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Why Riemannian Geometry is a Necessary Tool for Us ?

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Differential Geometric Approach

Fifteen ago, we introduced the differential geometrical approach

It has become a basic tool to cope with

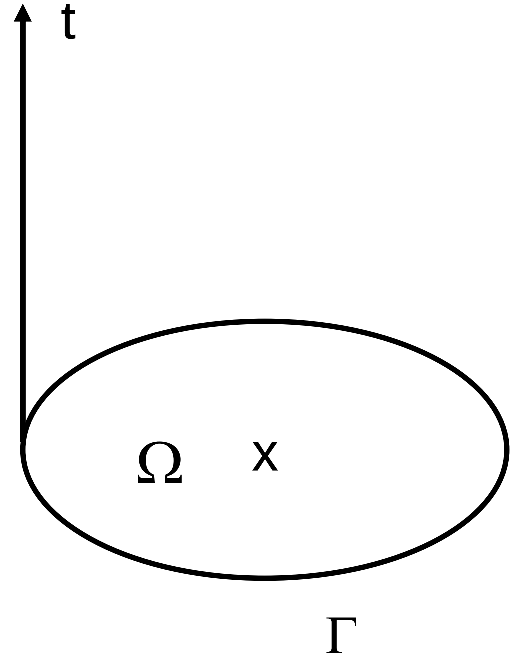
(1) control problems with variable coefficients
(2) modeling and control of shells
(3) control problems with nonlinear elasticity

Control of the Wave Equation With Variable Coefficients

1. Extension Method
2. Dual Methods. Ho's Multipliers. Lions' open question
3. Geometric Optics Methods
4. Pseudo-differential Methods
5. Operator Theory Methods
6. Differential Geometric Approach

Wave Equation with Variable Coefficents

$$u_{tt} = div A(x) \nabla u \quad (0,\infty)\times\Omega$$



$A(x)$ a matrix by the material

- If $A(x)$ is a constant matrix, it is a constant coefficient problem.

- If $A(x)$ varies with the material variable x , then it is of variable coefficients.
- $(u(t,x), u_t(t,x))$ state of the wave

Exact Controllability

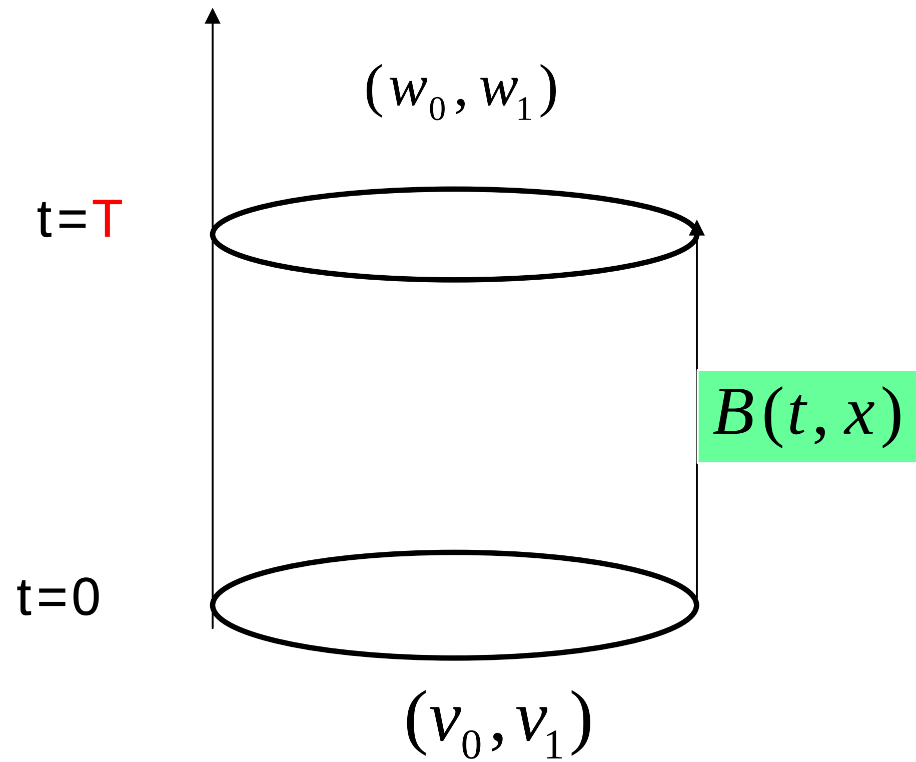
Given initial state (v_0, v_1)

terminal state (w_0, w_1)

To find out a control time T and a control boundary function $B(t,x), x \in \Gamma$ such that

$$(u(0,x), u_t(0,x)) = (v_0, v_1)$$

$$(u(T,x), u_t(T,x)) = (w_0, w_1)$$



Models for Wave Equations

A big jump

- Level I: For the constant coefficient,

$$u_{tt} = \Delta u, \quad \Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial^2 x_k}$$

A big jump

- Level II: For the variable coefficient,

$$u_{tt} = div A(x) \nabla u$$

- Level III: For nonlinear elasticity,

$$u_{tt}(x) = \sum_{i=1}^{n} a_{ij}(x, \nabla u) u_{x_i x_i} + b(x, \nabla u)$$

Difficulty varies with models

1. Extension Methods

Pioneer Results

D. L. Russell, W. Littman: obtained a control time T and a boundary control function $B(t,x)$ in 1978

An essential assumption: Constant Coefficients

Extension Methods

- Extend the initial data from Ω to the whole space R^n
- To obtain solutions by the formula of the Cauchy problem
- Restrict solutions to the boundary to have

  a control

2 Duality Method

- Kalman(1963): Controllability is equivalent to observability (linear system)
- Observability is described by an inequality
- S. Dolecki and D. L. Russell(1977) presented an abstract frame of PDES. They did not apply their theory to the wave equation

2. Dual Method

- J. L. Lions: exact controllability = observability inequality by the dual method in 1988: For solutions $\phi(t,x)$

$$\phi_{tt}(x) = divA(x)\nabla\phi, \phi|_{\Gamma} = 0, \phi(0) = \phi_0, \phi_t(0) = \phi_1$$

prove

$$c_1 E(0) \leq \int_0^T \int_{\Gamma} (\frac{\partial \phi}{\partial \nu})^2 d\Gamma dt \leq c_2 E(0)$$

$$E(0) = \int_{\Omega} (\phi_1^2 + |\nabla \phi_0|^2) dx$$

2. Dual Method

- L. F. Ho proved observability inequality by: Multiplier technique in 1986 for constant coefficients

  His methods were so simple that surprised the community of this area. Lions used it immediately to other systems, for example plates with constant coefficients.

  The same multipliers were firstly used by Morawetz C. in scattering problems in 1950's.

2. Dual method

- Lions' inequality was uncheckable for variable coefficients

- Lions' open question (1988):
  Is it true in the case of variable coefficients ?

3. Geometric Optics Methods

- This has been developed by PDEs which is used in study of control of the wave equation:

  Geometric optics theory, Hormander

- J. Ralston (1982): A necessary condition for EC is: any ray initiated inside the domain will hit the boundary after a finite time (Geometric optics conditions (GOC))

- C. Bardos, G. Lebeau and J. Rauch (1992): The GOC is also sufficient for EC

Can we know that GOC is true or not ?

3. Geometric Optics Methods

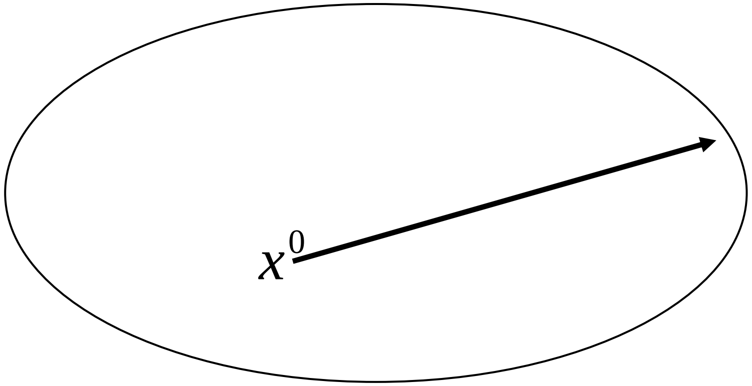
Waves propagate along rays . Then GOC means that boundary exact controllability holds if and only if "all waves from the domain will go to the boundary".

The key question is: What conditions can guarantee that all waves go to the boundary ?

3. Geometric Optics Methods

For constant coefficients

a ray is a straight line



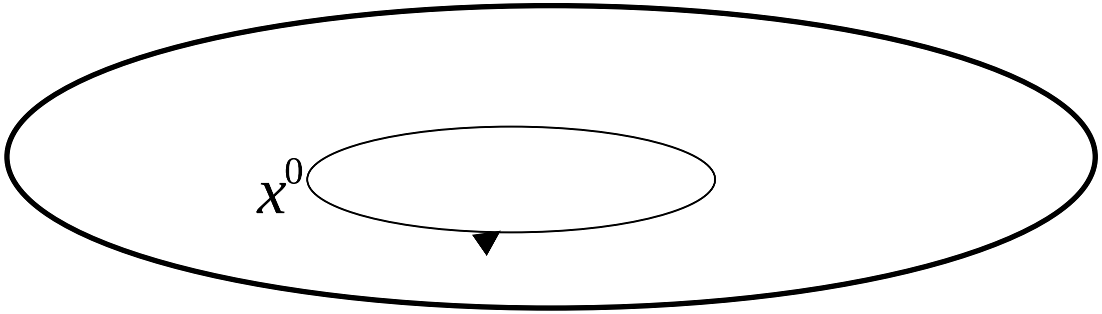
Waves go along rays at a fixed speed

then obviously, GOC is true

3. Geometric Optics Methods

For variable coefficients,

A ray is a (real curve) solution to a nonlinear ODE. There is no way to prove if the GOC is true or not



Geometric Optics Methods

GOC changed a problem of linear PDEs into one of nonlinear ODEs

After GOC, Lions' question was still open.

4. Pseudo-differential Method

- D. Tataru(1994): If there is a pseudo-convex function on the domain, then the EC is true.
- For constant coefficients, a pseudo-convex function exists.
- For variable coefficients, there is no way to check the existence of a pseudo-convex function.

5. Operator Theory Methods

- A. Wyler (1994): If there is a vector field $H=(h_1,h_2,\cdots,h_n)$ such that $(p_{ij})>0$, then EC is true where

$$p_{ij}=\sum_{k=1}^{n}\left(a_{ik}\frac{\partial h_j}{\partial x_k}+a_{jk}\frac{\partial h_i}{\partial x_k}-\frac{\partial a_{ij}}{\partial x_k}h_k\right), \quad A(x)=(a_{ij}(x))$$

- For constant coefficients, let $H=x$. Then $(p_{ij})=(\delta_{ij})>0$.
- For variable coefficients, the above is again uncheckable

EC is Open for Variable Coefficients

- Lion's inequality was only verified by Ho's multipliers for constant coefficients .
- GOC was only checked for constant coefficients.
- Pseudo-convexity was only checked for constant coefficients.
- Wyler's condition is only checked for constant coefficients.

EC was still open for Variable Coefficients

Checkability is Necessary

- Equivalent conditions:

  Exact controllability $\Leftrightarrow$ observability inequality(1988)

  Exact controllability $\Leftrightarrow$ the (GOC) (1982, 1992)

- For application, checkable conditions are necessary

  Equivalent conditions: To understand the problem

  Checkable conditions: To understand and to use in application

Differential Geometric Approach

Peng-Fei Yao (1999)--Thanks to Lagnese

The original goal was by the Riemannian geometry to give checkable condition to the controllability of the wave equation for variable coefficients.

Riemannian Geometric Condition

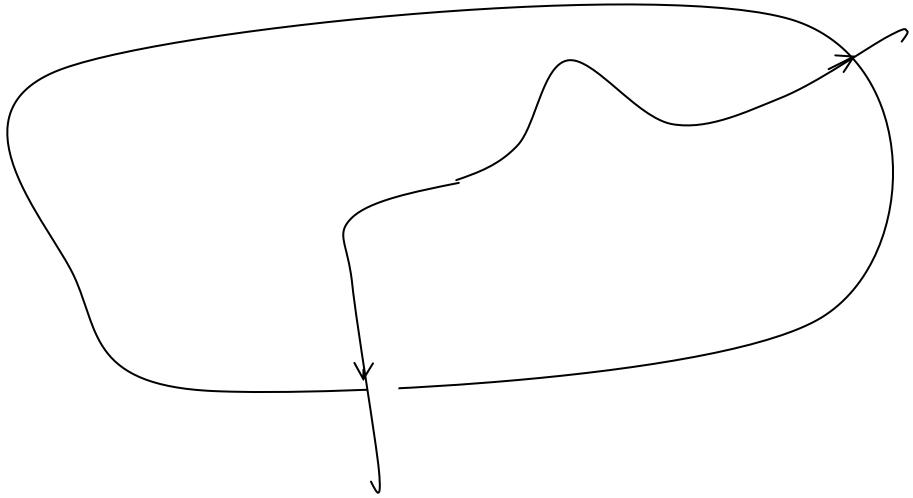
Riemannian geometrical approach says:

If there is an escape vector field, then controllability holds true.

Fortunately, this assumption is checkable by curvature

RGC for EC checkable by Curvature

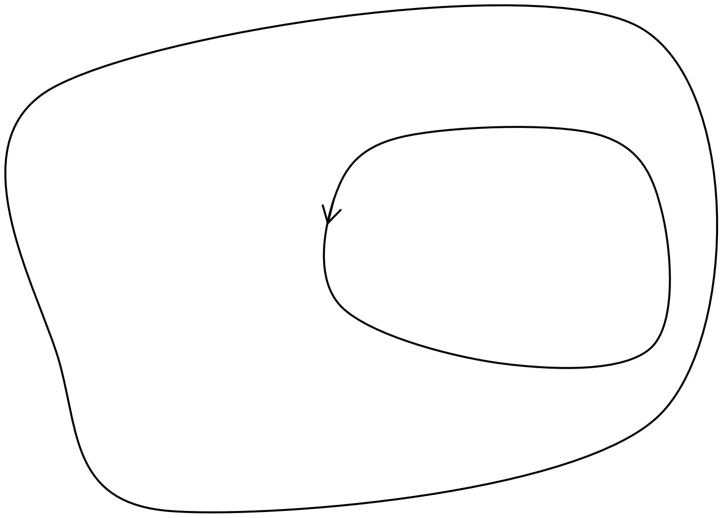
An escape vector field can guarantee all waves go away outside of the domain along rays



RGC for EC checkable by Curvature

Without escape vector fields a ray may circle inside the domain, for example,

$$A(x) = \begin{pmatrix} (1+|x|^2)^2 & 0 \\ 0 & (1+|x|^2)^2 \end{pmatrix}$$



RGC for EC checkable by Curvature

Escape vector fields is also a topic in Riemannian geometry, checkable by curvature

Riemannian Geometry

- The heart of Riemannian geometry is a study of geodesics, solutions to a nonlinear ODE
- Geodesics are rays in our problem. This is the bridge between Riemannian geometry and exact controllability

Riemannian Geometry

There are two ways that outdo the classical geometry:

- It gives global information by curvature
- It yields a technique (the Bochner technique) to simplify local computation.

Riemannian Geometry

- Normally the classical analysis tells us if something is locally true or not (by coordinates)

- Gauss, Riemann says: If you want to know whether it is true globally, just check curvature

  Curvature controls everything, including controllability

Curvature is defined locally but it tells us information globally

Conclusion

Exact controllability is a global problem. The Riemannian geometry is the only theory which is capable of providing the global information by curvature.

Thank You For Your Time