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Transmission problem of Schrödinger and wave equation with viscous damping

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A B S T R A C T

In this paper, we consider the transmission problem of a Schrödinger equation with a viscous damped wave equation which acts as a controller of the system. We show that the system operator generates a C_0-semigroup of contractions in the energy state space, and the system is well-posed. By giving the asymptotic expressions of the eigenvalues of the system, we know they all locate in the left hand side of the complex plane. It follows that the C_0-semigroup generated by the system operator achieves strong stability when the feedback gain is real.

1. Introduction

In the past few decades, there have been extensive literature on the control of Schrödinger equation (see [1–4]). In [2], the collocated boundary control is designed to exponentially stabilize the Schrödinger system

$$\begin{cases} w_t(x,t) + iw_{xx}(x,t) = 0, & x \in (0,1),\ t > 0, \\ w_x(1,t) = 0, & t \geq 0, \\ w(0,t) = U(t), & t \geq 0, \\ Y(t) = w_x(0,t), & t \geq 0, \end{cases} \tag{1.1}$$

where $U(t)$ is the control input and $Y(t)$ is the output observation. When $U(t) = -icY(t)$, where $c > 0$ is a positive constant, the authors in [2] showed that the system operator of the closed-loop system generates an exponentially stable semigroup in the energy space; and the eigenvalues approach a vertical line parallel to the imaginary axis. It is also known that, there has been much interest in studying the transmission problems, that is, the vibrational propagation over bodies consisting of two physically different types of

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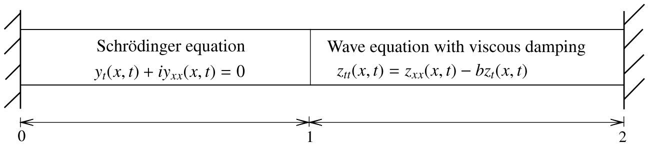

Fig. 1. Block diagram for the transmission between Schrödinger–Wave system.

materials [5,6]. The transmission problem to hyperbolic equations was studied in [5], the authors proved the existence and regularity of solutions for the linear problem. In [6], the authors study the wave propagations over materials consisting of elastic and viscoelastic components, and they proved that the dissipation produced by the viscoelastic part is strong enough to ensure the decay of the solution.

We are thus inspired and interested in studying the transmission problem of the Schrödinger equation with a viscous damped wave equation, to show through vibration and energy exchanging in the intersection, the whole system could finally be stable. In other words, we replace the static feedback in [2] by dynamic feedback governed by a wave equation with viscous damping and view it as a controller. The transmission problem of Schrödinger–wave system (as shown in Fig. 1) is written as follows:

$$\begin{cases} y_t(x,t) + iy_{xx}(x,t) = 0, & 0 < x < 1,\ t > 0, \\ z_{tt}(x,t) = z_{xx}(x,t) - bz_t(x,t), & 1 < x < 2,\ t > 0, \\ y(0,t) = z(2,t) = 0, & t \geq 0, \\ y(1,t) = kz_t(1,t), & t \geq 0, \\ z_x(1,t) = iky_x(1,t), & t \geq 0, \end{cases} \tag{1.2}$$

where $b > 0$. The two equations are connected at $x = 1$ and fixed at each end.

By introducing the following transformation

$$\begin{cases} w(x,t) = y(1-x,t), & 0 < x < 1,\ t > 0, \\ u(x,t) = z(x+1,t), & 0 < x < 1,\ t > 0, \end{cases} \tag{1.3}$$

then (1.2) becomes

$$\begin{cases} w_t(x,t) + iw_{xx}(x,t) = 0, & 0 < x < 1,\ t > 0, \\ u_{tt}(x,t) = u_{xx}(x,t) - bu_t(x,t), & 0 < x < 1,\ t > 0, \\ w(1,t) = u(1,t) = 0, & t \geq 0, \\ w(0,t) = ku_t(0,t), & t \geq 0, \\ u_x(0,t) = ikw_x(0,t), & t \geq 0. \end{cases} \tag{1.4}$$

The energy function for (1.4) is given by

$$E(t) = \frac{1}{2}\int_0^1 \left[|w(x,t)|^2 + |u_x(x,t)|^2 + |u_t(x,t)|^2\right] dx. \tag{1.5}$$

In this paper, we analyze the spectrum for the connected system (1.4). The system is well-posed and we give the asymptotic expressions of the eigenvalues. We can see that this kind of design is effective, since it moves the eigenvalues of the Schrödinger and wave equation into the left hand side of the complex plane. Finally, the strong stability of the system is achieved.

The rest of this paper is organized as follows. Section 2 is devoted to present the well-posedness of the system. In Section 3 we give the spectral and stability analysis of the system. The conclusion remarks are shown in Section 4.

2. Well-posedness of system (1.4)

We consider system (1.4) in the energy space $\mathcal{H} = L^2(0,1) \times H_E^1(0,1) \times L^2(0,1)$, where $H_E^1(0,1) = \{g \in H^1(0,1) | g(1) = 0\}$. The norm in $\mathcal{H}$ is induced by the inner product

$$\langle X_1, X_2 \rangle = \int_0^1 \left[f_1(x)\overline{f_2(x)} + g_1'(x)\overline{g_2'(x)} + h_1(x)\overline{h_2(x)} \right] dx, \tag{2.1}$$

where $X_s = (f_s, g_s, h_s) \in \mathcal{H}$, $s = 1, 2$. Define the system operator of (1.4) by

$$\begin{cases} \mathcal{A}(f,g,h) = (-if'',\ h,\ g'' - bh\ ), \quad \forall\, (f,g,h) \in D(\mathcal{A}), \\ D(\mathcal{A}) = \left\{ (f,g,h) \in \mathcal{H}, \mathcal{A}(f,g,h) \in \mathcal{H}, \left| \begin{array}{l} f(1) = 0, \\ f(0) = kh(0), \\ g'(0) = ikf'(0) \end{array} \right. \right\}. \end{cases} \tag{2.2}$$

Then (1.4) can be written as an evolution equation in $\mathcal{H}$:

$$\begin{cases} \dfrac{dX(t)}{dt} = \mathcal{A}X(t), \quad t > 0, \\ X(0) = X_0, \end{cases} \tag{2.3}$$

where $X(t) = (w(\cdot,t), u(\cdot,t), u_t(\cdot,t))$.

Theorem 2.1. *Let $\mathcal{A}$ be given by (2.2). Then $\mathcal{A}^{-1}$ exists, and hence $0 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. Moreover $\mathcal{A}$ is dissipative in $\mathcal{H}$ and $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions in $\mathcal{H}$.*

Proof. For any given $(f_1, g_1, h_1) \in \mathcal{H}$, by

$$\mathcal{A}(f,g,h) = (-if'', h, g'' - bh) = (f_1, g_1, h_1), \tag{2.4}$$

we have

$$\begin{cases} f''(x) = if_1(x), \quad h(x) = g_1(x), \quad g''(x) - bh(x) = h_1(x), \\ f(1) = g(1) = 0, \quad f(0) = kh(0), \quad g'(0) = ikf'(0), \end{cases} \tag{2.5}$$

which gives

$$f'(x) = f'(0) + i\int_0^x f_1(s)ds,$$

by the boundary condition $f(1) = 0$, we get

$$f(x) = f'(0)(x-1) - i\int_0^x (1-x)f_1(\xi)d\xi - i\int_x^1 (1-\xi)f_1(\xi)d\xi. \tag{2.6}$$

Substituting $f(0) = kh(0) = kg_1(0)$ into (2.6), we have

$$f'(0) = -kg_1(0) - i\int_0^1 (1-\xi)f_1(\xi)d\xi.$$

So that

$$g'(0) = ikf'(0) = -ik^2 g_1(0) + k\int_0^1 (1-\xi)f_1(\xi)d\xi.$$

Then

$$g'(x) = g'(0) + \int_0^x [bg_1(s) + h_1(s)]ds,$$

by the boundary condition $g(1) = 0$ we have

$$g(x) = g'(0)(x-1) - (1-x)\int_0^x [bg_1(\xi) + h_1(\xi)]d\xi - \int_x^1 (1-\xi)[bg_1(\xi) + h_1(\xi)]d\xi.$$

To sum up, the solution of (2.5) gives

$$\begin{cases} f(x) = f'(0)(x-1) - i\int_0^x (1-x)f_1(\xi)d\xi - i\int_x^1 (1-\xi)f_1(\xi)d\xi, \\ g(x) = g'(0)(x-1) - (1-x)\int_0^x [bg_1(\xi) + h_1(\xi)]d\xi - \int_x^1 (1-\xi)[bg_1(\xi) + h_1(\xi)]d\xi, \\ h(x) = g_1(x), \qquad f'(0) = -kg_1(0) - i\int_0^1 (1-\xi)f_1(\xi)d\xi, \\ g'(0) = -ik^2 g_1(0) + k\int_0^1 (1-\xi)f_1(\xi)d\xi. \end{cases}$$

Hence, we get the unique solution $(f, g, h) \in D(\mathcal{A})$ to equation (2.4), thus $\mathcal{A}^{-1}$ exists. Now we show that $\mathcal{A}$ is dissipative in $\mathcal{H}$. Let $X = (f, g, h) \in D(\mathcal{A})$. Then we have

$$\begin{aligned} \langle \mathcal{A}X, X\rangle &= \langle(-if'', h, g'' - bh), (f, g, h)\rangle \\ &= \int_0^1 (-if'')\overline{f}dx + \int_0^1 h'\overline{g'}dx + \int_0^1 (g'' - bh)\overline{h}dx \\ &= -if'\overline{f}\Big|_0^1 + i\int_0^1 |f'|^2 dx + \int_0^1 h'\overline{g'}dx + g'\overline{h}\Big|_0^1 - \int_0^1 g'\overline{h'}dx - b\int_0^1 |h|^2 dx \\ &= -if'(1)\overline{f}(1) + if'(0)\overline{f}(0) + i\int_0^1 |f'|^2 dx + \int_0^1 h'\overline{g'}dx + \overline{h}(1)g'(1) \\ &\quad - \overline{h}(0)g'(0) - \int_0^1 \overline{h'}g'dx - b\int_0^1 |h|^2 dx \\ &= if'(0)\overline{f}(0) + i\int_0^1 |f'|^2 dx + \int_0^1 h'\overline{g'}dx - g'(0)\overline{h}(0) - b\int_0^1 |h|^2 dx - \int_0^1 \overline{h'}g'dx \\ &= -b\int_0^1 |h|^2 dx + \left(i\int_0^1 |f'|^2 dx + \int_0^1 h'\overline{g'}dx - \int_0^1 \overline{h'}g'dx\right) \end{aligned}$$

and

$$\mathrm{Re}\langle \mathcal{A}X, X\rangle = -b\int_0^1 |h|^2 dx \le 0. \tag{2.7}$$

Hence $\mathcal{A}$ is dissipative and $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions in $\mathcal{H}$ by the Lumer–Phillips theorem [7]. The proof is complete. □

3. Spectral analysis and stability

In this section, we first consider the eigenvalue problem of (1.4). Let $\mathcal{A}X = \lambda X$, where $0 \neq X = (f, g, h) \in D(\mathcal{A})$, then f, g, h satisfy:

$$\begin{cases} f''(x) - i\lambda f(x) = 0, \qquad h(x) = \lambda g(x), \\ g''(x) = (\lambda^2 + b\lambda)g(x) = \lambda(\lambda + b)g(x), \\ f(1) = g(1) = 0, \\ f(0) = kh(0), \qquad g'(0) = ikf'(0). \end{cases} \tag{3.1}$$

Let $p(\lambda) = \lambda^2 + b\lambda$, (3.1) changes to

$$\begin{cases} f''(x) = i\lambda f(x), \qquad g''(x) = p(\lambda)g(x), \\ f(1) = g(1) = 0, \\ f(0) = \lambda k g(0), \qquad g'(0) = ikf'(0). \end{cases} \tag{3.2}$$

We can get

$$f(x) = c_1 \sinh(\sqrt{i\lambda})(1-x), \qquad g(x) = c_2 \sinh(\sqrt{p(\lambda)})(1-x), \tag{3.3}$$

where c_1 and c_2 are constants. Substituting these into the boundary conditions of (3.2), we have

$$\begin{cases} c_1 \sinh(\sqrt{i\lambda}) = c_2 k\lambda \sinh(\sqrt{p(\lambda)}), \\ c_2\sqrt{p(\lambda)}\cosh(\sqrt{p(\lambda)}) = ikc_1\sqrt{i\lambda}\cosh(\sqrt{i\lambda}), \end{cases} \tag{3.4}$$

i.e.

$$\begin{cases} c_1 = \dfrac{c_2 k\lambda \sinh(\sqrt{p(\lambda)})}{\sinh(\sqrt{i\lambda})}, \\ c_2\left[ik^2\lambda\sqrt{i\lambda}\sinh(\sqrt{p(\lambda)})\cosh(\sqrt{i\lambda}) - \sqrt{p(\lambda)}\cosh(\sqrt{p(\lambda)})\sinh(\sqrt{i\lambda})\right] = 0. \end{cases} \tag{3.5}$$

Then (3.2) has the nontrivial solution if and only if

$$ik^2\lambda\sqrt{i\lambda}\sinh(\sqrt{p(\lambda)})\cosh(\sqrt{i\lambda}) - \sqrt{p(\lambda)}\cosh(\sqrt{p(\lambda)})\sinh(\sqrt{i\lambda}) = 0$$

have solutions. Hence, we get the following lemma immediately.

Lemma 3.1. *Let $\mathcal{A}$ be defined by (2.2), and let*

$$\Delta(\lambda) = ik^2\lambda\sqrt{i\lambda}\sinh(\sqrt{p(\lambda)})\cosh(\sqrt{i\lambda}) - \sqrt{p(\lambda)}\cosh(\sqrt{p(\lambda)})\sinh(\sqrt{i\lambda}). \tag{3.6}$$

Then

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} | \Delta(\lambda) = 0\}. \tag{3.7}$$

Lemma 3.2. *Let $\mathcal{A}$ be defined by (2.2). Then for each $\lambda \in \sigma_p(\mathcal{A})$, we have* $\operatorname{Re}\lambda < 0$.

Proof. By Theorem 2.1, since $\mathcal{A}$ is dissipative, we have for each $\lambda \in \sigma(\mathcal{A})$, $\operatorname{Re}\lambda \le 0$. So we only need to show there is not any eigenvalue on the imaginary axis. Let $\lambda = \pm i\mu^2 \in \sigma_p(\mathcal{A})$ with $\mu \in \mathbb{R}^+$ and $X = (f, g, h) \in D(\mathcal{A})$ be its associated eigenfunction of $\mathcal{A}$. Then by (2.7), we have

$$\operatorname{Re}\langle \mathcal{A}X, X\rangle = -b\int_0^1 |h|^2 dx = 0.$$

Hence $h(x) = 0$. By the second equations of (3.1), we have $g(x) = 0$. Then by the first equation of (3.1) and its boundary conditions we have:

$$\begin{cases} f''(x) = i\lambda f(x), \\ f(0) = f'(0) = f(1) = 0. \end{cases}$$

A direct computation yields $f(x) = 0$. Hence, $X = (f, g, h) = 0$. Therefore, there is no eigenvalue on the imaginary axis. These complete the proof. □

Proposition 3.1. *Let $\mathcal{A}$ be defined by (2.2), and let $\Delta(\lambda)$ be given by (3.6). The eigenvalues of $\mathcal{A}$ have the following asymptotic expressions:*

$$\lambda_{1n} = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4n^2\pi^2}}{2} + \mathcal{O}(n^{-1/2}), \quad n \in \mathbb{N}, \tag{3.8}$$

and

$$\lambda_{2n} = \left(n - \frac{1}{2}\right)^2 \pi^2 i + \mathcal{O}(n^{-1}), \quad n \in \mathbb{N}. \tag{3.9}$$

Proof. By $\Delta(\lambda) = 0$, we have

$$\begin{aligned} 4\Delta(\lambda) = {} & ik^2\lambda\sqrt{i\lambda}(e^{\sqrt{p(\lambda)}} - e^{-\sqrt{p(\lambda)}})(e^{\sqrt{i\lambda}} + e^{-\sqrt{i\lambda}}) \\ & - \sqrt{p(\lambda)}(e^{\sqrt{p(\lambda)}} + e^{-\sqrt{p(\lambda)}})(e^{\sqrt{i\lambda}} - e^{-\sqrt{i\lambda}}) = 0. \end{aligned} \tag{3.10}$$

Let $\lambda = \rho^2$, since Re $\lambda < 0$, then $\rho \in [\frac{\pi}{4}, \frac{3\pi}{4}]$. When $\rho \in \mathcal{S}_1 = (\frac{5\pi}{8}, \frac{3\pi}{4}]$, (3.10) becomes

$$i\sqrt{i}k^2(e^{2\sqrt{p(\lambda)}} - 1) + \mathcal{O}(\lambda^{-\frac{1}{2}}) = 0. \tag{3.11}$$

Since the solution of $e^{\sqrt{p(\lambda)}} - e^{-\sqrt{p(\lambda)}} = 0$ goes to

$$\sqrt{p(\lambda)} = -n\pi i, \quad \textit{i.e. } p(\lambda) = -n^2\pi^2, \ n \in \mathbb{N}, \tag{3.12}$$

that is

$$\lambda^2 + b\lambda = -n^2\pi^2, \quad n \in \mathbb{N},$$

which has solutions of the form

$$\hat{\lambda}_n = -\frac{b}{2} - \frac{\sqrt{b^2 - 4n^2\pi^2}}{2}, \quad n \in \mathbb{N}.$$

Applying Rouché's theorem [8], we get the solution of (3.11)

$$\lambda_{1n} = -\frac{b}{2} - \frac{\sqrt{b^2 - 4n^2\pi^2}}{2} + \mathcal{O}(n^{-1/2}), \quad n \in \mathbb{N}. \tag{3.13}$$

When $\rho \in \mathcal{S}_2 = [\frac{\pi}{4}, \frac{3\pi}{8})$, (3.10) becomes

$$i\sqrt{i}k^2(e^{\sqrt{p(\lambda)}} - e^{-\sqrt{p(\lambda)}})(e^{\sqrt{i\lambda}} + e^{-\sqrt{i\lambda}}) + \mathcal{O}(\lambda^{-\frac{1}{2}}) = 0.$$

The solution of

$$(e^{\sqrt{p(\lambda)}} - e^{-\sqrt{p(\lambda)}})(e^{\sqrt{i\lambda}} + e^{-\sqrt{i\lambda}}) = 0$$

goes to

$$e^{\sqrt{p(\lambda)}} - e^{-\sqrt{p(\lambda)}} = 0 \tag{3.14}$$

or

$$e^{\sqrt{i\lambda}} + e^{-\sqrt{i\lambda}} = 0. \tag{3.15}$$

The process of solving (3.14) is similar to what we did when $\rho \in \mathcal{S}_1$. So we have

$$\lambda_{2n_1} = -\frac{b}{2} + \frac{\sqrt{b^2 - 4n^2\pi^2}}{2} + \mathcal{O}(n^{-1/2}), \quad n \in \mathbb{N}. \tag{3.16}$$

(3.15) is equivalent to $e^{2\sqrt{i\lambda}} = -1$, and using Rouché's theorem again we get

$$\lambda_{2n_2} = \left(n - \frac{1}{2}\right)^2 \pi^2 i + \mathcal{O}(n^{-1}), \quad n \in \mathbb{N}. \tag{3.17}$$

When $\rho \in \mathcal{S}_2 = [\frac{3\pi}{8}, \frac{5\pi}{8}]$, (3.10) has no solution. The proof is complete. □

We summarize the above analysis and the main results into the following theorem.

Theorem 3.1. *The eigenvalue problem of (3.1) has the following asymptotic expressions:*

$$\lambda_{1n} = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4n^2\pi^2}}{2} + \mathcal{O}(n^{-1/2}), \quad n \in \mathbb{N}, \tag{3.18}$$

in particular,

$$\text{Re } \lambda_{1n} \to -\frac{1}{2}b < 0, \quad \text{as } |n| \to \infty, \tag{3.19}$$

that is, Re $\lambda = -\frac{1}{2}b$ *is the asymptote of the eigenvalues* λ_{1n} *given by (3.18); and*

$$\lambda_{2n} = \left(n - \frac{1}{2}\right)^2 \pi^2 i + \mathcal{O}(n^{-1}), \quad n \in \mathbb{N}, \tag{3.20}$$

when $|n| \to \infty$*, it approaches the imaginary axis from the left.*

Remark 3.1. From Lemma 3.2, we know that the real part of the eigenvalues are all negative, so 0 is not in the point spectrum. When $|n| \to \infty$, the imaginary axis is the asymptote of the second branch of eigenvalues.

Definition 3.1. A C_0-semigroup $T(t)$ is called strongly stable, if

$$\lim_{t\to\infty} \|T(t)x\| = 0.$$

Theorem 3.2. *Let* $\mathcal{A}$ *be defined by (2.2). Then the system (1.4) achieves strong stability.*

Proof. Since from Theorem 2.1, Lemmas 3.1 and 3.2, we know that $\mathcal{A}$ generate a C_0-semigroup $e^{\mathcal{A}t}$, and when $\lambda \in \sigma(\mathcal{A})$, Re $\lambda < 0$. Also, the asymptotic expressions of the eigenvalues shown in Theorem 3.1. We then get that system (1.4) is strongly stable. □

4. Concluding remarks

In this article, we study the transmission problem of a Schrödinger and wave equation with viscous damping. By detailed spectral analysis, we achieve that the eigenvalues all have negative real parts and approach vertical lines parallel to the imaginary axis. Furthermore, we can get the conclusion that although the viscous damping is weak, it still be able to ensure the strong stability of the whole system. More interesting results appear when we discuss the feedback gain $k \in \mathbb{C}$. That is, when $k = bi,\ 0 \neq b \in \mathbb{R}$, we recalculate

$$\text{Re}\langle \mathcal{A}X, X\rangle = \text{Re}\,\langle(-if'', h, g'' - bh), (f, g, h)\rangle = if'(0)\overline{h}(0)(\overline{k} - k) - b\int_0^1 |h|^2 dx$$

to see that $\mathcal{A}$ is not dissipative anymore. So the system cannot be stable when k is a pure imaginary number, although there is no difference in the expressions of the eigenvalues. Hence, more precisely, the system achieves strong stability when the feedback gain k is a real number.

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