)} t_0 C_{b0} (2M + 1)^2 + C_{\varepsilon_{b0}}$. It follows from (42) and (46) that

\[
\|\phi(t)\|_{H} \leq m(t) + \nu L_0 \int_{0}^{t} e^{-\beta (t-s)} \|\phi(s)\|_{H} ds, \\
\text{for all } t > t_0 \text{ and } 0 < \varepsilon_1, \varepsilon_2 < \varepsilon_{b0}^*.
\]

---

Figure 3. The state of closed loop system in the presence of external disturbance. (a) Displacement $u(x, t)$, (b) velocity $u_t(x, t)$ and (c) velocity $\omega(t)$. 

where $C$ is a constant and independent of $\tilde{d}_1$, and

\[
(u \odot v) = \begin{cases} 
\quad u(t), & 0 \leq t \leq \tau, \\
\quad v(t - \tau), & t > \tau.
\end{cases}
\]
Applying Gronwall’s Lemma to the above inequality, we have

\[ \|\phi(t)\|_H \leq m(t) + \int_0^t e^{-\beta(t-r)} m(s) \exp \left\{ \int_r^t v L_0 e^{-\beta(t-r)} \, dr \right\} \, ds \]

\[ \leq m(t) + e^{\nu L_0} \int_0^t e^{-\beta(t-r)} m(s) \, ds \]

\[ = m(t) + e^{\nu L_0} \left[ L_0 e^{-\beta t} \|\phi(0)\|_H + L_0 e^{-\beta t} \|\phi(0)\|_H \right] \]

\[ + L_0 e^{-\beta t} \|\phi(0)\|_H \]

\[ + L_0 e^{-\beta t} \|\phi(0)\|_H + L_0 e^{-\beta t} \|\phi(0)\|_H \]

\[ = m(t) + e^{\nu L_0} \left[ L_0 e^{-\beta t} \|\phi(0)\|_H \right] \]

\[ + L_0 e^{-\beta t} \|\phi(0)\|_H + L_0 e^{-\beta t} \|\phi(0)\|_H \]

\[ = m(t) + e^{\nu L_0} \left[ L_0 e^{-\beta t} \|\phi(0)\|_H \right] \]

\[ + L_0 e^{-\beta t} \|\phi(0)\|_H + L_0 e^{-\beta t} \|\phi(0)\|_H \]

\[ = m(t) + e^{\nu L_0} \left[ L_0 e^{-\beta t} \|\phi(0)\|_H \right] \]

\[ + L_0 e^{-\beta t} \|\phi(0)\|_H + L_0 e^{-\beta t} \|\phi(0)\|_H \]

where \( C_1 \) is a positive constant. Passing to the limit as \( t \to \infty \) in (47), we finally obtain

\[ \lim_{t \to \infty} \|\phi(t)\|_H \leq C_1 e^{\nu L_0} + C_2 \epsilon_0. \]

Since \( \epsilon_0, \nu \) are arbitrary sufficiently small, we deduce that

\[ \lim_{t \to \infty} \|\phi(t)\|_H = 0 \text{ in } \mathcal{H}. \]

Finally, collecting these two steps, we conclude the desired result that the solution \((u(\cdot, t), u_t(\cdot, t), \omega)\) of system (41) asymptotically tends to equilibrium point \((0, 0, \hat{\omega})\) as \( t \to \infty \), \( \epsilon_1, \epsilon_2 \to 0 \). The proof is complete. \( \square \)

5. Numerical simulation

In this section, some simulation results are given to illustrate the effect of our controls design (6) and (37) for the disk–beam system (1). The numerical results of system (39) are obtained by the finite difference method. Let \( I_d = \frac{3}{2}, \gamma = 5, \alpha = 2, \hat{\omega} = 3, \) the initial displacement \( u(x, 0) = \frac{1}{20} x^2 \) and the initial velocity \( u_t(x, 0) = 1 \)

6. Conclusions

This paper concerns with the stability characteristics of the ADRC for the rotating disk–beam system with the input disturbances. By the ADRC, we first estimate the disturbances and then cancel the disturbances in the feedback loop. The collocated feedback control is applied to suppress the rotating beam and the torque control is forced to steer the disk to the desired rotating angular velocity. In the energy state space, it is shown that the feedback control laws are robust to the external disturbances when the angular velocity of the disk is less than the square root of the first nature frequency of the beam, and the vibration of the beam can be suppressed while the disk rotating with a desired angular velocity in presence of the disturbances. Numerical simulations are presented.
to show the convergence of both state and approximated disturbances.

The nonlinear feedback controls and the output feedback stabilisation for the rotating disk–beam system with the input disturbances are still open. They will be considered in future.

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**Appendix A. Proof of Theorem 4.1**

In this appendix, we present the proof of Theorem 4.1. It is pointed out that Laousy et al. (1996) have showed that if the angular velocity $\hat{\omega} < \sqrt{EI/\rho}$ (or 3 for EI = $\rho = 1$), the semigroup $e^{A_{\mu} t}$ is exponentially stable in $H$. In this appendix, we improve this best desired value to $\sqrt{\mu_{1}} \simeq 3.516$ (EI = $\rho = 1$) and show that if the angular velocity $\hat{\omega} < \sqrt{\mu_{1}}$, the semigroup $e^{A_{\mu} t}$ is exponentially stable in $H$. Before going on the proof of Theorem 4.1, we recall the well-known Huang result (see Huang, 1985):

**Lemma A.1**: A strongly continuous semigroup of contractions $e^{\lambda t}$ on a Hilbert space $H$ is exponentially stable if and only if

$$\sup\{\| (i \mu I - A_{\mu})^{-1}\|_{H}; \mu \in \mathbb{R} \} < \infty, \quad (A.1)$$

and $\{ i \mu : \mu \in \mathbb{R} \} \subset \rho(A_{\mu})$.

**Proof of Theorem 4.1**: First, we show $\{ i \mu : \mu \in \mathbb{R} \} \subset \rho(A_{\mu})$. In Lemma 2.1, it has showed $A_{\mu}^{-1}$ exists. Suppose that there is a non-zero $\mu$ such that $i \mu \in \sigma(A_{\mu})$, that is, there exist $\Phi = (y, z) \in D(A_{\mu})$ with $\|\Phi\|_{H} = 1$ such that $A_{\mu} \Phi = i \mu \Phi$. A simple computation finds

$$0 = \text{Re}(i \mu \Phi, \Phi) = \text{Re}(A_{\mu} \Phi, \Phi) = -\alpha z_{x}^{2}(1), \quad (A.2)$$

and $\{ i \mu : \mu \in \mathbb{R} \} \subset \rho(A_{\mu})$.

Next, we are going to show

$$\sup\{\| (i \mu I - A_{\mu})^{-1}\|_{H}; \mu \in \mathbb{R} \} < \infty, \quad (A.2)$$

Suppose that (A.2) does not hold. Therefore, there exist a sequence of real numbers $\mu_{n} \to \infty$, and $\Phi_{n} = (y^{n}, z^{n}) \in D(A_{\mu})$, satisfying

$$\left\{ \begin{array}{l}
\| (i \mu_{n} I - A_{\mu}) \Phi_{n}\|_{H} \to 0, \quad \text{as } n \to \infty,
\\
\| \Phi_{n}\|_{H} = \| y^{n}\|_{l^{2}} + \| z^{n}\|_{l^{2}} = 1, \\
\| y^{n}\|_{l^{2}} = \int_{0}^{1} y^{n}_{xx} dx - \hat{\omega}^{2} y^{n} dx.
\end{array} \right. \quad (A.3)$$

These together with (8) imply that as $n \to \infty$,

$$\left\{ \begin{array}{l}
f^{n} \triangleq i \mu_{n} y^{n} - z^{n} \to 0, \quad \text{in } H_{2}^{2}(0, 1),
\\
g^{n} \triangleq i \mu_{n} z^{n} + y^{n}_{xxxx} - \hat{\omega}^{2} y^{n} \to 0, \quad \text{in } L^{2}(0, 1),
\\
y(0) = y^{n}(0) = y_{xx}(1) = 0,
\\
y_{xx}(1) = -\alpha z_{x}(1).
\end{array} \right. \quad (A.4)$$

From (A.3), we have $\| y^{n}\|_{*}$ and $\| z^{n}\|_{l^{2}}$ are uniformly bounded, that is, there exist constants $C_{1}, C_{2} > 0$, such that

$$\| y^{n}\|_{*} < C_{1}, \quad \| z^{n}\|_{l^{2}} < C_{2}. \quad (A.5)$$

One can check that

$$\| (i \mu_{n} I - A_{\mu}) \Phi_{n}\|_{H} \geq \alpha(z_{x}(1))^{2}. \quad (A.6)$$

By (A.3), we have $|z^{n}_{x}(1)| \to 0$, as $n \to \infty$. By the trace theorem, this together with the first equation of (A.4) yields that

$$|\mu_{n} z_{x}(1)| \to 0, \quad \text{as } n \to \infty. \quad (A.7)$$

Furthermore, the last boundary condition of (A.4) yields

$$|y_{xx}(1)| \to 0, \quad \text{as } n \to \infty. \quad (A.8)$$

From the second equation of (A.4), we have

$$\left\{ \begin{array}{l}
y_{xxx}(0) = -\int_{0}^{1} y^{n}_{xxx} dx
\\
y_{xxx}(1) = -\int_{0}^{1} (-i \mu_{n} z^{n} + g^{n} + \hat{\omega}^{2} y^{n}) dx,
\\
x_{n}^{-1} e^{-\epsilon_{n}} |y^{n}_{xxx}(0)| \leq k_{1} x_{n}^{-1} e^{-\epsilon_{n}} \left( |y^{n}\|_{*} + g_{n}^{n}\|_{l^{2}} \right).
\end{array} \right. \quad (A.9)$$

where $k_{1}$ is a constant. Multiplying the second equation of (A.4) by $x$ and using integration by parts, to yield

$$\int_{0}^{1} (-i \mu_{n} z^{n} + g^{n} + \hat{\omega}^{2} y^{n}) x dx = \int_{0}^{1} y^{n}_{xxx} x dx = y^{n}_{xx}(0) - y^{n}_{xx}(1),$$
and

\[
e^{-\xi_n} y^n_{xx}(0) = e^{-\xi_n} \int_0^1 (-i\mu_n z^n + g^n + \hat{\omega}^2 y^n) dx + y^n_{xx}(1) \leq k_2(e^{-\xi_n} ||y^n||_* + e^{-\xi_n} ||g^n||_{L^2} + e^{-\xi_n} ||\mu_n|| ||z^n||_{L^2}) + e^{-\xi_n} |y^n_{xx}(1)|,
\]

where \(k_2\) is a constant. Now, using (A.8)–(A.10) and the fact \(||y^n||_*\) and \(||z^n||_{L^2}\) are uniformly bounded, we get

\[
\xi_n^{-1} e^{-\xi_n} |y^n_{xx}(0)| \to 0, \quad e^{-\xi_n} |y^n_{xx}(0)| \to 0, \quad \text{as} \quad n \to \infty.
\]

From the first two equations of (A.4), we get

\[
y^n_{xxxx} - (\hat{\omega}^2 + \mu_n^2) y^n = g^n + i\mu_n f^n. \tag{A.12}
\]

Let \(\xi_n = (\hat{\omega}^2 + \mu_n^2)^{\frac{1}{2}}\). Multiplying (A.12) by \(\xi_n^{-1} e^{\xi_n (x-1)}\) and using integration by parts to yield

\[
-\xi_n^{-1} y^n_{xx}(0) e^{-\xi_n} - y^n_{xx}(1) e^{-\xi_n} + \xi_n y^n_{xx}(0) e^{-\xi_n} - \xi_n^{-2} y^n(1) = \xi_n^{-1} \int_0^1 (i\mu_n f^n + g^n) e^{\xi_n (x-1)} dx.
\]

Clearly, as \(n \to \infty\),

\[
\xi_n^{-1} \left| \int_0^1 e^{\xi_n (x-1)} g^n dx \right| \leq k_3 \xi_n^{-1} ||g^n||_{L^2} \to 0,
\]

\[
i\mu_n \xi_n^{-1} \left| \int_0^1 e^{\xi_n (x-1)} f^n dx \right| \leq k_4 ||f^n||_{L^2} \to 0,
\]

where \(k_3\) and \(k_4\) are two positive constants. Finally, inserting (A.7), (A.8) and (A.11) into (A.13), we have

\[
\xi_n^2 |y^n(1)| = \sqrt{\mu_n^2 + \hat{\omega}^2} |y^n(1)| \to 0, \quad \text{as} \quad n \to \infty.
\]

(A.14)

Multiply (A.12) by \(y^n\) and use integration by parts twice, to yield

\[
- [y^n_{xx}(1)]^2 - 2y^n_{xx}(1) y^n_x(1) - \left| \sqrt{\hat{\omega}^2 + \mu_n^2} y^n(1) \right|^2 + 3 \int_0^1 (y^n_x)^2 dx + \int_0^1 \left| \sqrt{\hat{\omega}^2 + \mu_n^2} y^n \right|^2 dx \tag{A.15}
\]

\[
= 2 \int_0^1 (i\mu_n f^n + g^n) y^n_x dx.
\]

Applying the Hölder inequality, we get,

\[
2 \int_0^1 (i\mu_n f^n + g^n) y^n_x dx \leq C_3 [||y^n||_{L^2} ||y^n||_* + ||f^n||_* ||\mu_n y^n||_{L^2} + ||f^n||_* ||\mu_n y^n||_{L^2}] + C_4 [||g^n||_{L^2}]
\]

where \(C_3\) is a positive constants. The above inequality together with (A.4), (A.5) and (A.14), implies that

\[
2 \int_0^1 (i\mu_n f^n + g^n) y^n_x dx \to 0, \quad \text{as} \quad n \to \infty. \tag{A.16}
\]

Finally, inserting (A.7), (A.8), (A.14), (A.16) into (A.15), we get

\[
||y^n||_* \to 0, \quad ||\mu_n y^n||_{L^2} \to 0, \quad ||z^n||_{L^2} \leq ||\mu_n y^n||_{L^2} + ||f^n||_{L^2} \to 0, \quad n \to \infty. \tag{A.17}
\]

which contradicts (A.3). The proof of Theorem 4.1 follows from the two steps and Lemma A.1. The proof is complete. \(\square\)