Dynamic Boundary Stabilization of a Schrödinger Equation Through a Kelvin-Voigt Damped Wave Equation

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Abstract In this paper, we study an interconnected system of a Schrödinger and a wave equation with Kelvin-Voigt (K-V) damping, where the K-V damped wave equation performs as a dynamic feedback controller. We show that the system operator generates a C_0-semigroup of contractions in the energy state space, and the system is well-posed. By detailed spectral analysis, we know that the spectral of the system operator composes of two parts, point spectrum and continuous spectrum. Moreover, the points in the spectra all have negative real parts. It follows that the C_0-semigroup generated by the system operator achieves asymptotic stability.

1 Introduction

There has been much interests in the problem of dynamic feedback controller, which is modeled by coupled partial differential equations (PDEs) or interconnected PDEs. This kind of problems are quite challenging, yet have became attractive to researchers these days. Control design and stability analysis for such systems have become active over the past 5 years, see [8, 9] and the references therein. In [9], an interconnected system of Schrödinger equation and heat equation is carefully studied, which replaces the static feedback by dynamic feedback governed by a heat equation. It shows the exponential stability of the system and the Gevery class property of the semigroup. The boundary and internal stabilizations for Schrödinger equation are considered in [6], where the stability of the systems is achieved by using multiplier techniques. Two kinds of boundary controllers for Schrödinger equation are concerned in [2], which shows that a simple proportional collocated boundary controller can exponentially stabilize the system but the decay rate cannot be prescribed, while the backstepping method can ensure it to have arbitrary decay rate.

It is also known that viscoelastic materials have been widely used in engineering and lots of researchers have put much efforts to analyze the dynamic behavior of

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vibration for elastic structures with viscoelasticity over the past several decades. Kelvin-Voigt (K-V) damping is one of those most commonly used viscoelastic model, due to its easy and huge applications in modern technology. And there has been an abundance of literature on the study of elastic system with viscoelastic damping. In [3, 4], it is shown that local K-V damping can ensure the exponential stability of both a string and an Euler-Bernoulli beam system when the material parameter is smooth enough at the interface. However, if the smooth condition cannot be satisfied, the string system is non-exponentially stable even if the material parameter is a constant. Passive control of a wave equation with internal K-V damping is studied in [1]. Results there reveal that the spectrum of the system operator consists of point spectrum and continuous spectrum, due to the fact that the resolvent of the system operator for a viscoelastic system is not compact anymore.

In this paper, we present a dynamic input/output feedback controller, which feeds the K-V damped wave equation and Schrödinger equation into each other through the boundary. The coupled Schrödinger-wave system (as shown in Fig. 1) is written as follows:

$$\begin{cases} y_t(x,t) + iy_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ z_{tt}(x,t) - z_{xx}(x,t) - \alpha z_{xxt}(x,t) = 0, & 1 < x < 2, t > 0, \\ y(0,t) = z(2,t) = 0, & t \geq 0, \\ y(1,t) = kz_t(1,t), & t \geq 0, \\ \alpha z_{xt}(1,t) + z_x(1,t) = -iky_x(1,t), & t \geq 0, \\ y(x,0) = y_0(x), & 0 < x < 1, \\ z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x), & 1 < x < 2, \end{cases} \tag{1}$$

where $\alpha > 0$, $k \neq 0$. The two equations are coupled at $x = 1$ with interconnected conditions and fixed at each end.

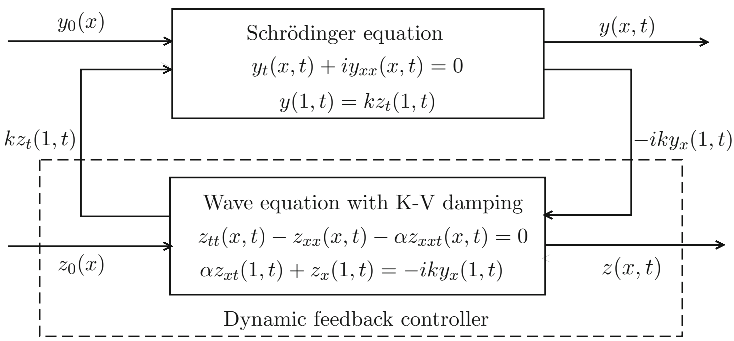

Fig. 1 Block diagram for the dynamic boundary feedback of the coupled system

By introducing the following transformation

$$\begin{cases} w(x,t) = y(1-x,t),\ 0 < x < 1, t > 0, \\ u(x,t) = z(x+1,t),\ 0 < x < 1, t > 0, \end{cases} \tag{2}$$

then (1) becomes

$$\begin{cases} w_t(x,t) + iw_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ u_{tt}(x,t) - u_{xx}(x,t) - \alpha u_{xxt}(x,t) = 0, & 0 < x < 1, t > 0, \\ w(1,t) = u(1,t) = 0, & t \geq 0, \\ w(0,t) = ku_t(0,t), & t \geq 0, \\ \alpha u_{xt}(0,t) + u_x(0,t) = ikw_x(0,t), & t \geq 0, \end{cases} \tag{3}$$

Accordingly, the initial conditions for system (3) are $w(x,0) = w_0(x), u(x,0) = u_0(x), u_t(x,0) = u_1(x), 0 < x < 1$.

The energy function for (3) is given by

$$E(t) = \frac{1}{2}\int_0^1 \left[|w(x,t)|^2 + |u_x(x,t)|^2 + |u_t(x,t)|^2\right] dx. \tag{4}$$

In this paper, we analyze the spectrum of (3) in which the system operator has no compact resolvent. We first set up the system operator and show it generates a C_0-semigroup of contractions, and the system is well-posed. By detailed spectral analysis, we obtain that the residual spectrum is empty and the continuous spectrum contains only one negative point. Moreover, all the eigenvalues of the system lie in the open left half plane. Therefore, this controller design moves the eigenvalues of the Schrödinger and wave equations into the second quadrant. It follows that the C_0-semigroup generated by the system operator achieves asymptotic stability.

2 Well-Posedness of System (3)

We consider system (3) in the energy space $\mathscr{H} = L^2(0,1) \times H_E^1(0,1) \times L^2(0,1)$, where $H_E^1(0,1) = \{g \in H^1(0,1) | g(1) = 0\}$. The norm in $\mathscr{H}$ is induced by the inner product

$$\langle X_1, X_2\rangle = \int_0^1 \left[f_1(x)\overline{f_2(x)} + g_1'(x)\overline{g_2'(x)} + h_1(x)\overline{h_2(x)}\right] dx, \tag{5}$$

where $X_s = (f_s, g_s, h_s) \in \mathscr{H}$, $s = 1, 2$. Define the system operator of (3) by

$$\begin{cases} \mathscr{A}(f,g,h) = (-if'',\ h,\ (g' + \alpha h')'\ ),\ \forall\ (f,g,h) \in D(\mathscr{A}), \\ D(\mathscr{A}) = \left\{ (f,g,h) \in \mathscr{H}, \mathscr{A}(f,g,h) \in \mathscr{H}, \left| \begin{array}{l} g' + \alpha h' \in H^1(0,1), \\ f(1) = 0,\ f(0) = kh(0), \\ g'(0) + \alpha h'(0) = ikf'(0) \end{array} \right. \right\}. \end{cases} \tag{6}$$

Then (3) can be written as an evolution equation in $\mathscr{H}$:

$$\begin{cases} \dfrac{dX(t)}{dt} = \mathscr{A}X(t),\quad t > 0, \\ X(0) = X_0, \end{cases} \tag{7}$$

where $X(t) = (w(\cdot,t), u(\cdot,t), u_t(\cdot,t))$.

Theorem 1 *Let $\mathscr{A}$ be given by (6). Then $\mathscr{A}^{-1}$ exists, and hence $0 \in \rho(\mathscr{A})$, the resolvent set of $\mathscr{A}$. Moreover $\mathscr{A}$ is dissipative in $\mathscr{H}$ and $\mathscr{A}$ generates a C_0-semigroup $e^{\mathscr{A}t}$ of contractions in $\mathscr{H}$.*

Proof For any given $(f_1, g_1, h_1) \in \mathscr{H}$ by,

$$\mathscr{A}(f,g,h) = (-if'', h, (g' + \alpha h')') = (f_1, g_1, h_1), \tag{8}$$

we have

$$\begin{cases} f''(x) = if_1(x),\quad h(x) = g_1(x),\quad (g'(x) + \alpha h'(x))' = h_1(x), \\ f(1) = g(1) = 0,\quad f(0) = kh(0),\quad g'(0) + \alpha h'(0) = ikf'(0),\quad g_1(1) = 0, \end{cases} \tag{9}$$

and the solution of (9) is given by

$$\begin{cases} f(x) = f'(0)(x-1) - i\displaystyle\int_0^x (1-x)f_1(\xi)d\xi - i\int_x^1 (1-\xi)f_1(\xi)d\xi, \\ g(x) = ik(x-1)f'(0) - \alpha g_1(x) - \displaystyle\int_x^1 (1-\xi)h_1(\xi)d\xi\ + (x-1)\int_0^x h_1(\xi)d\xi, \\ h(x) = g_1(x) \\ f'(0) = -i\displaystyle\int_0^1 (1-\xi)f_1(\xi)d\xi - kg_1(0). \end{cases} \tag{10}$$

Hence, we get the unique solution $(f, g, h) \in D(\mathscr{A})$ to equation (8), thus $\mathscr{A}^{-1}$ exists. Now we show that $\mathscr{A}$ is dissipative in $\mathscr{H}$. Let $X = (f, g, h) \in D(\mathscr{A})$. Then we have

$$\begin{aligned}
\langle \mathscr{A}X, X\rangle &= \big\langle(-if'', h, (g' + \alpha h')'), (f, g, h)\big\rangle \\
&= \int_0^1 (-if'')\overline{f}dx + \int_0^1 h'\overline{g'}dx + \int_0^1 (g' + \alpha h')'\overline{h}dx \\
&= -if'\overline{f}\Big|_0^1 + i\int_0^1 |f'|^2 dx + \int_0^1 h'\overline{g'}dx + (g' + \alpha h')\overline{h}\Big|_0^1 - \int_0^1 (g' + \alpha h')\overline{h'}dx \\
&= -if'(1)\overline{f}(1) + if'(0)\overline{f}(0) + i\int_0^1 |f'|^2 dx + \int_0^1 h'\overline{g'}dx + (g'(1) + \alpha h'(1))\overline{h}(1) \\
&\quad -(g'(0) + \alpha h'(0))\overline{h}(0) - \int_0^1 \overline{h'}g'dx - \alpha\int_0^1 |h'|^2 dx \\
&= if'(0)\overline{f}(0) + i\int_0^1 |f'|^2 dx + \int_0^1 h'\overline{g'}dx - ikf'(0)\overline{h}(0) - \alpha\int_0^1 |h'|^2 dx \\
&\quad - \int_0^1 \overline{h'}g'dx \\
&= -\alpha\int_0^1 |h'|^2 dx + \left(i\int_0^1 |f'|^2 dx + \int_0^1 h'\overline{g'}dx - \int_0^1 \overline{h'}g'dx\right)
\end{aligned} \tag{11}$$

and

$$\mathrm{Re}\langle \mathscr{A}X, X\rangle = -\alpha\int_0^1 |h'|^2 dx \le 0. \tag{12}$$

Hence $\mathscr{A}$ is dissipative and $\mathscr{A}$ generates a C_0-semigroup $e^{\mathscr{A}t}$ of contractions in $\mathscr{H}$ by the Lumer-Phillips theorem [7]. The proof is complete. □

3 Spectral Analysis

In this section, we consider the eigenvalue problem of (3). Let $\mathscr{A}X = \lambda X$, where $0 \neq X = (f, g, h) \in D(\mathscr{A})$, then f, g, h satisfy:

$$\begin{cases}
f''(x) - i\lambda f(x) = 0, \\
h(x) = \lambda g(x), \\
(1 + \alpha\lambda)g''(x) - \lambda^2 g(x) = 0, \\
f(1) = g(1) = 0, \ f(0) = kh(0), \\
\alpha h'(0) + g'(0) = ikf'(0).
\end{cases} \tag{13}$$

Let $p(\lambda) = 1 + \alpha\lambda$, when $p(\lambda) \neq 0$ i.e. $\lambda \neq -\dfrac{1}{\alpha}$, (13) changes to

$$\begin{cases} f''(x) = i\lambda f(x), \\ g''(x) = \dfrac{\lambda^2}{1+\alpha\lambda} g(x) = \dfrac{\lambda^2}{p(\lambda)} g(x), \\ f(1) = g(1) = 0, \ f(0) = \lambda k g(0), \\ (1+\alpha\lambda) g'(0) = p(\lambda) g'(0) = ikf'(0). \end{cases} \tag{14}$$

We can get

$$f(x) = a_1 e^{\sqrt{i\lambda}x} + b_1 e^{-\sqrt{i\lambda}x}, \quad g(x) = c_1 e^{\sqrt{\frac{\lambda^2}{p(\lambda)}}x} + d_1 e^{-\sqrt{\frac{\lambda^2}{p(\lambda)}}x}, \tag{15}$$

where a_1, b_1, c_1 and d_1 are constants. Substituting these into the boundary conditions of (14), we have

$$\begin{cases} a_1 e^{\sqrt{i\lambda}} + b_1 e^{-\sqrt{i\lambda}} = 0, \\ c_1 e^{\sqrt{\frac{\lambda^2}{p(\lambda)}}} + d_1 e^{-\sqrt{\frac{\lambda^2}{p(\lambda)}}} = 0, \\ a_1 + b_1 = \lambda k (c_1 + d_1), \\ p(\lambda)\sqrt{\frac{\lambda^2}{p(\lambda)}}(c_1 - d_1) = ik\sqrt{i\lambda}(a_1 - b_1). \end{cases} \tag{16}$$

Then (14) has the nontrivial solution if and only if the characteristic equation $\det \Delta(\lambda) = 0$, where

$$\Delta(\lambda) = \begin{bmatrix} e^{\sqrt{i\lambda}} & e^{-\sqrt{i\lambda}} & 0 & 0 \\ 0 & 0 & e^{\sqrt{\frac{\lambda^2}{p(\lambda)}}} & e^{-\sqrt{\frac{\lambda^2}{p(\lambda)}}} \\ 1 & 1 & -k\lambda & -k\lambda \\ ik\sqrt{i\lambda} & -ik\sqrt{i\lambda} & -p(\lambda)\sqrt{\frac{\lambda^2}{p(\lambda)}} & p(\lambda)\sqrt{\frac{\lambda^2}{p(\lambda)}} \end{bmatrix}. \tag{17}$$

Lemma 1 *Let $\mathscr{A}$ be defined by (6). Then for each $\lambda \in \sigma_p(\mathscr{A})$, we have* $\mathrm{Re}\lambda < 0$.

Proof By Theorem 1, since $\mathscr{A}$ is dissipative, we have for each $\lambda \in \sigma(\mathscr{A})$, $\mathrm{Re}\lambda \le 0$. So we only need to show there is not any eigenvalue on the imaginary axis. Let $\lambda = \pm i\mu^2 \in \sigma_p(\mathscr{A})$ with $\mu \in \mathbb{R}^+$ and $X = (f, g, h) \in D(\mathscr{A})$ be its associated eigenfunction of $\mathscr{A}$. Then by (12), we have

$$\mathrm{Re}\langle \mathscr{A}X, X\rangle = -\alpha \int_0^1 |h'|^2 dx = 0.$$

Hence $h'(x) = 0$. By the second and third equations of (13), we have $h(x) = g(x) = 0$. Then by the first equation of (13) and its boundary conditions we have:

$$\begin{cases} f''(x) = i\lambda f(x), \\ f(0) = f'(0) = f(1) = 0. \end{cases}$$

A direct computation yields $f(x) = 0$. Hence, $X = (f, g, h) = 0$. Therefore, there is no eigenvalue on the imaginary axis. This completes the proof. □

Proposition 1 *Let $\mathscr{A}$ be defined by (6). Then $\lambda = -\frac{1}{\alpha} \notin \sigma_p(\mathscr{A})$.*

Proof When $\lambda = -\frac{1}{\alpha}$, then $p(\lambda) = 0$. From the third and second equation of (13), we can easily get $g(x) = h(x) = 0$. Following the same manner as the proof of Lemma 1, we have $f(x) = 0$. Hence, $X = (f, g, h) = 0$. So that $\lambda = -\frac{1}{\alpha} \notin \sigma_p(\mathscr{A})$. □

Theorem 2 *Let $\mathscr{A}$ be defined by (6). The eigenvalues of $\mathscr{A}$ have the following asymptotic expressions:*

(i) When $\lambda \to -\frac{1}{\alpha}$, the asymptotic eigenvalue is given by:

$$\lambda_{2n} = -\frac{1}{\alpha} - \frac{1}{n^2\pi^2\alpha^3} + \frac{\sqrt{\alpha}A}{n^4\pi^4\alpha^5} + \mathscr{O}(n^{-5}), \tag{18}$$

where

$$A = \frac{2}{B(1 + e^{\frac{2i\sqrt{i}}{\alpha}})}[\frac{1 - B}{\sqrt{\alpha}} - \frac{1 + B}{\sqrt{\alpha}}e^{\frac{2i\sqrt{i}}{\alpha}}], \quad B = \frac{k^2\sqrt{i}}{\alpha^2}. \tag{19}$$

(ii) When $|\lambda| \to \infty$, there are two families of eigenvalues given by:

$$\lambda_{1n} = -(n\pi + \frac{\theta}{2})|\ln r| + \left[(n\pi + \frac{\theta}{2})^2 - \frac{\ln^2 r}{4}\right]i + \mathscr{O}(n^{-1}), \tag{20}$$

$$\lambda_{3n} = -\alpha(n\pi + \frac{\varphi}{2})^2 + \frac{\alpha}{4}\ln^2 r + \frac{1}{\alpha} - \left[\alpha(n\pi + \frac{\varphi}{2})\ln r\right]i + \mathscr{O}(n^{-1}). \tag{21}$$

Here r, θ and φ are three constants given by

$$0 < r = \frac{\sqrt{\alpha^2 + k^8}}{\alpha + \sqrt{2\alpha}k^2 + k^4} < 1, \quad \ln r < 0, \;\; \ln r^{-1} = -\ln r > 0, \tag{22}$$

$$\theta = \begin{cases} \arctan \frac{\sqrt{2\alpha}k^2}{\alpha - k^4}, & \alpha - k^4 > 0, \\ \frac{\pi}{2}, & \alpha = k^4, \\ \pi - \arctan \frac{\sqrt{2\alpha}k^2}{k^4 - \alpha}, & \alpha - k^4 < 0, \end{cases} \tag{23}$$

and

$$\varphi = \begin{cases} \arctan \frac{\sqrt{2\alpha}k^2}{k^4 - \alpha}, & k^4 - \alpha > 0, \\ \frac{\pi}{2}, & \alpha = k^4, \\ \pi - \arctan \frac{\sqrt{2\alpha}k^2}{\alpha - k^4}, & k^4 - \alpha < 0. \end{cases} \tag{24}$$

Moreover,

$$\mathrm{Re}\lambda_{1n}, \;\; \mathrm{Re}\lambda_{3n} \to -\infty, \;\; \text{when } n \to \infty. \tag{25}$$

Proof Due to space limitation, we give the outline of the proof here. From (17), and let $s(\lambda) := \sqrt{\frac{\lambda^2}{p(\lambda)}} \in \mathbb{C}$, a direct computation gives,

$$\begin{aligned} \det \Delta(\lambda) &= \left[e^{-\sqrt{i\lambda}} e^{s(\lambda)} - e^{\sqrt{i\lambda}} e^{-s(\lambda)}\right]\left[p(\lambda)s(\lambda) + ik^2\lambda\sqrt{i\lambda}\right] \\ &\quad + \left[e^{-\sqrt{i\lambda}} e^{-s(\lambda)} - e^{\sqrt{i\lambda}} e^{s(\lambda)}\right]\left[p(\lambda)s(\lambda) - ik^2\lambda\sqrt{i\lambda}\right]. \end{aligned} \tag{26}$$

Let $\det \Delta(\lambda) = 0$, asymptotic expressions of the eigenvalues can be achieved.

(I) When $\lambda \to -\frac{1}{\alpha}$, let $\varepsilon = \lambda + \frac{1}{\alpha}$, then $\varepsilon \to 0$. We get λ_{2n} as shown in (18) and (19).

(II) When $\lambda \to \infty$, we consider $\lambda := i\rho^2, \; 0 \le \arg \rho \le \frac{\pi}{2}$. We then get λ_{1n} and λ_{3n} (given by (20) and (21)), when ρ belongs to $0 \le \arg \rho \le \frac{\pi}{8}$ and $\frac{\pi}{8} < \arg \rho \le \frac{3\pi}{8}$, respectively. There is no high frequency solution in $\frac{3\pi}{8} < \arg \rho \le \frac{\pi}{2}$. □

Proposition 2 *Let $\mathscr{A}$ be defined by (6). Then its adjoint operator $\mathscr{A}^*$ has the following form:*

$$\begin{cases} \mathscr{A}^*(f, g, h) = (if'', \ -h, \ -(g' - \alpha h')' \,), \ \forall \ (f, g, h) \in D(\mathscr{A}^*), \\ D(\mathscr{A}^*) = \left\{ (f, g, h) \in \mathscr{H}, \mathscr{A}^*(f, g, h) \in \mathscr{H} \left| \begin{array}{l} g' - \alpha h' \in H^1(0, 1), \\ f(1) = 0, \ f(0) = kh(0), \\ g'(0) - \alpha h'(0) = ikf'(0) \end{array} \right. \right\}. \end{cases} \tag{27}$$

Proposition 3 *Let $\mathscr{A}$ is defined by (6). Then $\sigma(\mathscr{A}) = \{-\frac{1}{\alpha}\} \bigcup \sigma_p(\mathscr{A})$.*

Proposition 4 *Let $\mathscr{A}$ be defined by (6). Then $\sigma_r(\mathscr{A}) = \emptyset$, and $\sigma_c(\mathscr{A}) = \{-\frac{1}{\alpha}\}$.*

Proof From Propositions 1 and 3, we have $-\frac{1}{\alpha} \notin \sigma_p(\mathscr{A})$ and $\{-\frac{1}{\alpha}\} \bigcup \sigma_p(\mathscr{A}) = \sigma(\mathscr{A})$. The desired results will be got if $-\frac{1}{\alpha} \notin \sigma_r(\mathscr{A})$. Now we suppose $-\frac{1}{\alpha} \in \sigma_r(\mathscr{A})$, then $-\frac{1}{\alpha} \in \sigma_p(\mathscr{A}^*)$. By $\mathscr{A}^* X = -\frac{1}{\alpha} X$, where $X = (f, g, h) \in D(\mathscr{A}^*)$, we get

$$\begin{cases} f''(x) - \dfrac{i}{\alpha} f(x) = 0 \\ h(x) = \dfrac{1}{\alpha} g(x) \\ -(g'(x) - \alpha h'(x))' = -\dfrac{1}{\alpha} h(x) \\ f(1) = g(1) = 0, \ f(0) = kh(0), \\ \alpha h'(0) - g'(0) = -ikf'(0). \end{cases} \tag{28}$$

From the second and third equation of (28), we get $g(x) = h(x) = 0$. Then $f(x)$ satisfy

$$\begin{cases} f''(x) - \dfrac{i}{\alpha} f(x) = 0, \\ f(0) = f'(0) = f(1) = 0. \end{cases} \tag{29}$$

Simple computation shows that $f(x) = 0$. This implies that $X = (f, g, h) = 0$, which is a contradiction. So $-\frac{1}{\alpha} \notin \sigma_r(\mathscr{A})$. The proof is complete. □

4 Asymptotic Stability of System (3)

Definition 1 A C_0-semigroup $T(t)$ is called asymptotically (strongly) stable, if

$$\lim_{t\to\infty} ||T(t)x|| = 0.$$

Theorem 3 *([5]) Let $T(t)$ be a uniformly bounded C_0-semigroup on a Banach space X and let A be its generator. If*

$$\sigma(A) \bigcap i\mathbb{R} \subset \sigma_c(A),$$

and $\sigma_c(A)$ is countable, then $T(t)$ is asymptotically stable.

Theorem 4 *Let $\mathscr{A}$ be defined by (6). Then the system (3) achieves asymptotic stability.*

Proof Since from Lemma 1 and Proposition 3, we know that when $\lambda \in \sigma(\mathscr{A})$, $\text{Re}\lambda < 0$. So we have

$$\sigma(\mathscr{A}) \bigcap i\mathbb{R} = \emptyset \subset \sigma_c(\mathscr{A}) = \left\{-\frac{1}{\alpha}\right\},$$

and $\sigma_c(\mathscr{A})$ is obviously countable. Then system (3) achieves asymptotic stability by Theorem 3. □

5 Conclusions

In this paper, we use a wave equation with K-V damping to be a dynamic feedback controller for a Schrödinger equation. We give the asymptotic expression of the eigenvalues, and also the exact composition of the spectrum of the system operator. At last, the asymptotic stability of the system was achieved.

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