Bang-bang properties of minimal time and minimal norm controls for some evolution equations

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Let X and U denote our state and control spaces. Here, X is a real Hilbert space, with its inner product and norm $\langle \cdot, \cdot \rangle_X$ and $\| \cdot \|_X$; U is another real Hilbert space, with its inner product and norm $\langle \cdot, \cdot \rangle_U$, $\| \cdot \|_U$. Let $A : D(A) \subset X \to X$ generate a C_0-semigroup $\{S(t)\}_{t \geq 0}$ in X. Let $B \in \mathcal{L}(U, X)$.

Consider two controlled equations:

$$\begin{cases} y'(t) = Ay(t) + Bu(t),\ t > 0, \\ y(0) = y_0 \end{cases} \tag{1.1}$$

and

$$\begin{cases} y'(t) = Ay(t) + Bv(t),\ 0 < t \leq T, \\ y(0) = y_0. \end{cases} \tag{1.2}$$

Here, $y_0 \in X$, $T > 0$, $u \in L^\infty(\mathbb{R}^+; U)$ and $v \in L^\infty(0, T; U)$. Denote by $y(\cdot; y_0, u)$ and $\hat{y}(\cdot; y_0, v)$ the solutions of (1.1) and (1.2), respectively.

For each $M > 0$, we define a constant set of controls:

$$\mathcal{U}^M \triangleq \{u \in L^\infty(\mathbb{R}^+; U) \ : \ \|u(t)\|_U \leq M, \text{ a.e. } t \in \mathbb{R}^+\}.$$

First, consider the minimal time control problem $(TP)^{M,y_0}$, with $M > 0$ and $y_0 \in X \setminus \{0\}$:

$$T(M, y_0) \triangleq \inf\{t > 0 \ : \ \exists\, u \in \mathcal{U}^M \text{ s.t. } y(t; y_0, u) = 0\}.$$

In $(TP)^{M,y_0}$, $T(M, y_0)$ is called the minimal time; $u^* \in \mathcal{U}^M$ is called a minimal time control if $y(T(M, y_0); y_0, u^*) = 0$; $\hat{u} \in \mathcal{U}^M$ is an admissible control if $y(\hat{t}; y_0, \hat{u}) = 0$ for some $\hat{t} > 0$. We say $(TP)^{M,y_0}$ has the bang-bang property if any minimal time control u^* verifies that $\|u^*(t)\|_U = M$ for a.e. $t \in (0, T(M, y_0))$.

Next, consider the minimal norm control problem $(NP)^{T,y_0}$, with $T > 0$ and $y_0 \in X\backslash\{0\}$:

$$N(T, y_0) \triangleq \inf\{\|v\|_{L^\infty(0,T;U)} \ : \ \hat{y}(T; y_0, v) = 0\}.$$

In $(NP)^{T,y_0}$, $N(T, y_0)$ is called the minimal norm; v^* is a minimal norm control if $\hat{y}(T; y_0, v^*) = 0$ and $\|v^*\|_{L^\infty(0,T;U)} = N(T, y_0)$; $\hat{v} \in L^\infty(0, T; U)$ is an admissible control if $\hat{y}(T; y_0, \hat{v}) = 0$. We say $(NP)^{T,y_0}$ has the bang-bang property if any minimal norm control v^* verifies that $\|v^*(t)\|_U = N(T, y_0)$ for a.e. $t \in (0, T)$.

We introduce two kinds of reachable subspaces:

$$\mathcal{R}_T \triangleq \{y_T = \hat{y}(T;0,v) \ : \ v \in L^\infty(0,T;U)\};$$

$$\mathcal{R}_T^0 \ \triangleq \ \{\hat{y}(T;0,v) \ : \ v \in L^\infty(0,T;U), \ \lim_{s\to T} \|v\|_{L^\infty(s,T;U)} = 0\}.$$

For each $T > 0$ and $y_T \in \mathcal{R}_T$, we define another kind of minimal norm problem $(NP)^{y_T}$:

$$\|y_T\|_{\mathcal{R}_T} \triangleq \inf\{\|v\|_{L^\infty(0,T;U)}; \ \hat{y}(T;0,v) = y_T\}.$$

(One can verify that $\|\cdot\|_{\mathcal{R}_T}$ defines a norm on $\mathcal{R}_T$, and also on $\mathcal{R}_T^0$.)

In $(NP)^{y_T}$, $\|y_T\|_{\mathcal{R}_T}$ is called the optimal norm; v^* is an optimal control if $\hat{y}(T;0,v^*) = y_T$ and $\|v^*\|_{L^\infty(0,T;U)} = \|y_T\|_{\mathcal{R}_T}$; $\hat{v} \in L^\infty(0,T;U)$ is an admissible control if $\hat{y}(T;0,\hat{v}) = y_T$. We say $(NP)^{y_T}$ has the bang-bang property if any optimal control v^* verifies that $\|v^*(t)\|_U = \|y_T\|_{\mathcal{R}_T}$ for a.e. $t \in (0,T)$.

Relation between $(NP)^{T,y_0}$ and $(NP)^{y_T}$:

- $(NP)^{T,y_0}$ has admissible controls $\Leftrightarrow -S(T)y_0 \in \mathcal{R}_T$.
- When $-S(T)y_0 \in \mathcal{R}_T$, $(NP)^{-S(T)y_0}$ and $(NP)^{T,y_0}$ have same optimal controls.
- When $-S(T)y_0 \in \mathcal{R}_T$, it holds that $\|y_T\|_{\mathcal{R}_T} = N(T,y_0)$.

We give the following two assumptions:

(H1) (Observability estimate) $\forall\, T>0$, $\exists\, C(T)>0$ s.t.

$$\|S^*(T)z\|_X \le C(T)\int_0^T \|B^*S^*(T-t)z\|_U \mathrm{d}t,\ \forall\, z\in X;$$

(H2) For each $T>0$, if $f\in Y_T$ and $f=0$ over a subset $E\subset(0,T)$ of positive measure, then $f\equiv 0$ over $(0,T)$. Here,

$$Y_T \triangleq \overline{\{B^*S^*(T-\cdot)z;\ z\in X\}}^{\|\cdot\|_{L^1(0,T;U)}},$$

with the norm $\|\cdot\|_{Y_T}\triangleq\|\cdot\|_{L^1(0,T;U)}$.

The above $(H3)$ is equivalent to the L^∞-null controllability of (A,B).
The above $(H4)$ says roughly that any function in Y_T has a unique continuation from measurable sets.

Notice that (i) A function in Y_T may not be able to expressed as $B^*\varphi$, with φ a solution of the adjoint equation over $(0,T)$; (ii) When (A,B) is the L^∞-null controllable, any function in Y_T can be expressed as $B^*\varphi$, where φ is a solution of the adjoint equation over $(0,T)$, with an initial datum in "a quite larger space".

Notice that for each $y_0 \in X \setminus \{0\}$, the map $T \to N(T, y_0)$ is decreasing. Thus, we can set

$$N(\infty, y_0) \triangleq \lim_{t\to\infty} N(t, y_0).$$

The main results are presented in the following theorem:

Theorem 1

Suppose that (H1) and (H2) hold. Let $y_0 \in X \setminus \{0\}$.
(i) $(TP)^{M,y_0}$ *has the bang-bang property iff* $M > N(\infty, y_0)$*;*
(ii) For each $T > 0$, $(NP)^{T,y_0}$ *has the bang-bang property;*
(iii) If $y_T \in \mathcal{R}_T^0$, *then* $(NP)^{y_T}$ *has the bang-bang property.*

- When $y_T \in \mathcal{R}_T \setminus \mathcal{R}_T^0$, we do not know if the result (iii) in above theorem holds.

In general, there are two ways to derive the bang-bang property for minimal time control problems for time-invariant systems.

The first way is the use of the null controllability from measurable sets. In the paper An abstract bang-bang principle and time-optimal boundary control of the heat equation, SIAM J. Control Optim., 1997, V. Mizel and T. Seidman pointed out that the bang-bang property for minimal time controls can be derived from the L^∞-null controllability from measurable sets. That is, for each $y_0 \in X$, each $T > 0$, and each subset $E \subset (0, T)$ of positive measure, there is a control $v \in L^\infty(0, T; U)$, with its support over E, so that $\hat{y}(T; y_0, v) = 0$. We call such controllability as E-controllability.

How to get the bang-bang property of $(TP)^{M,y_0}$ from E-controllability?

Let t^*, u^* and y^* be the optimal time, an optimal control and the optimal state (corresponding to u^*) for $(TP)^{M,y_0}$, respectively. Then $y^*(t^*) = 0$ and $\|u^*\|_{L^\infty(\mathbb{R}^+;U)} \leq M$. Supposed, by contradiction, that $\exists$ a subset $e \subset (0, t^*)$ of positive measure and $\varepsilon_0 > 0$ s.t. $\|u^*(t)\|_U \leq M - \varepsilon_0$ for a.e. $t \in e$. By the "E-controllability", when $\delta > 0$ is small enough, $\exists\, v$, with

$$\chi_{(\mathbb{R}^+ \cap e^c)} v = 0; \ \ \|\chi_e v\|_{L^\infty(\mathbb{R}^+;U)} \leq \varepsilon_0/2,$$

s.t. v drives the solution from $\big(y_0 - y^*(\delta)\big)$ at $t = \delta$ to 0 at $t = t^*$. Now set $u_1(\cdot) \triangleq (u^* + \chi_e v)(\cdot + \delta)$. Clearly, $u_1 \in \mathcal{U}^M$. Then by the translation invariance in time of the system, we find that

$$
\begin{aligned}
& y(t^* - \delta; y_0, u_1) \\
= \;& S(t^* - \delta) y_0 + \int_0^{t^* - \delta} S(t^* - \delta - t) B(u^* + \chi_e v)(t + \delta) \mathrm{d}t \\
= \;& S(t^* - \delta) y_0 + \int_\delta^{t^*} S(t^* - t) B(u^* + \chi_e v)(t) \mathrm{d}t \\
= \;& \Big[S(t^* - \delta) y^*(\delta) + \int_\delta^{t^*} S(t^* - t) B u^*(t) \mathrm{d}t \Big] + \\
& \Big[S(t^* - \delta) \big(y_0 - y^*(\delta) \big) + \int_\delta^{t^*} S(t^* - t) B(\chi_e v)(t) \mathrm{d}t \Big] \\
= \;& 0.
\end{aligned}
$$

This contradicts the optimality of t^*. So u^* is a bang-bang control.

However, the above method does not work for time varying systems, because of the lack of the translation invariance in time. It deserves to mention that the bang-bang property of $(NP)^{T,y_0}$ for time varying heat equations still can be derived from the "E-controllability", since its proof does not need the above invariance. See

- K-D Phung and G. Wang, *An observability estimate for parabolic equations from a measurable set in time and its applications*, J. Eur. Math. Soc., 2013.

Consider $(TP)^{M,y_0}$, where Equation (1.1) is replaced by

$$y'(t) = A(t)y(t) + Bu(t),\ t > 0;\ \ y(0) = y_0. \qquad (2.1)$$

Here $A(t) = A + D(t)$, with A a generator of a C_0-semigroup on X, $D(\cdot) \in L^1(0,T;U)$, and here $B \in \mathcal{L}(U,X)$.
To get the bang-bang property, we need
Condition (H): For each $T > 0$, it holds that $Y_T = Z_T$. Here, Z_T is the space of all functions $B^*\varphi(\cdot) \in L^1(0,T;U)$ that for each $\hat{t} \in (0,T)$, there is $z_{\hat{t}} \in X$ s.t. $\varphi(\cdot) = \varphi(\cdot; z_{\hat{t}}, \hat{t})$, where $\varphi(\cdot; z_{\hat{t}}, \hat{t})$ solves the equation: $\varphi_t + A^*\varphi = 0$ in $(0,\hat{t})$; $\varphi(\hat{t}) = z_{\hat{t}}$.

For the time varying system (2.1), we proved that Condition (H) $+$ the bang-bang property for $(NP)^{T,y_0}$ implies the bang-bang property for $(TP)^{M,y_0}$. Consequently, Condition (H) $+$ E-controllability implies the bang-bang property for $(TP)^{M,y_0}$. See
G. Wang, Y. Xu and Y. Zhang, Attainable subspaces and the bang-bang property of time optimal controls for heat equations, SIAM J. Control Optim., 2015.
However, we do not know if Condition (H) holds even for heat equations with time-varying lower terms in general. So we are trying to go another way to get the bang-bang property for $(TP)^{M,y_0}$. (The aforementioned another way is the second way which will be introduced on the next page.)
So far, we can get the bang-bang only for time-invariant case, via the above-mentioned "another way". BUT, we feel that we are on the right way to approach the bang-bang property for $(TP)^{M,y_0}$, with time-varying systems.

About the null controllability from measurable set, we mention the following papers:

• V. Mizel and T. Seidman, *An abstract bang-bang principle and time-optimal boundary control of the heat equation*, SIAM J. Control Optim., 1997. They built up the "E-controllability" for the 1-D boundary controlled heat equation.

• G. Wang, *L^∞-null controllability for the heat equation and its consequences for the time optimal control problem*, SIAM J. Control Optim., 2008. In this paper, the "E-controllability" was built up for the n-dimensional internal controlled heat equation.

- G. Wang and C. Zhang, *Observability inequalities from measurable sets for some evolution equations*, arXiv: 1406.3422, (2014).

This paper presents that the "E-controllability" can be implied by the L^2-exact null controllability over each interval $[0, T]$ with a cost like $e^{C(1+\frac{1}{T^k})}$, $k > 0$, $T > 0$, under the framework of our talk with $\{S(t)\}_{t\geq 0}$ analytic.

- J. Apraiz, L. Escauriaza, G.Wang and C. Zhang, Observability inequalities and measurable sets, J. Eur. Math. Soc. 16 (2014).

This paper shows the null controllability from measurable sets in both time and space variables for the heat equation.

The second way is the use of the Pontryagin maximum principle and the unique continuation from measurable sets in time. The key of this way is to derive the Pontryagin maximum principle. We would like to mention that the maximum principle may not hold for these problems, in general. (See Example 1.4 on Page 132 in the book: X. Li and J. Yong, OPTIMAL CONTROL THEORY FOR INFINITE DIMENSIONAL SYSTEMS, Birkhäuser Boston 1995.)

In 1999, H. Fattorini studied the Pontryagin maximum principle for both the minimal time and the minimal norm control problems, with an initial state ζ and a target state $\bar{y}$, for the system:

$$\begin{cases} y'(t) = Ay(t) + v(t), t > 0, \\ y(0) = y_0. \end{cases} \tag{2.2}$$

In our framework, this corresponding to the case that $U = X$ and $B = I_X$. He first realized the following fact:

$$D(A) \hookrightarrow \mathcal{R}_T \text{ for all } T > 0. \tag{2.3}$$

Then, with the aid of (2.3), he divided the dual space of $\mathcal{R}_T$ into the regular part and the singular part. After that, he proved that if $\bar{y}-S(T^*)\zeta \in \overline{D(A)}$, then $\bar{y}-S(T^*)\zeta$ and $B^{\infty}_{M}(T^*)$ can be separated by a hyperplane (in $\mathcal{R}_{T^*}$), with a regular normal vector. Here T^* is the minimal time and $B^{\infty}_{M}(T^*)$ is the ball in $\mathcal{R}_{T^*}$, centered at the origin and of radius $M > 0$.

Finally, with the help of the aforementioned separating property, he obtained the Pontryagin maximum principle. It is worth mentioning that (2.3) holds because $U = X$ and $B = I_X$.
In the paper: J. Lohéac and M. Tucsnak, Maximum principle and bang-bang property of time optimal controls for Schrödinger type systems, SIAM J. Control Optim., 2013, the authors successfully used this way to get the bang-bang property for the minimal time controls for the Schrödinger equation.

We approach the bang-bang properties via the second way. The sketch of the proof of the main theorem is as follows.

Step 1. We prove that when (H1) holds, for each $y_T \in \mathcal{R}_T^0$, there exists $f^* \in Y_T \setminus \{0\}$ so that

$$\langle v^*(t), f^*(t)\rangle_U = \max_{\|w\|_U \leq \|y_T\|_{\mathcal{R}_T}} \langle w, f^*(t)\rangle_U, \quad \text{a.e. } t \in (0,T) \quad (3.1)$$

for every optimal control v^* to $(NP)^{y_T}$. (The maximum principle for $(NP)^{y_T}$ with $y_T \in \mathcal{R}_T^0$.)

This is the main step. We will present the ideas to show (3.1) and explain why we can only do it for $y_T \in \mathcal{R}_T^0$, but not for $y_T \in \mathcal{R}_T$. Write $B_{\mathcal{R}_T}(0,M)$ and $B_{\mathcal{R}_T^0}(0,M)$ for the closed balls in $\mathcal{R}_T$ and $\mathcal{R}_T^0$, centered at the origin and of radius M, respectively.

To do this, we first prove that $(\mathcal{R}_T^0)^* = Y_T$ in the sense of linear isomorphism; then by using the Hahn-Banach Theorem, we separate y_T from $B_{\mathcal{R}_T^0}(0,M)$ in the space $\mathcal{R}_T^0$, by a a vector $f^* \in Y_T$; after that, by some relation between $B_{\mathcal{R}_T^0}(0,M)$ in $B_{\mathcal{R}_T}(0,M)$ (which will be explained later), we prove that the above f^* also separates y_T from $B_{\mathcal{R}_T}(0,M)$ in the space $\mathcal{R}_T$; finally, we verify (3.1), via two representation theorem as follows.

Theorem 2 (The representation theorem for Y_T^*)

Let $T > 0$. There is a linear isomorphism $\Phi_T : \mathcal{R}_T \to Y_T^*$ so that

$$\begin{aligned}\langle y_T, f\rangle_{\mathcal{R}_T, Y_T} &\triangleq \langle \Phi_T(y_T), f\rangle_{Y_T^*, Y_T} \\ &= \int_0^T \langle v(t), f(t)\rangle_U \mathrm{d}t, \ \forall\, y_T \in \mathcal{R}_T, \ \forall\, f \in Y_T,\end{aligned}$$

where v is any admissible control to $(NP)^{y_T}$.

Theorem 3 (The representation theorem for $(\mathcal{R}_T^0)^*$)

Let $T > 0$. Suppose that (H1) holds. Then there is a linear isomorphism $\Psi_T : Y_T \to (\mathcal{R}_T^0)^*$ so that

$$\begin{aligned}\langle f, y_T\rangle_{Y_T,\mathcal{R}_T^0} &\triangleq \langle \Psi_T(f), y_T\rangle_{(\mathcal{R}_T^0)^*,\mathcal{R}_T^0}\\ &= \int_0^T \langle f(t), v(t)\rangle_U \mathrm{d}t,\ \forall\, f \in Y_T,\ \forall\, y_T \in \mathcal{R}_T^0,\end{aligned}$$

where v is any admissible control to $(NP)^{y_T}$.

Two remarks are given in order: (i) When $y_T \in \mathcal{R}_T^0$, we do not know how to directly use the Hahn-Banach theorem to separate y_T from $B_{\mathcal{R}_T}(0, M)$ in $\mathcal{R}_T$ by $f^* \in Y_T$. What we did is to separate y_T from $B_{\mathcal{R}_T^0}(0, M)$ in $\mathcal{R}_T^0$ by $f^* \in Y_T$, and then prove that this f^* separates also y_T from $B_{\mathcal{R}_T}(0, M)$ in $\mathcal{R}_T$. (ii) When $y_T \in \mathcal{R}_T$, we can directly use the Hahn-Banach theorem to separate y_T from $B_{\mathcal{R}_T}(0, M)$ in $\mathcal{R}_T$, by a vector $\hat{f} \in (\mathcal{R}_T)^*$, but not in Y_T. Indeed, we have that for all $z_T \in B_{\mathcal{R}_T}(0, \|y_T\|_{\mathcal{R}_T})$, $\langle \hat{f}, y_T \rangle_{(\mathcal{R}_T)^*, \mathcal{R}_T} \geq \langle \hat{f}, z_T \rangle_{(\mathcal{R}_T)^*, \mathcal{R}_T}$. Unfortunately, we do not know how to represent $\langle \hat{f}, z_T \rangle_{(\mathcal{R}_T)^*, \mathcal{R}_T}$ in terms of controls like what we got in Theorem 3. So we are not able to use the above separation to get the maximum principle for $(NP)^{y_T}$. That is why we introduce the space $\mathcal{R}_T^0$.

Step 2. From (3.1) and the equivalence between $(NP)^{T,y_0}$ and $(NP)^{-S(T)y_0}$, we get that under the assumption (H1), for each $y_0 \in X \setminus \{0\}$, with $T^0(y_0) < T^1(y_0)$, and each $T \in \big(T^0(y_0), T^1(y_0)\big)$, there exists $f^* \in Y_T \setminus \{0\}$ so that

$$\langle u^*(t), f^*(t)\rangle_U = \max_{\|v\|_U \leq N(T,y_0)} \langle v, f^*(t)\rangle_U, \quad \text{a.e. } t \in (0,T) \qquad (3.2)$$

for every optimal control u^* to $(NP)^{T,y_0}$. (The maximum principle for $(NP)^{T,y_0}$.)

Step 3. With the aid of (H1), we prove that for each $y_0 \in X \setminus \{0\}$, with $T^0(y_0) < T^1(y_0)$, the map $T \to N(T, y_0)$ is decreasing and continuous. Then we prove that under the assumption (H1), for each $y_0 \in X \setminus \{0\}$, with $T^0(y_0) < T^1(y_0)$, and each M in $(N(\infty, y_0), N(T^0(y_0), y_0))$, there is $f^* \in Y_{T(M,y_0)} \setminus \{0\}$ so that

$$\langle u^*(t), f^*(t)\rangle_U = \max_{\|v\|_U \leq M} \langle v, f^*(t)\rangle_U, \text{ a.e. } t \in (0, T(M, y_0)) \quad (3.3)$$

for every optimal control u^* to $(TP)^{M,y_0}$.

Step 4. With the help of the assumption (H2), we get the bang-bang properties for three problems by (3.1), (3.2) and (3.3), respectively.

Several remarks are given in order: (i) In the case where $U = X$ and $B = I_X$, the condition $(H1)$ holds automatically.
(ii) (3.1), (3.2) and (3.3) are not the standard Pontryagin maximum principle, since in general, f^* cannot be expressed as $B^*\varphi$, where φ is a solution of the dual equation with an initial state in X.
(iii) The assumption that $B \in \mathcal{L}(U, X)$ can be relaxed to that $B \in \mathcal{L}(U, (D(A^*))^*)$ be an admissible control operator for $\{S(t)\}_{t\geq 0}$, i.e., for each $\hat{t} > 0$, there exists $C_1(\hat{t}) > 0$ so that

$$\| \int_0^{\hat{t}} S_{-1}(\hat{t} - \tau)Bu(\tau)\mathrm{d}\tau\|_X \leq C_1(\hat{t})\|u\|_{L^2(0,\hat{t};U)}$$

for all $u \in L^2_{\text{loc}}(\mathbb{R}^+; U)$.

(iv) The condition $(H1)$ is not essential for us to derive (3.1) (i.e., the maximum principle for $(NP)^{y_T}$ with $y_T \in \mathcal{R}_T^0$). $(H1)$ can be replaced by the following condition:

$(H3)$: There is $p_0 \in [2, \infty)$ so that $\mathcal{A}_{p_0}(T, \hat{t}) \subset \mathcal{A}_\infty(T, \hat{t})$ for all T, $\hat{t}$, with $0 < \hat{t} < T$, where

$$\mathcal{A}_{p_0}(T, \hat{t}) \triangleq \left\{ \hat{y}(T; 0, u) \ : \ u \in L^{p_0}(0, T; U), \, u|_{(\hat{t}, T)} = 0 \right\};$$

$$\mathcal{A}_\infty(T, \hat{t}) \triangleq \left\{ \hat{y}(T; 0, v) \ : \ v \in L^\infty(0, T; U), \, v|_{(0, \hat{t})} = 0 \right\}.$$

$(H3)$ roughly means that the functionality of a control supported on $(\hat{t}, T)$ can be replaced by that of a control supported on $(0, \hat{t})$.

We proved that $(H1)$ is stronger than $(H3)$.

Thank you !