J Dyn Control Syst (2016) 22:117–128
DOI 10.1007/s10883-014-9262-3

Sliding Mode Control to Stabilization of a Tip-Force Destabilized Shear Beam Subject to Boundary Control Matched Disturbance

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Received: 24 February 2014 / Revised: 16 October 2014 / Published online: 13 November 2014

Abstract In this paper, we are concerned with the boundary stabilization of a one-dimensional tip-force destabilized shear beam equation subject to boundary control matched disturbance. We use the sliding mode control (SMC) to deal with the disturbance. By the SMC approach, the disturbance is supposed to be bounded only. The existence and uniqueness of the solution for the closed-loop system is proved and the "reaching condition" is obtained.

Keywords Sheer beam equation · Sliding mode control · Stability · Boundary control · Disturbance rejection

Mathematics Subject Clasifications (2010) 93B52 · 93B51 · 93D15 · 35B37

1 Introduction

In this paper, we are concerned with boundary stabilization of a model of the undamped shear beam [1] with a destabilizing boundary condition. It consists of a wave equation coupled with a second order in space ordinary differential equation (ODE) or can be alternatively represented as a fourth order in space/second order in time partial differential equation (PDE). This makes it more complex than the Euler-Bernoulli model, similar in structure

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to the Rayleigh beam model [1, 2]. The destabilizing boundary condition is motivated by the physics of the atomic force microscopy (AFM), where the tip of the cantilever beam is destabilized by van der Waals forces acting between the tip and the material surface, we refer [3] for engineering interpretation of the beam equations.

There are many works contributed to the stabilization of beam equation. The examples can be found in [4–9] and the references therein. However, most of the control design for the beam equation are collocated control based on the passive principle and do not take the disturbance into account. The earlier non-collocated control design for the beam equation is [10]. Recently, a powerful backstepping method is introduced to stabilize the shear beam equation via completely non-collocated control [8]. Once again, the external disturbance is not considered in these works.

When the external disturbance enter the system from the boundary or the internal of the spatial domain, the new approach is needed to deal with the uncertainties. There are several different methods to deal with the uncertainties in system control. The sliding mode control (SMC) that is inherently robust is the most popular one that has been studied widely for both finite-dimensional systems and the infinite-dimensional counterparts. Recently, a boundary SMC controller for a one-dimensional heat, wave, Euler-Bernoulli beam, and Schrödinger equation with boundary input disturbance is designed in [11–14]. Another powerful method in dealing with the disturbance is the active disturbance rejection control (ADRC) method. The ADRC, as an unconventional design strategy, was first proposed by Han in the 1990s ([15]). It has been now acknowledged to be an effective control strategy for lumped parameter systems in the absence of proper models and in the presence of model uncertainty. The numerous applications have been carried out in the last decade (see e.g., [16]). Its convergence has been proved for lumped parameter systems in [17]. Other methods in dealing with uncertainty includes the Lyapunov function-based method, see [18, 19] and the references therein.

Motivated mainly by [13] and [9], we are concerned with, in this paper, the stabilization of a one-dimensional shear beam equation which is suffered from the unknown external disturbance on the input boundary by the SMC approach.

The system that we are concerned with is governed by the following PDEs:

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) - \alpha_x(x,t), \\ u_x(0,t) = \alpha(0,t) - q u_t(0,t), \\ u_x(1,t) = U_1(t) + d(t), \\ \alpha_{xx}(x,t) - b^2\alpha(x,t) + b^2 u_x(x,t) = 0, \\ \alpha_x(0,t) = 0, \\ \alpha(1,t) = U_2(t) \end{cases} \tag{1.1}$$

The state $u(x,t)$ represents the transversal displacement of the beam at time t and position x, and α is the angle due to bending. $q > 0$ is a constant number, $U_1(t)$ and $U_2(t)$ are the two control input through shear force, $d(t)$ is supposed to be bounded measurable (that is , $|d(t)| \le M$, for some $M > 0$ and all $t > 0$), which is the external disturbance at the control end.

The rest of the paper is organized as follows. Section 2 is devoted to the disturbance rejection by the SMC approach. The sliding mode control is designed and the existence and uniqueness of solution of the closed-loop system are proved. The finite time "reaching condition" is presented rigorously. Some concluding remarks are presented in Section 3.

2 Sliding Mode Control Approach

In order to proceed with the control design, we first need to write the model (1.1) in yet another form. To this end, we solve the ODE part of Eq. 1.1 as a two-point boundary value problem for α with boundary condition $\alpha_x(0,t)=0$.

$$\alpha(x,t)=\cosh(bx)\alpha(0,t)-b\int_0^x \sinh(b(x-s))u_x(s,t)ds. \tag{2.1}$$

Setting $x = 1$ in Eq. 2.1 and using the boundary condition $\alpha(1,t) = U_2(t)$, we can express $\alpha(0,t)$ in terms of u and U_2:

$$\alpha(0,t)=\frac{1}{\cosh(b)}U_2(t)+\frac{b}{\cosh(b)}\int_0^1 \sinh(b(1-s))u_x(s,t)ds \tag{2.2}$$

Next, we differentiate Eq. 2.1 in x and substitute the result into the first equation of Eq. 1.1. This way, instead of a wave equation coupled with a second-order ODE, we obtain a single hyperbolic partial integro-differential equation for u:

$$\begin{cases} u_{tt}(x,t) &= u_{xx}(x,t)-b^2\cosh(bx)u(0,t)+b^3\int_0^x \sinh(b(x-y))u(y,t)dy+b^2u(x,t) \\ &\quad -\dfrac{b\sinh(bx)}{\cosh(b)}\left[U_2(t)+b\int_0^1 \sinh(b(1-s))u_x(s,t)ds\right], \\ u_x(0,t) &= \dfrac{b\sinh(bx)}{\cosh(b)}\left[U_2(t)+b\int_0^1 \sinh(b(1-s))u_x(s,t)ds\right]-qu_t(0,t), \\ u_x(1,t) &= U_1(t)+d(t). \end{cases} \tag{2.3}$$

Since the backstepping control design [12] needs the PDE to be in a strict-feedback form (in other words, its right-hand side must be causal in x), we are going to use the control $U_2(t)$ to cancel the definite integral both in the domain and in the boundary condition:

$$U_2(t)=-b\int_0^1 \sinh(b(1-s))u_x(s,t)ds \tag{2.4}$$

We get the following PDE:

$$\begin{cases} u_{tt}(x,t) &= u_{xx}(x,t)+b^2u(x,t)-b^2\cosh(bx)u(0,t)+b^3\int_0^x \sinh(b(x-y))u(y,t)dy, \\ u_x(0,t) &= -qu_t(0,t), \\ u_x(1,t) &= U_1(t)+d(t). \end{cases} \tag{2.5}$$

We consider system (2.5) in the state space $\mathcal{H} = H^1(0,1)\times L^2(0,1)$. We introduce a transformation

$$w(x,t)=u(x,t)-\int_0^x k(x,y)u(y,t)dy-\int_0^x s(x,y)u_t(y,t)dy \tag{2.6}$$

to map (2.5) into the following system:

$$
\begin{cases}
w_{tt}(x,t) = w_{xx}(x,t), \\
w_x(0,t) = c_0 w_t(0,t), \\
w_x(1,t) = U_1(t) + d(t) + \dfrac{b^2}{2}u(1,t) - \displaystyle\int_0^1 k_x(1,y)u(y,t)dy, \\
\qquad -s(1,1)u_t(1,t) - \displaystyle\int_0^1 s_x(1,y)u_t(y,t)dy
\end{cases}
\tag{2.7}
$$

where $c_0 > 0$ is a design parameter. Note the crucial difference between the second equations of Eqs. 2.5 and 2.7, the destabilizing negative sign in the former and the stabilizing positive sign in the latter. The gain kernels $k(x, y)$ and $s(x, y)$ are given by the following PDEs:

$$
\begin{cases}
k_{xx}(x,y) = k_{yy}(x,y) + b^2 k(x,y) - b^3 \sinh(b(x-y)) \\
\qquad + b^3 \displaystyle\int_y^x k(x,\xi)\sinh(b(\xi - y))d\xi, \\
k(x,x) = -\dfrac{b^2}{2}x, \\
k_y(x,0) = -b^2\left[\cosh(bx) - \displaystyle\int_0^x k(x,y)\cosh(by)dy\right],
\end{cases}
\tag{2.8}
$$

$$
\begin{cases}
s_{xx}(x,y) = s_{yy}(x,y) + b^2 s(x,y) + b^3 \displaystyle\int_y^x s(x,\xi)\sinh(b(\xi-y))d\xi, \\
s(x,x) = -\dfrac{b^2}{2}x - c_0 - q, \\
s_y(x,0) = b^2 \displaystyle\int_0^x s(x,y)\cosh(by)dy - qk(x,0),
\end{cases}
\tag{2.9}
$$

which is obtained by substituting Eqs. 2.6 into 2.7 and matching the terms. Incidentally, this equation for $k(x, y)$ and $s(x, y)$ are in the same class as the one obtained in the control design for parabolic PDEs [20]. As shown in [20], the PDE Eqs. 2.8 and 2.9 have unique solution $k(x, y) \in C^2(\Omega)$ and $s(x, y) \in C^2(\Omega)$.

It can be solved either numerically or by using the following symbolic recursion:

$$
\begin{cases}
k(x,y) = \lim_{n\to\infty} k_n(x,y), \\
k_0(x,y) = -\dfrac{b}{2}[-\sinh(b(x-y)) + by\cosh(b(x-y))], \\
k_{n+1}(x,y) = k_0 + b^2 \displaystyle\int_{\frac{x-y}{2}}^{\frac{x+y}{2}}\int_0^{\frac{x-y}{2}} k_n(\sigma+s,\sigma-s)dsd\sigma + q\int_0^{x-y} k_n(\sigma,0)d\sigma \\
\qquad + b^2 \displaystyle\int_0^{\frac{x-y}{2}}\int_0^{\sigma} [2k_n(\sigma+s,\sigma-s) - k_n(\sigma,s)\cosh(bs)]dsd\sigma \\
\qquad + b^3 \displaystyle\int_{\frac{x-y}{2}}^{\frac{x+y}{2}}\int_0^{\frac{x-y}{2}}\int_{\sigma-s}^{\sigma+s} k_n(\sigma+s,\xi)\sinh(b(\xi-\sigma+s))d\xi dsd\sigma, \\
\qquad + 2b^3 \displaystyle\int_0^{\frac{x-y}{2}}\int_0^{\sigma}\int_{\sigma-s}^{\sigma+s} k_n(\sigma+s,\xi)\sinh(b(\xi-\sigma+s))d\xi dsd\sigma
\end{cases}
\tag{2.10}
$$

The first step of this recursion provides approximate control gain kernels, which are explicit:

$$\begin{cases} k_0(1,y) = -\frac{b}{2}[-\sinh(b(1-y)) + by\cosh(b(1-y))], \\ k_{0x}(1,y) = -\frac{b}{2}[-\cosh(b(1-y)) + by\sinh(b(1-y))], \\ k_0(1,1) = k(1,1) = -\frac{b^2}{2}. \end{cases} \tag{2.11}$$

It can be also solved either numerically or by using the following symbolic recursion :

$$\begin{cases} s(x,y) &= \lim_{n\to\infty} s_n(x,y), \\ s_0(x,y) &= -\frac{b}{2}[-\sinh(b(x-y)) + by\cosh(b(x-y))] - c_0 - q, \\ s_{n+1}(x,y) &= s_0 + b^2\int_{\frac{x-y}{2}}^{\frac{x+y}{2}}\int_0^{\frac{x-y}{2}} s_n(\sigma+s,\sigma-s)dsd\sigma + q\int_0^{x-y} s_n(\sigma,0)d\sigma \\ &\quad + b^2\int_0^{\frac{x-y}{2}}\int_0^{\sigma}[2s_n(\sigma+s,\sigma-s) - s_n(\sigma,s)\cosh(bs)]dsd\sigma \\ &\quad + b^3\int_{\frac{x-y}{2}}^{\frac{x+y}{2}}\int_0^{\frac{x-y}{2}}\int_{\sigma-s}^{\sigma+s} s_n(\sigma+s,\xi)\sinh(b(\xi-\sigma+s))d\xi dsd\sigma, \\ &\quad + 2b^3\int_0^{\frac{x-y}{2}}\int_0^{\sigma}\int_{\sigma-s}^{\sigma+s} s_n(\sigma+s,\xi)\sinh(b(\xi-\sigma+s))d\xi dsd\sigma. \end{cases} \tag{2.12}$$

The first step of this recursion provides approximate control gain kernels, which are explicit:

$$\begin{cases} s_0(1,y) = -\frac{b}{2}[-\sinh(b(1-y)) + by\cosh(b(1-y))] - c_0 - q, \\ s_{0x}(1,y) = -\frac{b}{2}[-\cosh(b(1-y)) + by\sinh(b(1-y))], \\ s_0(1,1) = s(1,1) = -\frac{b^2}{2} - c_0 - q. \end{cases} \tag{2.13}$$

Since $k(x,y), s(x,y) \in C^2(\Omega)$, the transformation (2.6) is bounded invertible, and therefore the system (2.5) with the below designed controller (2.26) dynamically behaves as Eq. 2.7.

Let us consider systems (2.7) in the state space $\mathcal{H} = H^1(0,1) \times L^2(0,1)$ with inner product given by $\forall (f_1, g_1)^\top, (f_2, g_2)^\top \in \mathcal{H}$,

$$\langle (f_1,g_1)^\top, (f_2,g_2)^\top\rangle = \int_0^1 [f_1'(x)\overline{f_2'(x)} + g_1(x)\overline{g_2(x)}]dx + f_1(1)\overline{f_2(1)}. \tag{2.14}$$

By virtue of the energy function of system (2.7),

$$E(t) = \frac{1}{2}\int_0^1 (w_t^2(x,t) + w_x^2(x,t))dx + \frac{1}{2}w^2(1,t),$$

we have

$$\dot{E}(t) = -c_0 w_t^2(0,t) + w_x(1,t)w_t(1,t) + w(1,t)w_t(1,t).$$

It is seen that in order to make non-increasing on the sliding surface $S_W(t)$ for system (2.7), which is a closed subspace of $\mathcal{H}$, it is natural to choose $S_W(t) = w(1,t)$ (so $w_t(1,t) = 0$), i.e.,

$$S_W = \{(f,g)^\top \in \mathcal{H} | f(1) = 0\}. \tag{2.15}$$

In this way, $\dot{E}(t) = -c_0 w_t^2(0,t) \le 0$ on S_W, and on S_W system (2.7) becomes

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \\ w_x(0,t) = c_0 w_t(0,t), \\ w(1,t) = 0. \end{cases} \tag{2.16}$$

Write system (2.16) as

$$\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = A \begin{pmatrix} w \\ w_t \end{pmatrix} \tag{2.17}$$

where A is given by

$$\begin{cases} A(f,g)^\top = (g, f''), \ \forall (f,g)^\top \in D(A), \\ D(A) = \left\{ (f,g)^\top \in \left(H^2(0,1) \times H^1(0,1)\right) \cap S_W \middle| \, f'(0) = c_0 g(0), g(1) = 0 \right\}. \end{cases} \tag{2.18}$$

By using the Lumer-Phillips theorem, it is direct to see that A generates a C_0-semigroup e^{At} on S_W, that is, for $(w(x,0)), w_t(x,0))^\top \in S_W$, there exists a unique solution $(w(\cdot,t), w_t(\cdot,t))^\top = e^{At}(w(\cdot,0), w_t(\cdot,0))^\top \in C[0,\infty; S_W)$ to system (2.16) ([21]), where the norm in S_W is induced norm of $\mathcal{H}$. Moreover, system (2.16) is exponentially stable in S_W, that is, there exist two positive constants $M_S, \omega_S > 0$ independent of initial value such that

$$\|(w(\cdot,t), w_t(\cdot,t))^\top\|_{\mathcal{H}} \le M_S e^{-\omega_S t} \|(w(\cdot,0), w_t(\cdot,0))^\top\|_{\mathcal{H}}. \tag{2.19}$$

Now, we are in a position to seek the finite time reaching condition for target system (2.7). Define the sliding mode function $S_W(t) = w(1,t)$ and differentiate $S_W(t)$ formally with respect to t to obtain

$$\begin{aligned} \dot{S}_W(t) &= w_t(1,t) = u_t(1,t) - \int_0^1 k(1,y)u_t(y,t)dy - s(1,1)(U_1(t) + d(t)) \\ &\quad + s(1,0)u_x(0,t) + \int_0^1 s_y(1,y)u_y(y,t)dy - b^2 \int_0^1 s(1,y)u(y,t)dy \\ &\quad + b^2 \int_0^1 s(1,y)\cosh(by)u(0,t)dy \\ &\quad - b^3 \int_0^1 s(1,y) \int_0^y \sinh(b(y-\xi))u(\xi,t)d\xi dy. \end{aligned} \tag{2.20}$$

where we have used the transformation (2.6). Design the feedback controller:

$$\begin{aligned} U_1(t) &= \frac{1}{s(1,1)} \Big\{ u_t(1,t) - \int_0^1 k(1,y)u_t(y,t)dy \\ &\quad + s(1,0)u_x(0,t) + \int_0^1 s_y(1,y)u_y(y,t)dy - b^2 \textstyle\int_0^1 s(1,y)u(y,t)dy \\ &\quad + b^2 \textstyle\int_0^1 s(1,y)\cosh(by)u(0,t)dy - b^3 \int_0^1 s(1,y) \int_0^y \sinh(b(y-\xi))u(\xi,t)d\xi dy \Big\} \\ &\quad + U_0(t) \end{aligned} \tag{2.21}$$

where U_0 is a new control. Then, we have

$$\dot{S}_W(t) = -s(1,1)U_0(t) - s(1,1)d(t). \tag{2.22}$$

Let

$$U_0(t) = -(M_0 + \eta)\frac{S_W(t)}{|S_W(t)|} \quad for \quad S_W(t) \neq 0. \tag{2.23}$$

Then,

$$\dot{S}_W(t) = s(1,1)(M_0+\eta)\frac{S_W(t)}{|S_W(t)|} - s(1,1)d(t) \tag{2.24}$$

Therefore,

$$S_W(t)\dot{S}_W(t) = s(1,1)(M_0+\eta)|S_W(t)| - s(1,1)S_W(t)d(t) \le -\eta|s(1,1)||S_W(t)| \tag{2.25}$$

which is just the finite time reaching condition, η is a positive number. Hence, the sliding mode controller is obtained as

$$\begin{aligned} U_1(t) &= \frac{1}{s(1,1)}\Big\{u_t(1,t) - \int_0^1 k(1,y)u_t(y,t)dy \\ &+s(1,0)u_x(0,t) + \int_0^1 s_y(1,y)u_y(y,t)dy - b^2\textstyle\int_0^1 s(1,y)u(y,t)dy \\ &+b^2\int_0^1 s(1,y)\cosh(by)u(0,t)dy - b^3\int_0^1 s(1,y)\int_0^y \sinh(b(y-\xi))u(\xi,t)d\xi dy\Big\} \\ &+(M_0+\eta)\frac{S_W(t)}{|S_W(t)|}. \end{aligned} \tag{2.26}$$

Under the control (2.26), the closed-loop of the target system (2.7) becomes

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \\ w_x(0,t) = c_0 w_t(0,t), \\ w_t(1,t) = s(1,1)(M_0+\eta)\dfrac{S_W(t)}{|S_W(t)|} - s(1,1)d(t) \triangleq \tilde{d}(t), t \ge 0. \end{cases} \tag{2.27}$$

The next result confirms the existence and uniqueness of the solution to Eq. 2.27 and the finite time reaching condition to the sliding mode surface S_W.

Proposition 2.1 *Suppose that d is measurable and $|d(t)| \leqslant M_0$ for all $t \geqslant 0$, and let S_W be defined by* Eq. 2.15. *Then for any $w(\cdot,0) \in \mathcal{H}$, $S_W(0) \neq 0$, there exists a $t_{max} > 0$ such that* Eq. 2.27 *admits a unique solution $w \in C(0, t_{max}; \mathcal{H})$ and $S_W(t) = 0$ for all $t \geqslant t_{max}$. Moreover, $S_W(t) = w(1,t)$ is continuous and monotone in* $[0, t_0]$.

Proof Define an operator $\mathcal{A}$ as follows:

$$\begin{cases} \mathcal{A}(f,g)^\top = (g, f''), \forall (f,g)^\top \in D(\mathcal{A}), \\ D(\mathcal{A}) = \Big\{(f,g)^\top \in H^2(0,1) \times H^1(0,1) \Big| \, f'(0) = cg(0), g(1) = 0\Big\}. \end{cases} \tag{2.28}$$

We claim that $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$. To this purpose, it suffices to show that $\mathcal{A}^*$, the adjoint operator of $\mathcal{A}$, generates a C_0-semigroup on $\mathcal{H}$.

A straightforward calculation shows that

$$\begin{cases} \mathcal{A}^*(\varphi,\psi)^\top = (-\psi, -\varphi''), \forall (\varphi,\psi)^\top \in D(\mathcal{A}^*), \\ D(\mathcal{A}^*) = \Big\{(\varphi,\psi)^\top \in H^2(0,1) \times H^1(0,1) \Big| \, \varphi'(0) = -c\psi(0), \psi(1) = 0\Big\}. \end{cases} \tag{2.29}$$

Take both sides of Eq. 2.27 with $(\varphi,\psi)^\top \in D(A^*)$ to get

$$\begin{aligned} &\frac{d}{dt}\left\langle \begin{pmatrix} w \\ w_t \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} w \\ w_t \end{pmatrix}, A^* \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &+\left\langle \begin{pmatrix} -\delta'(x-1) + \delta(x-1) \\ 0 \end{pmatrix} \tilde{d}(t), \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_{D(A^*)' \times D(A^*)} \end{aligned}$$

where $D(A^*)'$ is the dual of $D(A^*)$ with the pivot space $\mathcal{H}$. Then, system (2.27) can be written as

$$\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathcal{A}\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathcal{B}\tilde{d}(t),\ \ \mathcal{B} = \begin{pmatrix} -\delta'(x-1)+\delta(x-1) \\ 0 \end{pmatrix} \tag{2.30}$$

and $\delta(\cdot)$ is the Dirac disturbance.

We may suppose without loss of generality that $w(1,0) > 0$ since the proof for $w(1,0) < 0$ is similar. In this case, it follows from Eq. 2.27 that

$$\tilde{d}(t) = s(1,1)(M_0+\eta) - s(1,1)d(t). \tag{2.31}$$

Let $\mathcal{A}$ be defined by Eq. 2.28. For any $(f,g)^\top \in D(\mathcal{A})$, we have

$$\mathrm{Re}\langle \mathcal{A}(f,g)^\top, (f,g)^\top\rangle_{\mathcal{H}} = \int_0^1 [g'(x)\overline{f'(x)} + f''(x)\overline{g(x)}]dx + g(1)\overline{f(1)} = -c|g(0)|^2 \leq 0. \tag{2.32}$$

So, $\mathcal{A}$ is dissipative. Since by later (2.38), $1 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, it follows that $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{\mathcal{A}t}$ on $\mathcal{H}$ by the Lumer-Phillips theorem ([21, Theorem 4.3, p.14]).

Consider the dual system of Eq. 2.30 ,

$$\begin{cases} w^*_{tt}(x,t) = w^*_{xx}(x,t),\ \ x\in(0,1), t>0 \\ w^*_x(0,t) = cw^*_t(1,t),\ w^*_t(1,t) = 0,\ \ t\geq 0 \\ y(t) = w^*_x(1,t) + w^*(1,t), \end{cases} \tag{2.33}$$

We may consider, without loss of generality, the real solution of Eq. 2.33 only. Let

$$E(t) = \frac{1}{2}\int_0^1 \left[w^{*2}_x(x,t) + w^{*2}_t(x,t)\right]dx + \frac{1}{2}w^{*2}(1,t), \tag{2.34}$$

and

$$\rho(t) = \int_0^1 x w^*_x(x,t) w^*_t(x,t) dx \tag{2.35}$$

Then, $E(t) \leq E(0)$ and $|\rho(t)| \leq E(t)$ for all $t \geq 0$. Differentiate $\rho(t)$ with respect to t to yield

$$\dot{\rho}(t) = \frac{1}{2}w^*_x(1,t) - E(t) + \frac{1}{2}w^{*2}(1,t). \tag{2.36}$$

We thus have

$$\int_0^T \left[w^*_x(1,t) + w^*(1,t)\right]^2 dt \leq 2\int_0^T \left[w^{*2}_x + w^{*2}(1,t)\right] dt \leq 4(T+2)E(0). \tag{2.37}$$

A straightforward computation shows that ([22, p. 141–142])

$$
\begin{aligned}
&(I-\mathcal{A}^*)^{-1}\begin{pmatrix} f \\ g \end{pmatrix}\\
&=\begin{pmatrix} C_1e^x + C_2e^{-x} - \frac{1}{2}\int_0^x (e^{x-s}-e^{-(x-s)})(f(s)+g(s))ds \\ C_1e^x + C_2e^{-x} - \frac{1}{2}\int_0^x (e^{x-s}-e^{-(x-s)})(f(s)+g(s))ds - f \end{pmatrix}\\
&\mathcal{B}^*(I-\mathcal{A}^*)^{-1}\begin{pmatrix} f \\ g \end{pmatrix} = 2C_1e - \int_0^1 (e^{1-s}-e^{-(1-s)})(f(s)+g(s))ds,\ \forall \begin{pmatrix} f \\ g \end{pmatrix}\in\mathcal{H},\\
&\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} c-1 & c+1 \\ e & e^{-1} \end{pmatrix}^{-1}\begin{pmatrix} cf(0) \\ f(1)+\frac{1}{2}\int_0^1 (e^{1-s}-e^{-(1-s)})(f(s)+g(s))ds \end{pmatrix}
\end{aligned}
\tag{2.38}
$$

which shows that $\mathcal{B}^*(I-\mathcal{A}^*)^{-1}$ is bounded on $\mathcal{H}$. This, together with Eq. 2.37, shows that $\mathcal{B}^*$ is admissible for the C_0-semigroup $e^{\mathcal{A}^*t}$ generated by $\mathcal{A}^*$ ([23, Theorem 4.4.3, p.127]). Therefore, system (2.30) admits a unique weak solution.

Moreover, for any $T>0$, there exists a constant $C_T>0$ such that [24]

$$\|(w(\cdot,t),w_t(\cdot,t))^\top\|_{\mathcal{H}} \le C_T[\|(w(\cdot,0),w_t(\cdot,0))^\top\|_{\mathcal{H}} + \|\tilde{d}\|_{L^2(0,T)}],\ \forall t\in[0,T]. \tag{2.39}$$

Now, for $\tilde{d}$ defined by Eq. 2.31, since $H_0^1(0,T)$ is dense in $L^2(0,T)$, take $\tilde{d}_n\in H_0^1$ such that

$$\lim_{n\to\infty}\|\tilde{d}_n-\tilde{d}\|_{L^2(0,T)}=0. \tag{2.40}$$

Let $(w_n(\cdot,t),w_{nt}(\cdot,t))^\top$ be the solution of Eq. 2.30 corresponding to $\tilde{d}_n$, and the initial value $(w_n(\cdot,0),w_{nt}(\cdot,0))^\top\in D(\mathcal{A})$ ($D(\mathcal{A})$ is dense in $\mathcal{H}$) where

$$\lim_{n\to\infty}\|(w_n(\cdot,0),w_{nt}(\cdot,0))^\top-(w(\cdot,0),w_t(\cdot,0))^\top\|_{\mathcal{H}}=0 \tag{2.41}$$

It follows from Eq. 2.39 that

$$\lim_{n\to\infty}\|(w_n(\cdot,t),w_{nt}(\cdot,t))^\top-(w(\cdot,t),w_t(\cdot,t))^\top\|_{\mathcal{H}}=0 \tag{2.42}$$

uniformly in $t\in[0,T]$. By proposition 4.2.1 of [23, p. 120], we know that $(w_n(\cdot,t),w_{nt}(\cdot,t))^\top$ is the classical solution of Eqs. 2.30 or 2.27. Consequently,

$$w_{nt}(1,t)=\tilde{d}_n(t) \quad \text{or} \quad w_n(1,t)=w_n(1,0)+\int_0^t\tilde{d}_n(\tau)d\tau. \tag{2.43}$$

Passing to the limit as $n\to\infty$ in above equality, we obtain

$$w(1,t)=w(1,0)+\int_0^t\tilde{d}(\tau)d\tau,\ \forall t\in[0,T]. \tag{2.44}$$

Since T is arbitrary, we see that $S_W(t)=w(1,t)$ to Eq. 2.27 with $\tilde{d}$ defined by Eq. 2.31 is continuous in $(0,\infty)$ for any initial value in the state space. Furthermore, owing to Eq. 2.44, it has

$$
\begin{aligned}
S_W(t) &= w(1,t)=w(1,0)+\int_0^t\tilde{d}(\tau)d\tau=w(1,0)+\int_0^t[s(1,1)(M_0+\eta)-s(1,1)d(t)]\,ds\\
&\le w(1,0)-\eta t|s(1,1)|.
\end{aligned}
\tag{2.45}
$$

It is seen that $S_W(t)$ is decreasing in t. Since $w(1,0)>0$, there exists some $t_0>0$ such that $w(1,t)>0$ for $t\in[0,t_0)$ and $w(1,t)=0$ for all $t\ge t_0$. This completes the proof. □

Remark 2.1 When $w(1,0)<0$, Eq. 2.45 becomes

$$S_W(t) = w(1,t)=w(1,0)+\int_0^t \tilde{d}(\tau)d\tau = w(1,0)+\int_0^t [-s(1,1)(M_0+\eta)-s(1,1)d(t)]\,ds \geq w(1,0)+\eta t|s(1,1)|.$$

In this case, $S_W(t)$ is a continuous increasing function, and hence, there exists some $t_0>0$ such that $w(1,t)<0$ for $t\in[0;t_0)$ and $w(1,t)=0$ for all $t\geq t_0$. In particular, if $w(1,0)=S_W(0)=0$, then $t_0=0$.

Returning back to the system (2.5) under the transformation (2.6), feedback control (2.26), we obtain the main result of this section from Proposition

Theorem 2.1 *Suppose that d is measurable and $|d(t)|\leqslant M_0$ for all $t\geqslant 0$, and let S_U be the sliding mode function given by*

$$S_U(t) = u(1,t)-\int_0^1 k(1,y)u(y,t)dy-\int_0^1 s(1,y)u_t(y,t)dy. \tag{2.46}$$

Then, for any $(u(\cdot,0),u_t(\cdot,0))^\top\in\mathcal{H}$, $S_U(0)\neq 0$, there exists a $t_{max}>0$ such that the closed-loop system of Eq. 1.1 *under the feedback control* (2.26)

$$\begin{cases} u_{tt}(x,t)= u_{xx}(x,t)-\alpha_x(x,t),\\ u_x(0,t) = \alpha(0,t)-qu_t(0,t),\\ u_x(1,t) = \dfrac{1}{s(1,1)}\Big\{u_t(1,t)-\displaystyle\int_0^1 k(1,y)u_t(y,t)dy \\ \qquad +s(1,0)u_x(0,t)+\displaystyle\int_0^1 s_y(1,y)u_y(y,t)dy - b^2\textstyle\int_0^1 s(1,y)u(y,t)dy \\ \qquad +b^2\displaystyle\int_0^1 s(1,y)\cosh(by)u(0,t)dy-b^3\int_0^1 s(1,y)\textstyle\int_0^y \sinh(b(y-\xi))u(\xi,t)d\xi dy\Big\} \\ \qquad +(M_0+\eta)\dfrac{S_W(t)}{|S_W(t)|}+d(t).\\ \alpha_{xx}(x,t) - b^2\alpha(x,t)+b^2u_x(x,t)=0,\\ \alpha_x(0,t)=0,\\ \alpha(1,t) = -b\displaystyle\int_0^1 \sinh(b(1-s))u_x(s,t)ds \end{cases} \tag{2.47}$$

admits a unique solution $(u,u_t)\in C(0,t_{max};\mathcal{H})$ and $S_U(t)=0$ for all $t\geqslant t_{max}$. On the sliding mode surface $S_U(t)=0$, the system (1.1) *becomes*

$$\begin{cases} u_{tt}(x,t)=u_{xx}(x,t)-\alpha_x(x,t),\\ u_x(0,t)=\alpha(0,t)-qu_t(0,t),\\ u(1,t)-\displaystyle\int_0^1 k(1,y)u(y,t)dy-\int_0^1 s(1,y)u_t(y,t)dy=0,\\ \alpha_{xx}(x,t)-b^2\alpha(x,t)+b^2u_x(x,t)=0,\\ \alpha_x(0,t)=0,\\ \alpha(1,t)=-b\displaystyle\int_0^1 \sinh(b(1-s))u_x(s,t)ds \end{cases} \tag{2.48}$$

hence is exponentially stable in $\mathcal{H}$ with the decay rate $-c$.

3 Concluding Remarks

In this paper, we deal with the stabilization of a shear beam system which has disturbance on the input boundary. The sliding mode approach is adopted. By SMC approach, we can remove the restriction of the disturbance and the rejection of the disturbance can be achieved. The existence and uniqueness of the solution for the closed-loop system by SMC are proved. The reaching condition is presented without differentiation of the sliding mode function for which it may not exist for the weak solution of the closed-loop system.

Acknowledgements This work was supported by the National Natural Science Foundation of China, the Scientific Research Project of Beijing Municipal Commission of Education (grant no. KM 201511232018)

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