Feedback Stabilization of Systems with Delays–A survey

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Contents

Systems with delay

Example

Using positive feedback stabilizes the Oscillatory system

$$\ddot{y}(t) + k^2 y(t) = \gamma y(t-\tau)$$

Example

The elastic system with memory

$$\begin{cases} \frac{\partial^2 w(x,t)}{\partial t^2} = \Delta w(x,t) + \int_0^t g(t-s)\Delta w(x,s)ds = 0, \\ w(x,t) = 0, \quad x \in \partial\Omega \end{cases}$$

Example

The output with finite memory

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = \int_{t-\tau}^t Cx(\theta)d\theta \end{cases}$$

Classification

There are mainly three types of delays in systems: interior delays (or called the state delay), input delays (control delays) and output delays (measurement delays).

State Delay

The control system with interior delay (or memory)

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + \Phi(x_t) + Bu(t), \quad t > 0 \\ x(0) = x_0 \quad x(s) = v(s), s \in [-\tau, 0]; \end{cases}$$

Delay term

$$\Phi(f) = \int_{-\tau}^{0} d\eta(s) f(s)$$

where $\eta : [-\tau, 0] \to \mathcal{L}(\mathbb{X})$ of bounded variation.

Classification

The delay occurs in the signal input phase, it also is called the controller delay.

Input Delay

The control system with input delay (or memory)

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + B\Phi(u_t), \quad t > 0 \\ x(0) = x_0 \quad u(s) = \varphi(s), s \in [-\tau, 0]; \end{cases}$$

Delay term

$$\Phi(f) = \int_{-\tau}^{0} d\eta(s) f(s)$$

where $\eta : [-\tau, 0] \to \mathcal{L}(\mathbb{U})$ of bounded variation.

Classification

The delay occurs in the signal output phase, it also is called the observation delay.

Output Delay

The control system with output delay (or memory)

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t), \quad x(0) = x_0 \\ y(t) = \Phi(x_t), \quad t > 0 \\ y(s) = \psi(s), \ s \in [-\tau, 0] \end{cases}$$

Delay term

$$\Phi(f) = \int_{-\tau}^{0} d\eta(s) f(s)$$

where $\eta : [-\tau, 0] \to \mathcal{L}(\mathbb{X}, \mathbb{Y})$ of bounded variation.

State Delay

Problem of State Delay

State Delay and Stabilization

- Questions
- Stabilization

Questions

Question1

Let H be a Hilbert space, and A be a unbounded and linear operator in H, A_1 is another linear operator on H. Let A generate a C_0 semigroup on H of exponential stability. Let us consider the dynamic system

$$\dot{x}(t) = Ax(t) + A_1 x(t-\tau) \tag{1}$$

Under what conditions on A_1 and τ, is the system still stable?

Remark

Usually, a linear system of parabolic-type with interior delay has a stability margin. When the delay term is small enough in the sense of some norm, the system is still exponentially stable.

Questions

For the linear conservation system of hyperbolic-type, it is sensitive to small delay. Datko proposed the following question:

Datko Question

Let H be a Hilbert space, and A be a positive definite linear operator in H. Let $\mathbb{U}$ be another Hilbert space, and $C : \mathbb{U} \to H$ be a linear operator. Let us consider the dynamic system

$$\ddot{x}(t) + Ax(t) = Bu(t). \tag{2}$$

Suppose that the feedback operator $K : H \to U$ is bounded, the dynamic system with negative feedback control law

$$u(t) = -K\dot{x}(t) \tag{3}$$

is uniformly stable.

Question

Datko Question

Suppose the system is damped and has the following form

$$\ddot{x}(t) + D\dot{x}(t) + Ax(t) = Bu(t) \tag{4}$$

where D also is a nonnegative operator on H. When $u(t) = 0$, the system is uniformly stable. If the feedback control law (3) is placed by

$$u(t) = -K\dot{x}(t-\tau), \quad \tau > 0 \tag{5}$$

Whether do it improve the stability characteristics of (4)?

Reference

R. Datko, Two Questions Concerning the Boundary Control of Certain Elastic Systems, *Journal of Differential Equations*, Vol.92,1991,pp.27-44,

Following Datko Questions

X. J. Li, K. S. Liu's Result(1993):
A is positive definite, B is boundary control

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$$\dot{x}(t) + Ax(t) = -BB^*x(t-\tau)$$
the system is robustly for small delay.

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$$\ddot{x}(t) + Ax(t) = -BB^*x(t-\tau)$$
the system is sensitively for small delay

Reference

X. J. Li, K. S. Liu, the effect of small time delay in the feedback on boundary stabilization, Science in China (A), Vol.36, No.12, 1993, pp.1435-1443.

Following Datko Questions

Rebarbery and Townley's Result(1998):
A is unbounded and B is admissible, K is compact.

- If $-BK$ improves stability of e^{At}, then $\dot{x}(t) = Ax(t) - BKx(t-\tau)$ is robustly for small τ;
- U is finite dimensional and K is bounded, $-BK$ cannot be found to improve exponential stability.

Reference

R. Rebarbery, S. Townley, Robustness with respect to delays for exponential stability of distributed parameter systems, *SIAM J. Control Optim.*, Vol. 37, No. 1, 1998, pp. 230–244

Following Datko Question

R. X. Wang and Y. T. Wang's Result (2004)

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) - 2\alpha w_{xxt}(x,t) = 0, x \in (0,1), t > 0 \\ w(0,t) = w_{xx}(0,t) = 0, w_x(1,t) = 0 \\ w_{xxx}(1,t) = w_t(1,t-\varepsilon) \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), \\ w_t(x,s) = \varphi(x,s), s \in (-\varepsilon, 0) \end{cases} \tag{6}$$

If $0 < \alpha < 1$ and $0 < \varepsilon < \varepsilon_0$, the system is Robust stable;

Reference

R. X. Wang, Y. T. Wang, *The Well-posedness And Robust Stability Of Flexible Beam Systems With Respect To Small Delays In The Feedback Loop*, Master Thesis, Shanxi University, 2004.

Following Datko Question

Nicaise and Pignotti's Result(2014):
A positive definite and unbounded,$B_j : \mathbb{U}_j \to H_{-1}$,$j = 1, 2$, are A-admissible,

$$\begin{cases} \ddot{w}(t) = Aw(t) + B_1 B_1^* \dot{w}(t) + B_2 B_2^* \dot{w}(t-\tau), t > 0 \\ w(0) = w_0, \quad \dot{w}(0) = w_1, \\ B_2^* \dot{w}(s) = f(s), s \in (-\tau, 0). \end{cases} \tag{7}$$

Suppose that

- $e^{(A+B_1B_1^*)t}$ is exponentially stable.
- B_2 satisfies there exists $\alpha \in (0,1)$ such that

$$||B_2^* w||_{U_2} \leq \alpha ||B_1^* w||_{U_1}, \forall w \in D(A^{\frac{1}{2}})$$

then, the system (7) also is the exponentially stable.

Reference

S. Nicaise, C. Pignotti, Stabilization of second-order evolution equations with time delay, *Math. Control Signals Syst.*, Vol.26, 2014, pp.563-588.

Following Datko Question

Nicaise and Pignotti's Result (2015)

Suppose that

- e^{At} is exponentially stable;
- F satisfies Lipschitz condition $||Fx - Fy|| \leq L||x - y||$;
- B is bounded.

$$\begin{cases} U_t(t) = AU(t) + F(U(t)) + kBU(t-\tau), t > 0, \\ U(0) = U_0, \\ BU(s) = f(s), s \in (-\tau, 0), \end{cases}$$

where $A : D(A) \subset H \to H$ generates C_0 semigroup of exponential stability; $F : H \to H$ is a Lipschitz mapping, B is a bounded linear operator on H. They showed that if the C_0-semigroup describing the linear part is exponentially stable, then the system retains this good property when a suitable smallness condition on the time-delay feedback is satisfied.

Reference

S. Nicaise, C. Pignotti, Exponential stability of abstract evolution equa-

Survey

Theorem

Let A generates exponentially stable and contraction C_0 semigroup, $A_1 = BB^*$ where $B : Z \to H$. and Z be another Hilbert space. Suppose $D(A) \subset D(A_1)$.

- $A + A_1$ generates a contraction semigroup if and if $(A_1x, x) \leq -(A_Rx, x), \forall x \in D(A)$, where $A_R = \frac{A+A^*}{2}$ is an extension.
- $A + A_1$ generates a contraction semigroup of exponential stability if and if there exists $\alpha \in (0, 1)$ such that

$$(A_1x, x) \leq -\alpha(A_Rx, x), \quad \forall x \in D(A)$$

Survey

Theorem

Let A generates exponentially stable and contraction C_0 semigroup, $A_1 = BB^*$ where $B : Z \to H$. and Z be another Hilbert space. Suppose $D(A) \subset D(A_1)$.

- if $A + A_1$ generates a contraction semigroup of exponential stability, then

$$\dot{x}(t) = Ax(t) \pm A_1 x(t - \tau)$$

also is exponentially stable, and the decay rate λ and τ satisfy relation

$$\frac{\lambda}{2 - \alpha - \alpha e^{\lambda\tau}} = \inf_{||x||=1} |(A_R x, x)| = \mu_0.$$

Sketch of Proof

Lyapunov Functional

$$V(x(t),t) = e^{\lambda t}||x(t)||^2 + \int_{t-\tau}^{t} e^{\lambda(s+\tau)}||B^*x(s)||^2 ds.$$

$$\begin{aligned}\frac{dV}{dt} &= e^{\lambda t}[\lambda||x(t)||^2 + 2(A_r x(t), x(t)) + (1+e^{\lambda\tau})(A_1 x, x)] \\ &\quad -e^{\lambda t}||B^*x(t) \pm B^*x(t-\tau)||^2\end{aligned}$$

$e^{(A+A_1)t}$ is exponentially stable iff $(A_1 x, x) \le -\alpha(A_R x, x)$.

$$\frac{dV}{dt} \le e^{\lambda t}[\lambda||x(t)||^2 + [2-\alpha(1+e^{\lambda\tau})](A_1 x, x)]$$

the condition $\frac{\lambda}{2-\alpha-\alpha e^{\lambda\tau}} = \inf_{||x||=1}|(A_R x, x)| = \mu_0$ implys $\frac{dV}{dt} \le 0$.

Survey

Theorem

Let A generates exponentially stable and contraction C_0 semigroup, A_1 is linear operator. Suppose $D(A) \subset D(A_1)$.

- $A + A_1$ generates a C_0 semigroup of exponential stability, $\mu_0 = \inf_{||x||=1} |(A_R x, x)|$;
- there exists a constant $\alpha \in (0,1)$ such that $||A_1 x||^2 \leq \alpha|(A_R x, x)|, \forall x \in D(A)$
- $\mu_0 > \sqrt{\alpha}$.

then

$$\dot{x}(t) = Ax(t) \pm A_1 x(t-\tau)$$

also is exponentially stable, and the decay rate λ and τ has more complicated relationship.

Lyapunov Functional

$$V(x(t),t) = e^{\lambda t}||x(t)||^2 + \int_{t-\tau}^{t} e^{\lambda(s+\tau)}||A_1 x(s)||^2 ds.$$

Solvability Issue

Necessariness of solvability of delayed equations

When the differential equation involves time-delay in the equation or boundary condition, the study of solvability issue is completely necessary. Shang and Xu considered a 1D wave equation with the delay depending on position. They showed if the largest time delay equals to 1, the system equation is unsolvable.

Reference

Y. F. Shang, G. Q. Xu, The stability of a wave equation with delay-dependent position, *IMA Journal of Mathematical Control and Information* Vol. 28, 2011, pp.75-95.

Stabilization issue

The control issue of the system with state delay

Stabilization Issue

The linear system with state delay in Hilbert space:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1 x(t-\tau) + Bu(t), t > 0 \\ x(0) = x_0, \\ x(s) = f(s), s \in [-\tau, 0) \end{cases}$$

where $A^* = -A$, and A_1 is a linear operator in H (it might be unbounded), B is A-admissible. (A, B) is exactly controllable in finite time.

Find a state feedback control law such that the closed loop system is exponentially stable.

Stabilization issue

Feedback control law

$$u(t) = -B^*K_1x(t) - B^*K_2x(t-\tau)$$

with $(A - BB^*K_1)$ generates a C_0 semigroup of Exponential stability, and $||A_1 - BB^*K_2||_{H,H_{-1}} \leq \alpha$ is small enough.

Theorem

The closed loop system in Hilbert space:

$$\begin{cases} \dot{x}(t) = (A - BB^*K_1)x(t) + (A_1 - BB^*K_2)x(t-\tau), t > 0 \\ x(0) = x_0, \quad x(s) = f(s), s \in [-\tau, 0) \end{cases}$$

is exponentially stable.

Reference

G. Q. Xu, Feedback stabilization of the systems with delays, *in Preparing*,

Input Delay

Problem of Input Delay

Input Delay and Feedback Control Design

- Mathematical Model
- Existing Results (Collocated Feedback).
- Feedback Control design

Mathematical Model

Let us consider the following system

$$\begin{cases} \frac{\partial z(s,t)}{\partial t} = \frac{\partial z(s,t)}{\partial s}, \quad s \in (-\tau, 0), t > 0 \\ z(0,t) = u(t), t \geq 0 \\ z(s,0) = f(s), s \in (-\tau, 0) \end{cases} \tag{8}$$

where $u(t)$ is the control input and $f(x)$ is memory function. The output of the system is

$$v(t) = \int_{-\tau}^{0} d\nu(s) z(s,t)$$

where $\nu(x)$ is a bounded variation function and the integral is in the sense of the Riemann-Stieltjes integral.

Solution to (8)

$$z(s,t) = u(t+s), \; s \in (-\tau, 0)$$

Mathematical Model

Hence the output is of the following form

$$v(t) = \int_{-\tau}^{0} d\nu(s)u(t+s).$$

Therefore,

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bv(t), \\ \frac{\partial z(s,t)}{\partial t} = \frac{\partial z(s,t)}{\partial s}, \quad s \in (-\tau, 0), \\ z(0,t) = u(t), t \geq 0 \\ v(t) = \int_{-\tau}^{0} d\nu(s)z(s,t) \\ x(0) = x_0, \quad z(s,0) = f(s), s \in (-\tau, 0) \\ y(t) = Cx(t) \end{cases}$$

is equivalent to the following

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + B\int_{-\tau}^{0} d\nu(s)u(t+s), \\ x(0) = x_0, \quad u(s) = f(s), s \in (-\tau, 0) \\ y(t) = Cx(t) \end{cases}$$

Mathematical Model

Question for $v(t)$

$$v(t) = \int_{-\tau}^{0} d\nu(s) u(t+s).$$

Usually, $u(t)$ is in L^2_{loc}. For such a function the integral may not exist.

We make requirement:

Requirement

$\nu(s)$ takes the form

$$\nu(s) = \alpha H(s) + \beta H(s+\tau) + \gamma(s)\chi_{[-\tau,0]}$$

where α, β are constants, $H(s)$ is the Heaviside function, and $\gamma(s)$ is in Sobolev space $H^1(-\tau, 0)$.

Mathematical Model

Corresponding to the this form, the output is

$$v(t) = \alpha u(t) + \beta u(t-\tau) + \int_{-\tau}^{0} \gamma'(s)u(t+s)ds$$

where $\gamma'(s) \in L^2[-\tau, 0]$, this means that the output includes the distributed delays.

Special cases

- $\gamma(s) = 0$: then $v(t) = \alpha u(t) + \beta u(t-\tau)$, that means that the output has partial delay if $\alpha \neq 0$;
- $\gamma(s) \equiv 0$, $\alpha = 0$, $v(t)$ has the form $v(t) = \beta u(t-\tau)$, that means that the output has full delay if $\beta \neq 0$.
- $\gamma(s) \equiv 0$, $\beta = 0$, $v(t)$ has the form $v(t) = \alpha u(t)$ that means that the output has no delay if $\alpha \neq 0$.

Control Delay Issue

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + B[\alpha u(t) + \beta u(t-\tau) + \int_{-\tau}^{0} g(s)u(t+s)ds], \\ x(0) = x_0, \quad u(s) = f(s), s \in (-\tau, 0) \\ y(t) = Cx(t) \end{cases}$$

where $g(s) = \gamma'(s)$.

Existing Results

Control Problem of 1-D wave equation

Boundary control problem of 1-d wave equation

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1), t > 0 \\ w(0,t) = 0, \\ w_x(1,t) = v(t), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x),\ x \in (0,1). \end{cases} \tag{9}$$

where $v(t)$ is the exterior force.

Observation

The system observation is

$$y(t) = w_t(1,t) \tag{10}$$

Existing results

Positive result

If there is no input delay, i.e., $v(t) = u(t)$, under the collocated feedback control law

$$u(t) = -ky(t), \quad k > 0 \tag{11}$$

the closed-loop system decays exponentially.

Reference: J.-L. Lions,1982, Also see

J.-L. Lions, Exact controllability, stabilization and perturbations for distributed parameter system. SIAM Rev. 30 (1988) 1 - 68.

Negative results:

For any small delay time τ, if the system has the full input delay, i.e., $v(t) = u(t-\tau)$, then under the collocated feedback control law (11), the closed-loop system is unstable.

Existing results

References:

1. R. Datko, J. Lagness and M.P. Poilis, An example on the effect of time delays in boundary feedback stabilization of wave equations. SIAM J. Control Optim. 24 (1986) 152 - 156.
2. R. Datko, Not all feedback stabilized hyperbolic systems are robust with respect to small time delay in their feedbacks. SIAM J. Control Optim. 26 (1988) 697 - 713.

Comparing

For any finite-dimensional system, if the system is stable, then for small time delay τ, it also is stable. That means that there is a stable margin.

Question:

How we design anti-delay control?

Existing results

Case 1:

$$v(t) = \alpha u(t) + \beta u(t-\tau)$$

with $\alpha > 0$, $\beta > 0$.

If there is a input delay, then the system becomes

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1), t > 0 \\ w(0,t) = 0, \\ w_x(1,t) = \alpha u(t) + \beta u(t-\tau) \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \\ u(s) = f(s), \quad s \in (-\tau, 0) \end{cases} \tag{12}$$

where $\alpha > 0$, $\beta > 0$. The collocated observation $y(t) = w_t(1,t)$.

Collocated feedback law

take the collocated feedback law $u(t) = -y(t)$.

Existing results

Main result:

Assume that $\alpha > 0$, $\beta > 0$, under the feedback law (11), the closed-loop system is of the following properties

1) when $\alpha > \beta$, the closed-loop system is exponentially stable for any $\tau > 0$.

2) when $\alpha = \beta$, the closed-loop system at most is asymptotically stable, the stability depends on the τ-value.

3) when $\alpha < \beta$, the closed-loop system is unstable.

References:

G. Q. Xu, S. P. Yung and L. K. Li, Stabilization of wave systems with input delay in the boundary control, *ESAIM: Control Optim. Calc. Var.*, Vol.12, (2006), pp. 770–785.

$\frac{1}{2}$-stable rule

How to understand this result

Rewrite $v(t) = \alpha u(t) + \beta u(t-\tau)$ into

$$v(t) = (\alpha + \beta)[\frac{\alpha}{\alpha+\beta}u(t) + \frac{\alpha}{\alpha+\beta}u(t-\tau)].$$

Then the above result can be read as $\frac{\alpha}{\alpha+\beta} > \frac{1}{2}$, the closed-loop system is stable, and $\frac{\alpha}{\alpha+\beta} < \frac{1}{2}$, then the closed-loop system is unstable.

$\frac{1}{2}$-stable rule

In $v(t)$, the weight of without delay part of control is larger than the delay part, the closed-loop system is exponentially stable, otherwise, the system is unstable.

Extended results about $\frac{1}{2}$-stable rule

High-dimensional wave

S. Nicaise, C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, SIAM Journal on Control and Optimization 45 (5) (2006) 1561 - 1585.

1-d wave networks

S. Nicaise, J. Valein, Stabilization of the wave equation on 1-d networks with a delay term in the nodal feedbacks, Networks and Heterogeneous Media 2 (3) (2007) 425 - 479.

High-dimensional wave with distributed delay

S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, *Differential and Integral Equations*, 21 (2008), 935–958.

1-d Euler-Bernoulli beam

J. Y. Park, Y. H. Kang, J. A. Kim, Existence and exponential stability for a Euler-Bernoulli beam equation with memory and boundary output feedback control term, *Acta Appl. Math.*, 104 (2008), 287–301.

Extended results about $\frac{1}{2}$-stable rule

Abstract evolution equation

E. M. Ait Benhassi, K. Ammari, S. Boulite, L. Maniar, Feedback stabilization of a class of evolution equations with delay, Journal of Evolution Equations 9 (2009) 103 - 121.

Abstract evolution equation

S. Nicaise and J. Valein, Stabilization of second order evolution equations with unbounded feedback with delay, *ESAIM Control, Optimization and Calculus of Variations*, 16, 2010, pp. 420–456.

Wave equation with interior input delay

S. Nicaise and C. Pignotti, Interior feedback stabilization of wave equations with time dependent delay, *Electron. J. Diff. Equ.*, Vol.2011, (2011), No.41, pp. 1-20.

Extended results about $\frac{1}{2}$-stable rule

1-d Timoshenko beam

Z. J. Han and G. Q. Xu, Exponential stability of Timoshenko beam system with delay terms in boundary feedbacks, *ESAIM: Control, Optimisation and Calculus of Variations*, 17 (2010), 552–574.

1-d Timoshenko beam

Z. J. Han, G. Q. Xu, Dynamical behavior of networks of non-uniform Timoshenko beams system with boundary time-delay inputs, *Networks and Heterogeneous Media*, Vol. 6, 2011, pp. 297–327.

1-d Timoshenko beam

B. Said-Houari, Y. Laskri, A stability result of a Timoshenko system with a delay term in the internal feedback, *Applied Mathematics and Computation*, Vol.217, 2010, pp.2857–2869.

1-d Timoshenko beam

M. Kirane, B. Said-Houari, M. N. Anwar, Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks, *Commun. Pure Appl. Anal.*, 10(2), 2011, pp.667-686.

Extended results about $\frac{1}{2}$-stable rule

1-d Timoshenko beam

B. Said-Houari and A. Soufyane, Stability result of the Timoshenko system with delay and boundary feedback, *IMA Journal of Mathematical Control and Information*, 2012, pp. 383-398

1-d Euler-Bernoulli beam

Y. F. Shang, G. Q. Xu and Y. L. Chen, Stability analysis of Euler-Bernoulli beam with input delay in the boundary control, *Asian Journal of Control*,14 (2012), 1–11.

Notice

This research shows that $\beta > 0$ is not necessary.

Extended results about $\frac{1}{2}$-stable rule

Extending to case 2:

$$v(t) = \alpha u(t) + \beta u(t-\tau) + \int_{-\tau}^{0} g(s)u(t+s)ds$$

How to understand these extension results

$$v(t) = \alpha u(t) + \beta u(t-\tau) + \int_{-\tau}^{0} g(s)u(t+s)ds$$

If $\alpha > \beta + \int_{-\tau}^{0} |g(s)|ds$, then under the collocated feedback control law, the closed-loop system is stable. This is a sufficient condition, but not necessary. It also is an extension version of $\frac{1}{2}$-stable rule.

1-d Euler-Bernoulli beam

Y. F. Shang, G. Q. Xu and Y. L. Chen, Stability analysis of Euler-Bernoulli beam with input delay in the boundary control, *Asian Journal of Control*,14 (2012), 1–11.

Questions

All results are under the collocated feedback control law. Therefore we have following questions:

Question 1

In the case of $v(t) = \alpha u(t) + \beta u(t-\tau)$, can one find out a new feedback control law such that, for any $\alpha, \beta \in \mathbb{R}$ and $|\alpha| + |\beta| \neq 0$, the closed-loop system is exponentially stable?

Question 2

If the system is a coupled system such as Timoshenko beam, the system has many input delays, can one find out a feedback control law that makes the closed loop system stable?

Question 3

In the case of distributed delay, i.e., $v(t) = \alpha u(t) + \beta u(t-\tau) + \int_{-\tau}^{0} \alpha'(s) u(t+s) ds$, find out a feedback control law and the least conditions that make the closed loop system stable.

Questions

Question 4

If the system has input delay and output delay, can one find out a feedback control law and the least conditions that make the closed loop system stable?

Question 5

In the multi-delays case, $v(t) = \alpha u(t) + \beta u(t-\tau_1) + \gamma u(t-\tau_2)$, can one find out a new feedback control law such that, for any $\alpha, \beta, \gamma \in \mathbb{R}$ and $|\alpha| + |\beta| + |\gamma| \neq 0$, the closed-loop system is exponentially stable?

Question 6

If above problems can be done, can one extend such control design to high-dimensional system?

Notice

There is a connection between Question 4 and Question 5.

Answer to Question 1

Euler-Bernoulli beam

Ying Feng Shang, Gen Qi Xu, Stabilization of an Euler - Bernoulli beam with input delay in the boundary control, System & Control Letter,Vol.61, 2012, pp.1069-1078

Euler-Bernoulli beam

Z. J. Han, G. Q. Xu, Output-based stabilization of Euler-Bernoulli beam with time-delay in boundary input, *IMA Journal of Mathematical Control and Information*, (2013), doi:10.1093/imamci/dnt030

1-D Wave equation

Han Wang, Gen Qi Xu, Exponential stabilization of 1-d wave equation with input delay, WSEAS Transactions on Mathematics, 12(10), 2013, pp.1001-1013.

Answer to Question 2

Timoshenko beam

Genqi Xu, Hongxia Wang, Stabilization of Timoshenko beam system with delay in the boundary control, International Journal of Control, 2013, *INT. J. Control*, **86**, (2013), pp. 1165-1178.

Remarks:

Note that the order of the system is, Euler-Bernoulli beam, 1-d wave equation, Timoskenko beam. What is the difference among them?

1) Euler-Bernoulli beam has larger spectral gap;

2) 1-d Wave equation has a constant spectral gap;

3) Timoshenko beam has coupled spectrum and smaller spectral gap, or has eigenvalue of multiplicity 2;

The main difficulty appears in the proof of stability of closed loop system. The detail please see the applicable method below.

Answer to Question 3

Euler-Bernoulli beam

Ying Feng Shang, Gen Qi Xu, Dynamic feedback control and exponential stabilization of a compound system, Journal of Mathematical Analysis and Applications, 422, (2015), 858 - 879

Timoshenko beam

X. F. Liu and G. Q. Xu, Exponential stabilization for Timoshenko beam with distributed delay in the boundary control, *Abstract and Applied Analysis*, (2013), doi:10.1155/2013/726794.

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X. F. Liu and G. Q. Xu, Output-based stabilization of Timoshenko beam with the boundary control and input distributed delay, Under Review

Idea to deal with the input delay

Step 1:

By a transform T, we translate the delay system into a system without delay.

Step 2: Control signal generating

For the system without delay, we adopt the collocated feedback approach to obtain a control signal.

Step 3: stability analysis of the closed loop system

For the system without delay, under the collocated feedback control law, we prove exponential stability of the closed loop system

Step 4: Comparing

Applying the control signal to original system, we prove this system also is exponentially stable.

Transform realization

Purpose:

Find a transform T that translate the delay system into a system without delay

Smith predictor

$$\begin{cases} \dot{x}(s,t) = Ax(s,t) + Bu(t+s) \\ x(0,t) = x(t) \end{cases}$$

it fits the pure delay problem

$$\dot{x}(t) = Ax(t) + Bu(t-\tau), \quad x(0) = x_0.$$

But it does not fit the problem of the form

$$\dot{x}(t) = Ax(t) + B_0u(t) + B_1u(t-\tau)$$

Transform realization

Artstein Transform:

Artstein Transform

$$z(t) = x(t) + \int_{t-\tau}^{t} e^{A(t-s-\tau)} B_1 u(s) ds$$

translates the delayed equation

$$\dot{x}(t) = Ax(t) + B_0 u(t) + B_1 u(t-\tau)$$

into a delay-independent system

$$\dot{z}(t) = Az(t) + (B_0 + e^{-A\tau} B_1) u(t)$$

However, in the infinite-dimensional system, $e^{-A\tau}$ might be a unbounded operator.

Reference

Z. Artstein, Linear Systems with Delayed Controls: A Reduction, *IEEE Transaction on Automatic Control*, Vol. AC-278, No.4, 1982,pp.869-879.

Transform realization

Partial state predictor

Purpose: Transform the delay system into a system without delay.

Partial state predictor

Consider the delayed equation

$$\dot{x}(t) = Ax(t) + B_0u(t) + B_1u(t-\tau).$$

We introduce the partial state predictor:

$$\begin{cases} \dot{x}(s,t) = Ax(s,t) + B_1u(t-\tau+s), s \in (0,\tau) \\ x(0,t) = x(t) \end{cases}$$

Set

$$p(t) = e^{A\tau}x(t) + \int_{t-\tau}^{t} e^{A(t-s)}B_1u(s)ds.$$

Then we have a delay-independent system

$$\dot{p}(t) = Ap(t) + (e^{A\tau}B_0 + B_1)u(t)$$

Transform realization

Extending to more general case:

$$\dot{x}(t) = Ax(t) + B_0u(t) + B_1u(t-\tau) + \int_{-\tau}^{0} G(s)u(t+s)ds$$

where $G(s)$ is operator-valued function defined on $[0,\tau]$.

Partial state predictor

We introduce the partial state predictor:

$$\begin{cases} \dot{x}(s,t) = Ax(s,t) + \left[B_1u(t+s-\tau) + \int_{t+s-\tau}^{t} G(t+s-r)u(r)dr\right], s \in (0 \\ x(0,t) = x(t) \end{cases}$$

whose solution is

$$x(s,t) = e^{As}x(t) + \int_0^s e^{A(s-\nu)}B_1u(t+\nu-\tau)d\nu$$

$$+ \int_0^s e^{A(s-\nu)}d\nu \int_\nu^\tau G(r)u(t+\nu-r)dr$$

Transform realization

Set

Transform

$$p(t) = x(\tau, t) = e^{A\tau}x(t) + \int_{t-\tau}^{t} \left[e^{A(\tau-s)}B_1 + \int_{t-s}^{\tau} e^{A(t-s-r+\tau)}G(r$$

Transform realization

Under this transform, the equation

$$\dot{x}(t) = Ax(t) + B_0u(t) + B_1u(t-\tau) + \int_{-\tau}^{0} G(s)u(t+s)ds$$

is translated into a delay-independent system

$$\dot{p}(t) = Ap(t) + [e^{A\tau}B_0 + B_1 + \int_{0}^{\tau} e^{A(\tau-s)}G(s)ds]u(t)$$

Transform realization

Case study

we consider the following compounded dynamical system

$$\begin{cases} z_t(s,t) = z_s(s,t), \quad s \in (-\tau, 0),\ t > 0 \\ w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ w_{xxx}(1,t) = \alpha z(0,t) + \beta z(-\tau,t) + \int_{-\tau}^{0} g(s) z(s,t) ds, \\ z(0,t) = u(t), \quad t \geq 0 \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \quad t > 0 \\ z(s,0) = f(s), \quad s \in (-\tau, 0) \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ x \in (0,1) \end{cases} \tag{13}$$

where $g \in L^2[-\tau, 0]$ with $||g(\cdot)||_{L^2} \neq 0$, α, β are real constants, and $u(t)$ is the external input.

Transform realization

Case study

By solving the first equation in (13), we know that the system (13) is equivalent to the following:

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ w_{xxx}(1,t) = \alpha u(t) + \beta u(t-\tau) + \int_{-\tau}^{0} g(\eta) u(t+\eta) d\eta, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \quad t > 0 \\ u(s) = f(s), \quad s \in (-\tau, 0) \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ x \in (0,1). \end{cases} \tag{14}$$

Obviously, this is a system of Euler-Bernoulli beam with distributed delay in input.

Assumption

Suppose that the state of the system is measurable, that is, $(w(x,t), w_t(x,t))$ can be measured.

Transform realization

Partial state predictor design

$$\begin{cases} \widehat{w}_{ss}(x,s,t)+\widehat{w}_{xxxx}(x,s,t)=0, \quad 0\le x\le 1,\ 0<s\le\tau, \\ \widehat{w}(0,s,t)=\widehat{w}_x(0,s,t)=\widehat{w}_{xx}(1,s,t)=0, \\ \widehat{w}_{xxx}(1,s,t)=\beta u(t+s-\tau)+\int_{-\tau}^{-s} g(\eta)u(t+s+\eta)d\eta \\ \widehat{w}(x,0,t)=w(x,t),\ \widehat{w}_s(x,0,t)=w_t(x,t) \end{cases} \tag{15}$$

Notice

The initial datum are the state of original system, the control is only the delay part.
Please note that in this auxiliary system we do not use the control information after t.

Transform realization

Transform

$$T(w) = p(x,t) = \widetilde{w}(x,\tau,t), \quad T(w_t) = q(x,t) = \widetilde{w}_s(x,\tau,t)$$

Case study

Using (14), we derived the following system:

$$\begin{cases} p_t(x,t) = q(x,t) - a(x)u(t), \quad 0 < x < 1,\ t > 0, \\ q_t(x,t) + p_{xxxx}(x,t) = -b(x)u(t), \\ p(0,t) = p_x(0,t) = p_{xx}(1,t) = 0, \\ p_{xxx}(1,t) = \beta u(t), \\ p(x,0) = E_0(w_0,w_1)(x) + \int_{-\tau}^0 a_0(x,r)f(r)dr, \\ q(x,0) = E_1(w_0,w_1)(x) - \int_{-\tau}^0 a_1(x,r)f(r)dr. \end{cases} \tag{16}$$

where $a_0(x,s), a_1(x,s)$, $a(x)$ and $b(x)$ are measurable functions and E_0 and E_1 are bounded linear operators on $H^2[0,1] \times L^2[0,1]$.

Remark:

(16) is a system of the control acting internal and boundary at same time. But it has no time delay.

Generation of control signal

Generating control signal

Purpose: Generating control signal by Feedback control design

Collocated feedback control design

To obtain a control signal, we consider the energy functional of system (16):

$$E(t) = \frac{1}{2}\int_0^1 \left[p_{xx}^2(x,t) + q^2(x,t)\right] dx$$

Direct calculation gives

$$\frac{dE(t)}{dt} \quad = \quad -u(t)\left[\beta q(1,t) + \int_0^1 q(x,t)b(x)dx + \int_0^1 p_{xx}(x,t)a''(x)dx\right]$$

For (16) we adopt the feedback control law

$$u(t) = U(p,q) = \beta q(1,t) + \int_0^1 q(x,t)b(x)dx + \int_0^1 p_{xx}(x,t)a''(x)dx. \quad (17)$$

This is the control signal used later.

Comparing

Closed loop system to (16)

$$\begin{cases} p_t(x,t) = q(x,t) - a(x)U(p,q), \quad 0 < x < 1,\ t > 0, \\ q_t(x,t) + p_{xxxx}(x,t) = -b(x)U(p,q), \\ p(0,t) = p_x(0,t) = p_{xx}(1,t) = 0, \\ p_{xxx}(1,t) = \beta U(p,q), \\ p(x,0) = E_0(w_0,w_1)(x) + \int_{-\tau}^{0} a_0(x,s)f(s)ds, \\ q(x,0) = E_1(w_0,w_1)(x) - \int_{-\tau}^{0} a_1(x,s)f(s)ds. \end{cases} \tag{18}$$

Original system with (17)

$$\begin{cases} w_t(x,t) = w_t(x,t), \quad 0 < x < 1,\ t > 0, \\ w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \\ w_{xxx}(1,t) = \alpha U(p,q)(t) + \beta U(p,q)(t-\tau) \\ \qquad + \int_{-\tau}^{0} g(s)U(p,q)(t+s)ds, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x). \end{cases} \tag{19}$$

Relation between both systems

Error

$$e(x,t) = w(x,t+\tau) - p(x,t), \quad \eta(x,t) = w_t(x,t+\tau) - q(x,t)$$

Error estimate

$$\begin{aligned} & ||p(\cdot,t) - w(\cdot,t+\tau)||_{H_E^2(0,1)} + ||q(\cdot,t) - w_t(\cdot,t+\tau)||_{L^2(0,1)} \\ \leq\ & 4\alpha^2 M_1^2 [E(t) - E(t+\tau)] \\ & +4(M_2^2 + M_3^2)\tau^2 \int_\tau^0 |g(r)|^2 dr [E(t-\tau) - E(t+\tau)]. \end{aligned}$$

Stability result

If the closed loop system (18) is exponentially stable, then the system (19) also is exponentially stable.
If the closed loop system (18) is asymptotically stable, then the system (19) also is asymptotically stable.

Test of exponential stability

Difficulty in Stability Test

Multiplier method fails to apply to the closed loop system!

Difficulty in Stability Test

The spectral method fails to apply to the closed loop system!

Feasible method

The duality method and system theory can be applied to the closed loop system!

Basic relation

If the system is exactly observable, then the dual system is exactly controllable. Hence the collocated feedback system is exponentially stable.

Dual system of (16)

The observation system corresponding to (16) is

$$\begin{cases} w_t(x,t) = v(x,t), \quad 0 < x < 1,\ t > 0, \\ v_t(x,t) = -w_{xxxx}(x,t), \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = w_{xxx}(1,t) = 0, \\ w(x,0) = w_0(x), \quad v(x,0) = v_0(x), \\ y(t) = \beta v(1,t) + \int_0^1 w_{xx}(x,t)a''(x)dx + \int_0^1 v(x,t)b(x)dx. \end{cases} \tag{20}$$

Lemma

Let the differential operator in $L^2[0,1]$ be defined by

$$\mathcal{L}z(x) = z^{(4)}(x), \mathcal{D}(\mathcal{L}) = \left\{ z(x) \in H^4(0,1) \;\middle|\; \begin{array}{l} z(0) = z'(0) = 0 \\ z''(1) = z'''(1) = 0 \end{array} \right\} \tag{21}$$

then eigenvalues of $\mathcal{L}$ are

$$0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots \tag{22}$$

and corresponding eigenfunctions $\varphi_n(x)$ are real functions and form a normalized orthogonal basis for $L^2[0,1]$.

Test of Observability

The spectral method for Exact observability

Let $\mathcal{H}$ be a Hilbert space, and let A_0 be a skew-adjoint operator with compact resolvent in $\mathcal{H}$, i.e., $A_0^* = -A_0$. Let $\sigma(A_0) = \{\lambda_k; k = \pm 1, \pm 2, \cdots\}$ and $\{\Phi_k\}_{k=-\infty}^{\infty}$ be the corresponding eigenvectors. Let $\mathbb{Y}$ be another Hilbert space, and let $C \in L(D(A_0), \mathbb{Y})$. If A_0 and C satisfy the following conditions:

(1) Spectral gap condition:

$$\inf_{k \neq m} |\lambda_k - \lambda_m| > 0,$$

(2) Boundedness condition:

$$0 < m = \inf_{k \in Z} ||C\Phi_k||_{\mathbb{Y}} < \sup_{k \in Z} |C\Phi_k||_{\mathbb{Y}} = M < \infty$$

then C is an admissible observable operator for A_0, and (A_0, C) is exactly observable in finite time.

Main result

Theorem

Let $\{\mu_n, n \in \mathbb{N}\}$ be given as in Lemma 1, and $\xi_n = \int_0^\tau e^{-i\sqrt{\mu_n}\eta} g(\eta-\tau)d\eta$, $n \in \mathbb{N}$. Then the following assertions hold:

1) If $\alpha, \beta \in \mathbb{R}$ satisfy the condition that $\inf_n |\beta + \alpha e^{-i\sqrt{\mu_n}\tau} + \xi_n| > 0$, then the system (18) decays exponentially.

2) If $\alpha, \beta \in \mathbb{R}$ satisfy condition $\beta + \alpha e^{-i\sqrt{\mu_n}\tau} + \xi_n = 0$ for some $n \in \mathbb{N}$, the system (18) is unstable.

3) If $\alpha = \beta = 0$ and $\xi_n \neq 0$ for all $n \in \mathbb{N}$, the system (18) is asymptotically stable. If for some n, $\xi_n = 0$, then the system is unstable.

Remark

The condition $\beta + \alpha e^{-i\sqrt{\mu_n}\tau} + \xi_n \neq 0$ cannot be improved. This is because if the equality holds, then the control of the from $e^{i\sqrt{\mu_n}t}$ is invalid.

Thank you!

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