Approximation of the controls for the beam equation with vanishing viscosity

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(joint work with Florin Bugariu and Sorin Micu)
What we study?

We consider a finite difference semi-discrete scheme for the approximation of the boundary controls of a 1-D equation modeling the transversal vibrations of a hinged beam.
Applications to built bridges
Applications to built bridges
Let us see the transversal vibrations for a beam!

Strategy: To have a good look along the beam?
Beijing 2008, Olympic Games, Two Gold Medals

We should ask them!
Please contact HE KEXIN, China!
Or ask SANDRA IZBAŞA, Romania!
The continuous model for beam equation

The boundary controlled transversal vibrations of a 1–D beam with hinged boundary conditions are modelled by the following equation

\[
\begin{align*}
    u''(t, x) + u_{xxxx}(t, x) &= 0 \quad \text{for} \quad (t, x) \in (0, T) \times (0, 1) \\
    u(t, 0) &= u(t, 1) = u_{xx}(t, 0) = 0 \quad \text{for} \quad t \in (0, T) \\
    u_{xx}(t, 1) &= v(t) \quad \text{for} \quad t \in (0, T) \\
    u(0, x) &= u^0(x) \quad \text{for} \quad x \in (0, 1) \\
    u'(0, x) &= u^1(x) \quad \text{for} \quad x \in (0, 1),
\end{align*}
\]

The vector \( \begin{pmatrix} u \\ u' \end{pmatrix} \) represents the state and \( v \) is the control acting on the extremity \( x = 1 \) of the beam.
Given $T > 0$ we say that (1) is null–controllable in time $T$ if, for every initial data \( \left( \begin{array}{c} u^0 \\ u^1 \end{array} \right) \in \mathcal{H} := H^1_0(0, 1) \times H^{-1}(0, 1) \), there exists a control $v \in L^2(0, T)$ such that the corresponding solution of (1) verifies

\[
\begin{align*}
u(T, x) &= u'(T, x) = 0 \quad (x \in (0, 1)).
\end{align*}
\]
From mathematical point of view: Bad news.

It is known that, due to the high frequency numerical spurious oscillations, the uniform (with respect to the mesh-size) controllability property of the semi-discrete model fails in the natural setting.

The convergence of the approximate boundary controls corresponding to initial data in the finite energy space cannot be guaranteed.
Good news!

We solve this deficiency by adding a vanishing numerical viscosity term, which will damp out these high frequencies.

We prove that, by adding a vanishing numerical viscosity, the uniform controllability property and the convergence of the scheme is ensured.
The observability inequality - the null controllability for the continuous case

A classical duality argument reduces the null-controllability property of (1) to the observability inequality

\[ \left\| \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}(0) \right\|^2_{\mathcal{H}} \leq C \int_0^T |\varphi_x(t,1)|^2 \, dt, \quad (3) \]

for any \( \begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix} \in \mathcal{H} \), \( \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} \) being the solution of the following homogeneous adjoint equation

\[
\begin{cases}
\varphi''(t, x) + \varphi_{xxxx}(t, x) = 0 & (t, x) \in (0, T) \times (0, 1) \\
\varphi(t, 0) = \varphi(t, 1) = \varphi_{xx}(t, 0) = \varphi_{xx}(t, 1) = 0 & t \in (0, T) \\
\varphi(T, x) = \varphi^0(x) & x \in (0, 1) \\
\varphi'(T, x) = \varphi^1(x) & x \in (0, 1).
\end{cases}
\quad (4)
\]

Boundary observability inequality (3) can be proved by using Fourier series/multipliers techniques (Lions (1988), Komornic-Loretti (2005)).
The semi-discrete model

Let $N \in \mathbb{N}^*$, $x_0 = 0 < x_1 = h < \ldots < x_j = jh < \ldots < x_{N+1} = 1$, where the mesh-size is $h = \frac{1}{N+1}$ and let two external points $x_{-1} = x_0 - h$ and $x_{N+2} = x_{N+1} + h$.

The semi–discretization of (1), for each $1 \leq j \leq N$, $t \in (0, T)$ is

\[
\begin{aligned}
&u''_j(t) + \frac{u_{j+2}(t) - 4u_{j+1}(t) + 6u_j(t) - 4u_{j-1}(t) + u_{j-2}(t)}{h^4} = 0 \\
u_0(t) = 0, \quad u_{N+1}(t) = 0 \\
u_{-1}(t) = -u_1(t), \quad u_{N+2}(t) = h^2v_h(t) - u_N(t) \\
u_j(0) = u^0_j(x), \quad u'_j(0) = u^1_j(x).
\end{aligned}
\]

(5)

The quantities $u_j(t)$ approximate $u(t, x_j)$,

If the initial data are regular, we shall choose

\[
u^0_j = u^0(jh), \quad u^1_j = u^1(jh) \quad (1 \leq j \leq N).
\]

(6)
We consider the following controllability property for (5): given $T > 0$ and \( \begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix} \) \( 1 \leq j \leq N \) \( \in \mathbb{C}^{2N} \), we look for a control \( v \in L^2(0,T) \) such that the corresponding solution \( \begin{pmatrix} u_j \\ u_j' \end{pmatrix} \) \( 1 \leq j \leq N \) of (5) verifies

\[
\begin{align*}
  u_j(T) &= u_j'(T) = 0 & (1 \leq j \leq N). \tag{7}
\end{align*}
\]

If this property is verified for every initial data \( \begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix} \) \( 1 \leq j \leq N \) \( \in \mathbb{C}^{2N} \), we say that (5) is \textit{null-controllable in time} \( T \).
The null–controllability property for (5) holds in any time $T > 0$.

Leon and Zuazua (ESAIM COCV, 2002)

For any $h > 0$, there exists a constant $C = C(T, h)$ such that

$$\left\| \left( \begin{array}{c} \varphi_j \\ \varphi_j' \end{array} \right) \right\|_{2}^{2} \leq C \int_{0}^{T} \left| \frac{\varphi_N(t)}{h} \right|^2 dt,$$

for any $\left( \begin{array}{c} \varphi_j^0 \\ \varphi_j^1 \end{array} \right) \in \mathbb{C}^{2N}$ and $\left( \begin{array}{c} \varphi_j \\ \varphi_j' \end{array} \right)_{1 \leq j \leq N}$ solution of the corresponding backward equation but

$$\lim_{h \to 0} \sup_{(\varphi, \varphi')} \frac{\left\| \left( \begin{array}{c} \varphi_j \\ \varphi_j' \end{array} \right) (0) \right\|_{1, -1}^{2}}{\int_{0}^{T} \left| \frac{\varphi_N(t)}{h} \right|^2 dt} = \infty.$$

(9)
In order to obtain a uniform observability inequality, two possibilities have been proposed and analyzed in Leon and Zuazua (ESAIM COCV, (2002)):

- The class of solutions has been restricted to a space in which the high frequencies have been filtered out. Under this assumption, the corresponding observability inequality becomes uniform and, consequently, the projection of the solution of (5) over this filtered space is controlled to zero uniformly;

- The observed quantity in the right side of (8) has been reinforced by introducing an extra term. This shows that an additional boundary control, which vanishes in limit, makes the system uniformly controllable.
The third possibility: the vanishing viscosity method

The bad spurious high frequencies introduced by the discretization process are responsible for the bad controllability properties of (5).

The third possibility

We introduce in the discrete equation (5) a numerical viscosity which vanishes in the limit.

Since this term damps out the high frequencies which are responsible for (9), we can expect that it will also help us to restore the desired uniform observability inequality and to improve the convergence properties of the discrete controls.
The new proposed perturbed problem

More precisely, for each $1 \leq j \leq N$ and $t \in (0,T)$ we consider

$$
\begin{align*}
    u''_j(t) + \frac{u_{j+2}(t) - 4u_{j+1}(t) + 6u_j(t) - 4u_{j-1}(t) + u_{j-2}(t)}{h^4} &= \varepsilon \frac{u'_{j+1}(t) - 2u'_j(t) + u'_{j-1}(t)}{h^2} \\
    u_{-1}(t) &= -u_1(t), \quad u_{N+2}(t) = h^2 v_h(t) - u_N(t) \\
    u_j(0) &= u_j^0(x), \quad u'_j(0) = u'_j(x)
\end{align*}
$$
The viscosity term

The ratio \( \varepsilon \frac{u'_{j+1}(t) - 2u'_j(t) + u'_{j-1}(t)}{h^2} \) represents a viscous term and the parameter \( \varepsilon \) depends on the step size \( h \) and verifies

\[
\lim_{h \to 0} \varepsilon(h) = 0. \tag{10}
\]

Note that the parameter \( \varepsilon \) should be chosen small, in order to preserve the convergence and the accuracy of the numerical scheme, but also sufficiently large to improve the observability properties of the system, damps out the spurious high frequencies.
Our analysis allows to obtain the minimal range of the parameter $\varepsilon$ answering to both desiderates. Indeed, we shall prove that

$$\varepsilon \geq \frac{h^2}{2T} \ln \frac{1}{h}$$

is a necessary condition for the uniform observability inequality.

Moreover, our main result ensures, for each $T > 0$, the existence of two positive constants $h_0$ and $c_0$ such that, for any $h \in (0, h_0)$ and $\varepsilon \in \left( c_0 h^2 \ln \left( \frac{1}{h} \right), h \right)$, our system is uniformly controllable in time $T$. 
More about vanishing viscosity

The artificial viscosity is a common tool in many numerical schemes.


- control problems for the wave equation: Micu (SICON (2008))

- a uniform controllability result in arbitrarily small time for Schrödinger equation: Micu and Rovenţa (ESAIM COCV (2012) and JOTA (2014))

- The corresponding result for the semi-discrete wave equation is still an open problem.
The numerical viscosity method can be easily implemented for more general domains and boundary conditions.

if the spectrum is not known explicitly and/or the control depends both on time and space variables is a challenging and still open problem.

The main problem in the boundary control approximation is mainly due to the bad numerical approximation of the normal derivative on the boundary of the high eigenfunctions.
We recall the discrete problem

In this paper we study the null–controllability of the following finite-difference space discretization equation:

\[
\begin{align*}
    u_j''(t) + \frac{u_{j+2}(t) - 4u_{j+1}(t) + 6u_j(t) - 4u_{j-1}(t) + u_{j-2}(t)}{h^4} &= \varepsilon \frac{u_{j+1}(t) - 2u_j'(t) + u_{j-1}(t)}{h^2} \\
    u_0(t) &= 0, \quad u_{N+1}(t) = 0 \\
    u_{-1}(t) &= -u_1(t), \quad u_{N+2}(t) = h^2 v_h(t) - u_N(t) \\
    u_j(0) &= u_j^0(x), \quad u_j'(0) = u_j^1(x),
\end{align*}
\]

where \(1 \leq j \leq N\), \(t \in (0, T)\).

(12)
Now, let us write (12) as an abstract Cauchy form by using the invertible and positive defined matrices $A_h, B_h \in \mathcal{M}_{N \times N}(\mathbb{R})$ given by

$$A_h = \frac{1}{h^2} \begin{pmatrix}
2 & -1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & -1 & 2 & -1 & 0 \\
0 & 0 & \ldots & 0 & -1 & 2 & -1 \\
0 & 0 & \ldots & 0 & 0 & -1 & 2 \\
\end{pmatrix} \quad \text{and} \quad B_h = A_h^2.$$
The matricial form of the equation

If we denote by

\[
U_h^0 = \begin{pmatrix} 0 \\ u_1^0 \\ u_2^0 \\ \vdots \\ u_N^0 \end{pmatrix}, \quad U_h^1 = \begin{pmatrix} 0 \\ u_1^1 \\ u_2^1 \\ \vdots \\ u_N^1 \end{pmatrix}, \quad U_h(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{pmatrix}, \quad F_h(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -v_h(t)/h^2 \end{pmatrix},
\]

then system (12) may be written vectorially as follows:

\[
\begin{cases}
U_h''(t) + B_h U_h(t) + \varepsilon A_h U_h'(t) = F_h(t) \quad t \in (0, T) \\
U_h(0) = U_h^0, \quad U_h'(0) = U_h^1.
\end{cases}
\]

(13)
The scalar products

Let us consider in $\mathbb{C}^N$ the following inner products

$$\langle f, g \rangle = h \sum_{k=1}^{N} f_k \overline{g}_k \quad (f = (f_k)_{1 \leq k \leq N} \in \mathbb{C}^N, \ g = (g_k)_{1 \leq k \leq N} \in \mathbb{C}^N),$$

$$(f, g)_1 = \langle A_h f, g \rangle \quad (f = (f_k)_{1 \leq k \leq N} \in \mathbb{C}^N, \ g = (g_k)_{1 \leq k \leq N} \in \mathbb{C}^N),$$

$$(f, g)_{-1} = \langle A_h^{-1} f, g \rangle \quad (f = (f_k)_{1 \leq k \leq N} \in \mathbb{C}^N, \ g = (g_k)_{1 \leq k \leq N} \in \mathbb{C}^N).$$

Finally, we define the discrete inner product in $\mathbb{C}^{2N}$

$$\langle (f^1, f^2), (g^1, g^2) \rangle_{1,-1} = (f^1, g^1)_1 + (f^2, g^2)_{-1} \quad (f^1, f^2, g^1, g^2 \in \mathbb{C}^N).$$
The discrete variational lemma

Lemma

Given $T > 0$, system (13) is null-controllable in time $T$ if, and only if, for any initial data $\begin{pmatrix} U_h^0 \\ U_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$ there exists $v_h \in L^2(0,T)$ which verifies

$$
\int_0^T v_h(t) \frac{W_{hN}(t)}{h} \, dt = \langle U_h^1, W_h(0) \rangle - \langle U_h^0, W'_h(0) - \varepsilon A_h W_h(0) \rangle,
$$

(14)

where $\begin{pmatrix} W'_h & W'_h \end{pmatrix}$ is the solution of homogeneous “adjoint” backward problem and $\begin{pmatrix} W_h^0 \\ W_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$. 
Now, if we put $Z = \begin{pmatrix} W_h(t) \\ W'_h(t) \end{pmatrix}$, then the homogeneous “adjoint” backward problem has the following equivalent vectorial form

$$\begin{aligned}
Z' &= A_h Z \\
Z(T') &= Z_T,
\end{aligned}$$

(15)

where the operator $A_h$ is given by $A_h = \begin{pmatrix} 0 & I \\ -A_h^2 & \varepsilon A_h \end{pmatrix}$. 

The matricial form of the adjoint problem
The eigenvalues and eigenvectors of $A_h$

The eigenvalues of $A_h$ are given by

$$ \mu_n = \frac{4}{h^2} \sin^2 \left( \frac{n\pi h}{2} \right) \quad (1 \leq n \leq N), \quad (16) $$

with the corresponding eigenvectors

$$ \phi_h^n = (\sin(knh\pi))_{1 \leq k \leq N} \in \mathbb{R}^N \quad (1 \leq n \leq N). \quad (17) $$
The eigenvalues and eigenvectors of $A_h$

**Lemma**

The eigenvalues of the operator $A_h$ are given by

$$\lambda_n = \mu |n| \frac{\varepsilon + i \text{sgn}(n) \sqrt{4 - \varepsilon^2}}{2} \quad (1 \leq |n| \leq N),$$

(18)

and the corresponding eigenvectors are

$$\Phi^n_h = \frac{1}{\sqrt{\mu |n|}} \begin{pmatrix} 1 \\ \lambda_n \end{pmatrix} \phi^n_h \quad (1 \leq |n| \leq N),$$

(19)

where $\phi^n_h$ are given by (17). Moreover, the vectors $(\Phi^n_h)_{1 \leq |n| \leq N}$ form a basis in $\mathbb{C}^{2N}$ and, for any $Z = \sum_{1 \leq |n| \leq N} a_n \Phi^n_h$, we have that

$$\left(1 - \frac{\varepsilon}{2}\right) \sum_{1 \leq |n| \leq N} |a_n|^2 \leq \|Z\|_{1,-1}^2 \leq \left(1 + \frac{\varepsilon}{2}\right) \sum_{1 \leq |n| \leq N} |a_n|^2.$$  

(20)
Theorem

Let $T > 0$. System (13) is null–controllable in time $T$ if, and only if, for any initial data $\begin{pmatrix} U^0_h \\ U^1_h \end{pmatrix} \in \mathbb{C}^{2N}$ of the form

$$
\begin{pmatrix} U^0_h \\ U^1_h \end{pmatrix} = \sum_{1 \leq |n| \leq N} a^0_{nh} \Phi^n_h,
$$

there exists a control $v_h \in L^2(0, T)$ such that for each $1 \leq |n| \leq N$

$$
\int_0^T v_h(t) e^{\lambda_n t} dt = \frac{(-1)^{n+1}}{2 \cos \left( \frac{n\pi h}{2} \right)} \left( \text{sgn}(n) i \sqrt{4 - \varepsilon^2} a^0_{nh} + \varepsilon a^0_{nh} + \varepsilon a^0_{-nh} \right).
$$

The moment problems have been, from the very beginning, one of the most successful method for controllability problems (see the books of Avdonin and Ivanov (1995), Coron (2007), Komornic and Loretti (2005), Russel (1978), Tucsnak and Weiss (2009)).
A sequence \((\theta_m)_{1 \leq |m| \leq N} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)\) is biorthogonal to the family of exponential functions \((e^{\lambda_n t})_{1 \leq |n| \leq N} \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right)\) if

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_m(t) e^{\lambda_n t} dt = \delta_{mn} \quad (1 \leq |m|, |n| \leq N).
\]
If \((\theta_m)_{1 \leq |m| \leq N}\) is a biorthogonal sequence to the family of exponential functions \((e^{\lambda_n t})_{1 \leq |n| \leq N}\) in \(L^2\left(-\frac{T}{2}, \frac{T}{2}\right)\), then a solution \(v_h\) of (22) is

\[
v_h(t) = \sum_{|n|=1}^{N} \frac{(-1)^{n+1} e^{-\lambda_n \frac{T}{2}}}{2 \cos \left(\frac{n\pi h}{2}\right)} \cdot \left(\text{sgn}(n) i \sqrt{4 - \varepsilon^2} a^0_{nh} + \varepsilon a^0_{nh} + \varepsilon a^0_{-nh}\right) \theta_n \left(t - \frac{T}{2}\right) \ (t \in (0, T)).
\]

(24)

The KEY POINT

To show that there exists a biorthogonal sequence \((\theta_m)_{1 \leq |m| \leq N}\) to the family \((e^{\lambda_n t})_{1 \leq |n| \leq N}\) in \(L^2\left(-\frac{T}{2}, \frac{T}{2}\right)\) and to evaluate its \(L^2\)–norm.
Construction of the biorthogonal sequence

- we evaluate a biorthogonal sequence to $\Lambda = (e^{\lambda n t})_{1 \leq |n| \leq N}$ in $L^2 \left(-\frac{T}{2}, \frac{T}{2}\right)$.

- we are interested not only on their existence, but also on their dependence of the parameters $h$ and $\varepsilon$ by evaluating its $L^2$--norm.

- we need to treat separately the biorthogonal sequences for the families $\Lambda_1 = (e^{\lambda n t})_{n \in F}$ and $\Lambda_2 = (e^{\lambda n t})_{n \in G}$.

- the exponents of $\Lambda_2$ have a gap at least $\gamma$ whereas those of $\Lambda_1$ are not so well separated.

- the cardinality of these two families is different: $\Lambda_2$ has a number of elements which goes to infinity when $h$ tends to zero whereas $\Lambda_1$ has a number of elements depending only on $\gamma$. 
A biorthogonal sequence to a family of exponentials
\[ \Lambda = (e^{\lambda_n t})_{n \geq 1}. \]

- We construct an entire function of exponential type, as a Weierstrass product, and we evaluate it on the real axis.

- By taking into account the estimates for the above Weierstrass product on the real axis, we construct another entire function, called multiplier, such that the product between this two functions is an entire function of arbitrarily small exponential type bounded on the real axis.

- This construction allows us to obtain a biorthogonal element \( \zeta_m \), through the inverse Fourier transform (firstly used by Paley, Wiener and Fattorini, Russell).
Different strategy from Seidman, Avdonin, Ivanov

Remark

We can use the results of Seidman, Avdonin, Ivanov (J. Fourier Anal. Appl., 2000) to deduce the existence of a biorthogonal sequence? For $\lambda^*$ belonging to $(\lambda_n)_{n \in \mathbb{G}_1}$ they introduce the counting function

$$ v(r) = \# \{ (\lambda_n)_{n \in \mathbb{G}} \mid |\lambda_n - \lambda^*| \leq r \}. \quad (25) $$

They have been used the following integrability condition for $v$, i.e.

$$ \int_{c}^{\infty} \frac{v(r)}{r^2} \, dr < \infty. \quad (26) $$

For our new family the counting function does not verify (26). Our technique is fundamentally different, it could be used to obtain biorthogonal sequence for a more general class of eigenvalues.
A biorthogonal sequence to $\Lambda = (e^{\lambda_n t})_{1 \leq |n| \leq N}$

**Theorem (Bugariu, Micu and Roventa, Math. Comp., 2015)**

Let $T > 0$. There exist $h_0, c_0 > 0$ such that, for any $h \in (0, h_0)$ and $\varepsilon \in (c_0 h^2 \ln \frac{1}{h}, h)$ there exists a biorthogonal sequence $(\theta_m)_{1 \leq |m| \leq N}$ to the family $(e^{\lambda_n t})_{1 \leq |n| \leq N}$ in $L^2 \left(-\frac{T}{2}, \frac{T}{2}\right)$ such that, for any finite sequence $(\alpha_m)_m$, we have that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{|m|=1}^{N} \alpha_m \theta_m(t) \right|^2 \ dt \leq C(T) \sum_{|m|=1}^{N} |\alpha_m|^2 e^{\frac{T}{2} |\Re(\lambda_m)|},$$

(27)

where $C(T)$ is a positive constant.
Theorem (Bugariu, Micu and Roventa, Math. Comp., 2015)

Let $T > 0$. There exist $h_0, c_0 > 0$ such that for any $h \in (0, h_0)$, $\varepsilon \in (c_0 h^2 \ln \frac{1}{h}, h)$ and any initial data

$$
\begin{pmatrix}
U_h^0 \\
U_h^1
\end{pmatrix} = \sum_{1 \leq |n| \leq N} a_{nh}^0 \Phi_h^n,
$$

such that there exists a constant $C > 0$ independent of $h$ and $\varepsilon$ with the property

$$
\left\| (a_{nh}^0)_n \right\|_{\ell_\infty} < C, \tag{28}
$$

there exists a control $v_h \in L^2(0, T)$ for problem (13) such that the family $(v_h)_h$ is uniformly bounded in $L^2(0, T)$. 
For any initial data \( \begin{pmatrix} U_0^0 \\ U_1^0 \\ U_0^1 \\ U_1^1 \end{pmatrix} \in \mathbb{C}^{2N} \) and \( t \in (0, T) \) the control for (13) is given by

\[
v_h(t) = \sum_{1 \leq |n| \leq N} \frac{(-1)^{n+1} e^{-\lambda_n T/2}}{2 \cos \left( \frac{n \pi h}{2} \right)} \cdot \left( \text{sgn}(n) i \sqrt{4 - \varepsilon^2} a_{nh}^0 + \varepsilon a_{nh}^0 + \varepsilon a_{-nh}^0 \right) \theta_n \left( t - \frac{T}{2} \right).
\]
By using the biorthogonal estimates we have that

$$\int_0^T |v_h(t)|^2 \, dt \leq C \sum_{|n|=1}^N \left| \frac{1}{\cos \left( \frac{n\pi h}{2} \right)} \right|^2 e^{-\frac{T}{2} |\Re(\lambda_n)|} \leq C,$$

where $C$ is a positive constant independent of $h$, $\varepsilon$ and $m$. Note that, the last inequality takes place since

$$e^{-\frac{T}{4} |\Re(\lambda_N)|} < h^2.$$
The main controllability result

**Theorem (Bugariu, Micu and Roventa, Math. Comp., 2015)**

Let $T > 0$ and $egin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H}$. There exist $h_0, c_0 > 0$ such that for any $h \in (0, h_0)$, $\varepsilon \in (c_0 h^2 \ln \frac{1}{h}, h)$ and any initial data $\begin{pmatrix} U^0_h \\ U^1_h \end{pmatrix} \in C^{2N}$ of the form (21) with the property

$$
(a^0_{nh})_n \overset{*}{\to} (a^0_n)_n \text{ in } l^\infty \text{ when } h \to 0,
$$

(30)

there exists a family of exact controls $(v_h)_h \subset L^2(0, T)$ for problem (12) which converges to a null-control for in $L^2(0, T)$ for the continuous problem.
Until now we have treated the **boundary controllability problem** and we have seen that the negative results are due to the **bad numerical approximation of the high eigenmodes**.

It is interesting to note that, in the case when the control acts in the interior of the domain, the uniform controllability property is ensured automatically for any initial data \((u^0, u^1)\) in the space

\[ H^2(0, 1) \cap H^1_0(0, 1) \times L^2(0, 1). \]
Numerical example: In this example we take $T = 1.5$, $\varepsilon = h$ and the initial data to be controlled

$$u^0(x) = 1 - |2x - 1|, \quad u^1(x) = 0 \quad (x \in (0, 1)).$$

We approximate the control and the corresponding controlled solution, for $N = 100$ and $\varepsilon = h$. We remark that, in this case, we have obtained a good approximation of the control and the solution goes to 0 at time $T$. 
Figure: Approximations of the control $\hat{v}_h$.

Figure: The controlled solution $U_h$ for $N = 100$ and $\varepsilon = h$. 
Numerical results

<table>
<thead>
<tr>
<th>N</th>
<th>25</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>|v_h|_{L^2} with \varepsilon = h</td>
<td>1.6059</td>
<td>1.7476</td>
<td>1.8238</td>
<td>1.8634</td>
<td>1.8835</td>
</tr>
<tr>
<td>|v_h|_{L^2} with \varepsilon = h^{1.5}</td>
<td>1.8451</td>
<td>1.8822</td>
<td>1.8961</td>
<td>1.9011</td>
<td>1.9029</td>
</tr>
<tr>
<td>|v_h|_{L^2} with \varepsilon = h^2 \ln(1/h)</td>
<td>1.8671</td>
<td>1.8923</td>
<td>1.9004</td>
<td>1.9029</td>
<td>1.9037</td>
</tr>
<tr>
<td>|v_h|_{L^2} with \varepsilon = 0</td>
<td>1.9066</td>
<td>1.9047</td>
<td>1.9041</td>
<td>1.9040</td>
<td>1.9040</td>
</tr>
</tbody>
</table>

Table: Example 1 - Numerical results for \|v_h\|_{L^2} obtained with different values of the parameters \varepsilon and N.

From the \(L^2\)-norms of the approximations presented in Table 1 we see that the viscous perturbation does affect the velocity of convergence of the numerical scheme and, in order to have better approximations, the parameter \varepsilon should be chosen as small as possible.
Example 1. In this example we take $T = 1.7$ and the initial data to be controlled are given by

$$u^0(x) = \sin(\pi x), \quad u^1(x) = 0 \quad (x \in (0, 1)).$$

Note that this particular initial condition belongs to $\mathcal{D}(A)$.

Two approximations of the control are presented for $N = 100$ and two different values of the viscosity parameter: $\varepsilon = 0$ and $\varepsilon = h$.

Sufficiently smooth initial data can be uniformly controlled, even if $\varepsilon = 0$. 
Figure: Example 1 - Two approximations of the control $v_h$: with $\varepsilon = 0$ (left) and with $\varepsilon = h$ (right).
Example 2. In this example we take $T = 2.3$ and the following initial data to be controlled

$$u^0(x) = \begin{cases} 16x^3 & \text{if } x \leq \frac{1}{2} \\ 16(1 - x)^3 & \text{if } x > \frac{1}{2}, \end{cases} \quad u^1(x) = 0 \quad (x \in (0, 1)).$$

Note that the initial data belong to $H = H^1_0(0, 1) \times H^{-1}(0, 1)$ but not to $D(A)$.

The space $H$ is the largest space of initial data which can be controlled with $L^2$—controls.
Figure: Example 2 - Controlled solution and the approximation of the control with $N = 100$ and $\varepsilon = h$. 
Numerical results

Figure: Example 2 - The first four iterations of the conjugate gradient method for the approximation of $v_h$ with $N = 100$ and $\varepsilon = 0$ (up) or $\varepsilon = h$ (down).
Example 3. In this example we take $T = 3$ and the initial data to be controlled are the following

$$u^0(x) = 1 - |2 - |4x - 1||$$
$$u^1(x) = 0$$

$x \in (0, 1))$.

The initial data belong to $\mathcal{H}$ but not to $\mathcal{D}(A)$. 
Numerical results

Figure: Example 3 - Error evolution in the conjugate gradient method with four different values of $\varepsilon$.

<table>
<thead>
<tr>
<th>N</th>
<th>$|v_h|_{L^2}$ with $\varepsilon = h$</th>
<th>25</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.5376</td>
<td>1.1518</td>
<td>1.6301</td>
<td>1.9209</td>
<td>2.099</td>
</tr>
<tr>
<td></td>
<td>$|v_h|_{L^2}$ with $\varepsilon = 4h^{1.5}$</td>
<td>0.7635</td>
<td>1.5728</td>
<td>1.9908</td>
<td>2.2008</td>
<td>2.3175</td>
</tr>
<tr>
<td></td>
<td>$|v_h|_{L^2}$ with $\varepsilon = 5h^2 \log(1/h)$</td>
<td>0.9746</td>
<td>1.7655</td>
<td>2.1201</td>
<td>2.2929</td>
<td>2.3819</td>
</tr>
</tbody>
</table>

Table: Example 3 - Numerical results for $\|v_h\|_{L^2}$
Numerical results

Figure: Example 3 - Controlled solution and the approximation of the control with $N = 400$ and $\varepsilon = h$. 
Conclusions

- the lack of convergence of the algorithm if $\varepsilon = 0$ and the initial data are not smooth enough.
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- a larger viscosity parameter $\varepsilon$ helps the convergence of the scheme in the non smooth case, but produces a slower convergence rate in the regular case.

- the amount of dissipation introduced in the system through the parameter $\varepsilon$ should be decided by taking into account the regularity of initial data to be controlled.
Thank you