Boundary Integral Operator and Its Applications

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at

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Outline

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Non-homogeneous boundary value problems

$$\begin{cases} u_t = P(x,D)u + f(x,t) \quad u(x,0) = \phi(x), \quad x \in \Omega, \quad t \in (0,T) \\ B_j(x,D) = g_j(x,t), \quad (x,t) \in \Gamma \times (0,T), \quad j = 1,2,\cdots,m-1. \end{cases}$$

$$\Omega \subset R^n, \qquad \Gamma = \partial\Omega$$

$P(x,D)$: $2m-$th order operator,

$B_j(x,D)$ is s_j-th order operator,

$0 \leq s_j \leq 2m-1, j = 1,2,\cdots m-1.$

Non-homogeneous boundary value problems

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Homogeneous boundary value problems

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Question

If the homogeneous boundary value problem admits a solution

$$u \in C([0, T); H^s(\Omega)),$$

what are

the optimal regularity conditions on boundary data g_j

for the non-homogenous boundary value problem to have a solution $u \in C([0, T); H^s(\Omega))$?

Non-Homogeenous Boundary Value Problems and Applications

Vol. I, Vol. II and Vol. III

(1968 French edition, 1972 English edition)

J.-L. Lions and E. Magnets

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Strategy to study non-homogeneous boundary value problems

(1) Study the homogeneous boundary value problem

(2) Homogenization

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Curious enough, the non-homogeneous boundary value problems do not seem to have undergone any systematic study for the cases considered in this Chapter, even for second-order hyperbolic operators (except, of course, where only "regular" data on the boundary are considered and where a "loss of regularity " in the results is accepted ...).

It was therefore necessary to take up this question from the beginning. We start with the proof of regularity results, which (although far from optimal; see the Remarks of section 6) seem to be new.

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Heat equation: homogeneous Dirichlet boundary

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$$\phi \in H^1(\Omega) \implies u \in C([0, T]; H^1(\Omega))$$

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$$u|_\Gamma \in L^2(0, T; H^1(\Gamma)) \cap H^1(0, T; L^2(\Gamma)).$$

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- True if $\Omega = (0, \infty)$;
- True if $\Omega = \mathbb{R}^n_+$ ($n > 1$), ϕ and ψ have compact support in Ω.;
- $u|_\Gamma \in H^{\frac{3}{4}}(\Sigma_T)$, but $u|_\Gamma \notin H^{\frac{3}{4}+\epsilon}(\Sigma_T)$, $\forall \epsilon > 0$ if $\Omega = \mathbb{R}^n_+$.
- $u \in H^{\frac{3}{4}-\epsilon}(\Gamma \times (0, T))$ if Ω is parallelepiped;
- $u \in H^{\frac{2}{3}}(\Omega \times (0, T))$ if Ω is a sphere.
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- $u \in H^{\frac{3}{5}}(\Gamma \times (0, T))$ in general.

Lasiecka & Triggiani (1989):

- True if $\Omega = (0, \infty)$;
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Nonlinear wave equations

Nonlinear Schrödinger equations

Nonlinear equations of the KdV type

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- Strichartz smoothing
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Outline

1. Introduction
2. KdV equation
   - Cauchy problem
   - Non-homogeneous boundary value problems in a quarter plane
3. KdV on a bounded domain
   - Smoothing properties
   - Boundary integral operator
4. Conclusion remarks

The Cauchy problem

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad u(x,0) = \phi(x), \quad x,\ t \in \mathbb{R}$$

Question

For what values of s, the Cauchy problem is well-posed in the space $H^s(\mathbb{R})$?

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Outline

The KdV equation posed on a quarter plane

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad 0 < x,\ t < \infty \\ u(x,0) = \phi(x), \quad u(0,t) = h(t) \end{cases}$$

Well-posedness

- Existence + Uniqueness + Continuous Dependence

$$(\phi, h) \in H^s(R^+) \times H^{s'}_{loc}(R^+) \to u \in C(0,T; H^s(R^+)).$$

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- **Bona and Wither** (1983, SIAM J. Math. Anal.)
  Let $k \geq 1$ be given integer. For any
  $$\phi \in H^{3k+1}(R^+) \qquad \text{and } h \in H^{k+1}_{loc}(R^+)$$
  satisfying certain standard compatibility conditions, the IBVP admits a unique solution $u \in L^\infty_{loc}(R^+; H^{3k+1}(R^+))$.
- **Bona and Wither** (1989, Diff. & Integral Eqns.)
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  to the space
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Question

For given $s \in R$ with $\phi \in H^s(R^+)$, what is the optimal value of s' such that when $h \in H^{s'}_{loc}(R^+)$ one has that the solution $u \in C([0,T]; H^s(R^+))$?

Answer: $s' = (s+1)/3$.

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & 0 < x, \ t < \infty \\ u(x,0) = \phi(x), & u(0,t) = h(t) \end{cases}$$

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Definition

The IVBP is said to be well-posed in the space $H^s(R^+)$ if for a given compatible pair

$$(\phi, h) \in H^s(R^+) \times H_{loc}^{\frac{s+1}{3}}(R^+)$$

the IVBP admits a unique solution $u \in C([0,T]; H^s(R^+))$ which depends on (ϕ, h) continuously.

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For what values of s, the IBVP is well-posed in the space $H^s(R^+)$?

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- How to solve the IBVP using the harmonic analysis based approach?
- Convert the IBVP to an equivalent integral equation

$$u(t) = W_0(t)\phi + W_{bdr}(t)h - \int_0^t W_0(t-\tau)(uu_x)(\tau)d\tau$$

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- $w(x,t) = W_{bdr}(t)h$ solves

$$\begin{cases} w_t + w_x + w_{xxx} = 0, & 0 \leq x,\ t < \infty \\ w(x,0) = 0, & w(0,t) = h(t). \end{cases}$$

- $v(x,t) = W_0(t)\phi$ solves

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$u(t) = W_0(t)\phi + W_{bdr}(t) - \int_0^t W_0(t-\tau)(uu_x)(\tau)d\tau$

- $w(x,t) = W_{bdr}(t)h$ solves

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$$u(t) = W_0(t)\phi + W_{bdr}(t) - \int_0^t W_0(t-\tau)(uu_x)(\tau)d\tau$$

- Extend the integral equation posed on the quarter plane $R^+ \times R^+$ to the whole plane $R \times R$.
- How?
- Find explicit integral representations of $W_0(t)\phi$ and $W_{bdr}(t)h$.

$u(t) = W_0(t)\phi + W_{bdr}(t) - \int_0^t W_0(t-\tau)(uu_x)(\tau)d\tau$

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$$u(x,t) = W_{bdr}(t)h$$

$$W_{bdr}(t)h = [U_b(t)h](x) + \overline{[U_b(t)h](x)}$$

where, for $x, \ t \geq 0$,

$$[U_b(t)h](x) = \frac{1}{2\pi} \int_1^{\infty} e^{it(\mu^3-\mu)} e^{-\left(\frac{\sqrt{3\mu^2-4}+i\mu}{2}\right)x} \tilde{h}(\mu) d\mu$$

with

$$\tilde{h}(\mu) = (3\mu^2 - 1) \int_0^{\infty} e^{-i\xi(\mu^3-\mu)} h(\xi) d\xi.$$

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Boundary integral operator

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Boundary integral operator

- The integral representation of $W_0(t)\phi$ is too complicated!
- For the Cauchy problem of the KdV equation posed on the whole line R:

$$u_t + u_x + u_{xxx} = 0, \quad u(x,0) = \psi(x), \qquad -\infty < x, \, t < \infty$$

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$$v_t + v_x + v_{xxx} = 0, v(x,0) = \tilde{\phi}(x), \quad x,\ t \in R$$

(ii) Let $q(t) := v(0,t)$ and solve

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KdV on a bounded domain

$$u_t + u_x + uu_x + u_{xxx} = 0, \qquad u(x,0) = \phi(x), \qquad x \in (0,L)$$

$$B_1 u = h_1(t), \qquad B_2 u = h_2(t), \qquad B_3 u = h_3(t)$$

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Under what conditions on a_{kj}, b_{kj}, is the IBVP well-posed?

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KdV on a bounded domain

$$u_t + u_x + uu_x + u_{xxx} = 0, \qquad u(x,0) = \phi(x), \qquad x \in (0, L)$$

$$B_1 u = h_1(t), \qquad B_2 u = h_2(t), \qquad B_3 u = h_3(t)$$

$$B_k u = \sum_{j=0}^{2} a_{kj} \partial_x^j u(0,t) + b_{kj} \partial_x^j u(L,t), \qquad k = 1, 2, 3,$$

Under what conditions on a_{kj}, b_{kj}, is the IBVP well-posed?

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Theorem 1 (Capistrano-Filho, Sun, Zhang): The IBVP is locally well-posed in $H^s(0,L)$ for any $s > -1$ with $\phi \in H^s(0,L)$,

$$h_1,\ h_2 \in H_{loc}^{\frac{s+1}{3}}(\mathbb{R}^+), \quad h_3 \in H_{loc}^{\frac{s}{3}}(\mathbb{R}^+).$$

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Theorem 2 (CSZ): The IBVP is locally well-posed in $H^s(0,L)$ for any $s > -1$ with $\phi \in H^s(0,L)$,

$$h_1 \in H^{\frac{s+1}{3}}_{loc}(\mathbb{R}^+), \quad h_2 \in H^{\frac{s-1}{3}}(\mathbb{R}^+), \quad h_3 \in H^{\frac{s}{3}}_{loc}(\mathbb{R}^+).$$

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Theorem 3 (CSZ): The IBVP is locally well-posed in $H^s(0,L)$ for any $s > -1$ with $\phi \in H^s(0,L)$,

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Theorem 4 (CSZ): The IBVP is locally well-posed in $H^s(0,L)$ for any $s > -1$ with $\phi \in H^s(0,L)$,

$$h_1 \in H_{loc}^{\frac{s-1}{3}}(\mathbb{R}^+), \quad h_2 \in H^{\frac{s-1}{3}}(\mathbb{R}^+), \quad h_3 \in H_{loc}^{\frac{s}{3}}(\mathbb{R}^+).$$

Outline

The key ingredients of the proofs

$$\begin{cases} u_t + u_{xxx} = 0, \quad u(x,0) = \phi(x), \quad x \in (0,L), \\ B_1 u = 0, \quad B_2 u = 0, \quad B_3 u = 0. \end{cases}$$

- Kato smoothing

$$\phi \in L^2(0,L) \implies u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L)).$$

- Sharp Kato smoothing (Hidden regularities):

$$\phi \in L^2(0,L) \implies \partial_x^j u \in L_x^\infty(0,L; H^{\frac{1-j}{3}}(0,T)), \quad j = 0,1,2.$$

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Outline

1. Introduction
2. KdV equation
   - Cauchy problem
   - Non-homogeneous boundary value problems in a quarter plane
3. KdV on a bounded domain
   - Smoothing properties
   - Boundary integral operator
4. Conclusion remarks

Boundary integral operator

$$\begin{cases} u_t + u_{xxx} = 0, \quad u(x,0) = 0, \quad x \in (0,L), \\ B_1 u = h_1(t), \quad B_2 u = h_2(t), \quad B_3 u = h_3(t). \end{cases}$$

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$$u = W_{bdr}(t)\vec{h}$$

The bridge to access Fourier analysis tools!

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Homogeneous boundary value problem

$$\begin{cases} u_t + u_{xxx} = 0, \quad u(x,0) = \phi(x), \quad x \in (0,L), \\ B_1 u = 0, \quad B_2 u = 0, \quad B_3 u = 0. \end{cases}$$

$$v_t + v_{xxx} = 0, \quad v(x,0) = \tilde{\phi}(x), \quad x \in \mathbb{R}, \ t \in \mathbb{R}$$

$v = W(t)\tilde{\phi}$, $\tilde{\phi}$ is the extension of ϕ from $(0,L)$ to $\mathbb{R}$.

$$u(x,t) = W(t)\tilde{\phi} - W_{bdr}(t)\vec{q}$$

$$q_1(t) = B_1 v, \quad q_2(t) = B_2 v, \quad q_3(t) = B_3 v.$$

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$$\begin{cases} u_t + u_{xxx} = f, \quad u(x,0) = 0, \quad x \in (0,L), \\ B_1 u = 0, \quad B_2 u = 0, \quad B_3 u = 0. \end{cases}$$

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$$\begin{cases} u_t + u_{xxx} = f, \quad u(x,0) = 0, \quad x \in (0,L), \\ B_1 u = 0, \quad B_2 u = 0, \quad B_3 u = 0. \end{cases}$$

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$$w = \int_0^t W(t-\tau)\tilde{f}(\tau)d\tau,$$

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Smootng properties of the boundary integral operator

$$u(x,t) := [W_{bdr}\vec{h}](x,t), \qquad \vec{h} = (h_1, h_2, h_3.)$$

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$$\begin{cases} \partial_t u = P(x,D)u + f(x,t) \quad u(x,0) = \phi(x), \quad x \in \Omega, \quad t \in (0,T) \\ B_j(x,D) = g_j(x,t), \quad (x,t) \in \Gamma \times (0,T), \quad j = 1, 2, \cdots, m-1, \end{cases}$$

$P(x, D)$: $m-$th order operator, $B_j(x, D)$ is s_j-th order operator,

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Curious enough, the non-homogeneous boundary value problems do not seem to have undergone any systematic study for the cases considered in this Chapter, even for second-order hyperbolic operators (except, of course, where only "regular" data on the boundary are considered and where a "loss of regularity " in the results is accepted ...).

It was therefore necessary to take up this question from the beginning. We start with the proof of regularity results, which (although far from optimal; see the Remarks of section 6) seem to be new.

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where $P(x,D)$ is $m-$th oder operator and $B_j(x,D)$ is s_j-th oder operator,

Question

What are the optimal regularity conditions on boundary data g_j for the IBVP to have a solution $u \in C([0,T); H^s(\Omega))$?

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Answer

$$g_j \in H^{r_j,s_j^*}(\Gamma\times(0,T)) := L^2(0,T;H^{s_j^*}(\Gamma))\cap H^{r_j}(0,T;L^2(\Gamma))$$

with

$$r_j \ge \frac{1}{2} - \frac{s-s_j-1/2}{m}, \qquad s_j^* \ge \frac{m-1}{2} + s - s_j, \quad j=1,2,\cdots,m$$

To solve

$$\begin{cases} \partial_t u = P(x,D)u + f(x,t), \quad u(x,0) = \phi(x), \quad x \in \Omega, \quad t \in (0,T) \\ B_j(x,D) = g_j(x,t), \quad (x,t) \in \Gamma \times (0,T), \quad j = 1, 2, \cdots, m-1, \end{cases}$$

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$$\partial_t w = P(x, D)w + \tilde{f}(x, t), \quad w(x, 0) = \tilde{\phi}(x), \quad x \in \mathbb{R}^n \quad t \in (0, T)$$

$$w(x, t) = W(t)\tilde{\phi} + \int_0^t (W(t-\tau)\tilde{f}(\tau)d\tau)$$

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Conclusion:

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