Recent Developments on Robust Regulation of Infinite-Dimensional Systems

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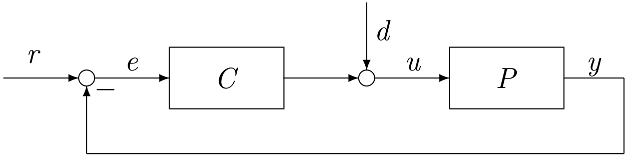
Robustness

- Robustness is a property that allows systems to maintain its functions despite external and internal perturbations.
- A system must be robust to function in unpredictable environments using unreliable components.
- Robustness is a fundamental feature of evolvable complex systems.
- Applications in economics.

Kitano: Biological Robustness, Nature (2004).
Sontag: Molecular Systems Biology and Control, European J. Control (2009).
Lars Peter Hansen, Thomas J. Sargent: Robustness (2007).

Feedback Control System

The closed-loop system with plant P, the reference signal r, the disturbance signal d, and the controller C.



- Output Regulation: Output will asymptotically track given reference signals and reject given disturbance signals.
- Robust Output Regulation: Regulation occurs despite perturbations in the system's parameters.

Introduction

- The Internal Model Principle (IMP) by Francis, Wonham and Davison: A stabilizing feedback controller solves the robust output regulation problem if and only if it incorporates a suitably reduplicated model (a p-copy) of the dynamics of the (reference and disturbance) signal generator.
- The aim of the talk is generalization of the IMP to infinite-dimensional (diffusion, vibration, time-delay) systems and infinite-dimensional signal generators (exosystems).

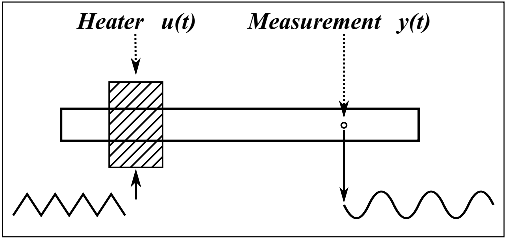
Previous work

- Structurally stable synthesis (Bhat 1976) (time-delay).
- IMP for systems with pseudorational impulse response, Yamamoto and Hara (1988)
- Robust control: Logemann, Townley, Pohjolainen, Hämäläinen (1982-).
- Regulator theory without robustness: Schumacher (1983), Byrnes et. al. (2000).
- New presentation for the finite-dimensional theory Huang (2004).
- Internal model for infinite-dimensional systems Immonen (2006,2007), Hämäläinen (2010), Paunonen (2010), jointly with Pohjolainen.

Previous work

- Polynomial stability and partial robustness (Paunonen, Pohjolainen (2012-2013)).
- Robust control for regular systems (Paunonen (2015))
- Frequency domain robust regulation for infinite-dimensional exosystems (Laakkonen & Pohjolainen (2015)).

Example:Heating a bar



$$\frac{\partial x}{\partial t} = \alpha \frac{\partial^2 x}{\partial z^2} + \beta x + b(z)u(t)$$

$$y(t) = \int_\Omega x(t,z)c(z)\,\mathrm{d}z.$$

Exosystem

Reference r and disturbance d signals are generated by the exosystem

$$\dot{v} = Sv, \qquad v(0) = v_0 \in W,$$

on a Hilbert space W. The operator S is given by

$$Sv = \sum_{n=-\infty}^{\infty} \mathrm{i}\omega_n \langle v, \phi_n \rangle \phi_n,$$

where $(\phi_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of W, $\inf_{n\neq k}|\omega_n - \omega_k| > 0$ and $|\omega_n| \to \infty$, as $n \to \pm\infty$.

The spectrum of S is a pure point spectrum on the imaginary axis with simple eigenvalues and spectral gap:

$$\sigma(S) = \sigma_p(S) = \{\, \mathrm{i}\omega_n \mid n \in \mathbb{Z} \,\}$$

Exosystem

Based on the spectrum $\sigma(S) = \{\, \mathrm{i}\omega_n \mid n \in \mathbb{Z} \,\}$ and a parameter α a scale of signal Hilbert spaces can be defined

$$W_\alpha = \left\{ v \in W \;\middle|\; \sum_{n=-\infty}^{\infty} (1+\omega_n^2)^\alpha |\langle v, \phi_n\rangle|^2 < \infty \right\}, \alpha \geq 0$$

$$W_0 = W, \quad W_1 = \mathcal{D}(S), \quad S \in \mathcal{L}(W_{\alpha+1}, W_\alpha)$$

$$\dot{v} = Sv, \quad v(0) = v_0 \in W_\alpha.$$

Bigger $\alpha \implies$ smoother signal v.

Exosystem

The signal $v(t)$ is given by

$$v(t) = T_S(t)v_0 = \sum_{n=-\infty}^{\infty} e^{i\omega_n t}\langle v_0, \phi_n\rangle \phi_n, \qquad v_0 \in W_\alpha.$$

Reference signal $r(t)$

$$r(t) = F_r v(t) = \sum_{n=-\infty}^{\infty} e^{i\omega_n t}\langle v_0, \phi_n\rangle F_r\phi_n, \qquad v_0 \in W_\alpha.$$

and disturbance signal $d(t) = ET_S(t)v_0$.

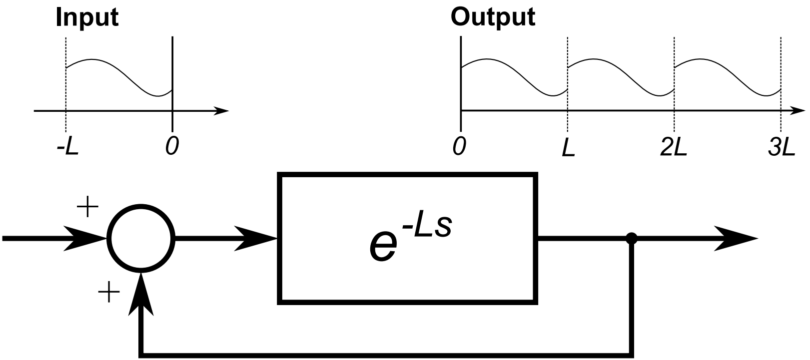
Smoothing conditions on E and F_r

Since the reference and (disturbance) signals are of the form

$$r(t) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega_n t} \langle v_0, \phi_n \rangle F_r \phi_n$$

- Smoothing can be added on the sequences $(F_r\phi_n)_{n\in\mathbb{Z}}$ and $(E\phi_n)_{n\in\mathbb{Z}}$ based on the system's transfer function at the infinity (Laakkonen (2011)).
- This limits the class of reference signals to be tracked and disturbance signals to be rejected.

Exosystem for Periodic Signals



Plant, exosystem, and controller

The plant can put into the standard form

$$\dot{x} = Ax + Bu + Ev, \qquad x(0) \in X$$
$$e = y - r = Cx + Du + Fv,$$

where v is generated by the exosystem.

- X and W_α are Hilbert spaces.The output and input spaces Y, U are p-dimensional.

Assumptions

- A generates C_0-semigroup.
- Other operators are bounded and E and F contain sufficient damping.
- The pair (A, B) is exponentially stabilizable.
- The pair (A, C) is exponentially detectable.
- The plant transfer function $P(s) = C(sI - A)^{-1}B + D$ is assumed to be invertible for $s \in \sigma_p(S)$ and $\sigma_p(S) \subset \rho(A)$.

Controller

The controller is given by

$$\dot{z} = \mathcal{G}_1 z + \mathcal{G}_2 e, \qquad z(0) \in Z$$
$$u = Kz,$$

on a Hilbert space Z, the output and input spaces Y, U are p-dimensional.

- $\mathcal{G}_1$ generates C_0-semigroup, K and $\mathcal{G}_2$ are bounded.
- $(\mathcal{G}_1, \mathcal{G}_2)$ satisfy $\mathcal{G}$-conditions i.e.
  (1) $\mathcal{N}(\mathcal{G}_2) = \{0\}$.
  (2) $\mathcal{R}(\mathcal{G}_2) \cap \mathcal{R}(\mathcal{G}_1 - \mathrm{i}\omega_n I) = \{0\} \quad \forall n \in \mathbb{Z}$.

Closed Loop System

The closed loop system consists of the plant and the controller

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e v, \qquad x_e(0) \in X_e \\ e &= C_e x_e + D_e v, \end{aligned}$$

on the Hilbert space $X_e = X \times Z$ where

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix}, \quad B_e = \begin{bmatrix} E \\ \mathcal{G}_2 F \end{bmatrix} \in \mathcal{L}(W_\alpha, X_e).$$

$$C_e = \begin{bmatrix} C & DK \end{bmatrix} \in \mathcal{L}(X_e, Y), \quad D_e = F \in \mathcal{L}(W_\alpha, Y).$$

Robust Output Regulation Problem

Definition

The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solves the Robust Output Regulation Problem (RORP) if

(i) The closed-loop system operator A_e generates a stable C_0-semigroup.

(ii) For all initial states $x_e(0) \in X_e$ and $v(0) \in W_\alpha$

$$\lim_{t\to\infty} [y(t) - r(t)] = \lim_{t\to\infty} e(t) = 0.$$

(iii) (i) and (ii) hold for a class of perturbations on the system parameters (A, B, C, D, E, F).

Dynamical steady-state operator H_{ss}

Assume that $\sigma(S) \subset \rho(A_e)$ and (formally) define

$$H_{\mathsf{ss}} v = \sum_{n=-\infty}^{\infty} \langle v, \phi_n \rangle R(\mathrm{i}\omega_n; A_e) B_e \phi_n.$$

For sufficiently smooth v, resolvent $R(\mathrm{i}\omega_n; A_e) = (i\omega_k I - A_e)^{-1}$ and B_e

$$H_{\mathsf{ss}} v = \sum_{n=-\infty}^{\infty} (1+\omega_n^2)^{\alpha/2} \langle v, \phi_n \rangle \frac{R(\mathrm{i}\omega_n; A_e) B_e \phi_n}{(1+\omega_n^2)^{\alpha/2}}.$$

So that $H_{\mathsf{ss}} \in \mathcal{L}(W_\alpha, X_e)$ satisfies $H_{\mathsf{ss}}(W_{\alpha+1}) \subset \mathcal{D}(A_e)$, and the Sylvester equation

$$H_{\mathsf{ss}} S - A_e H_{\mathsf{ss}} = B_e \quad \text{on} \quad W_{\alpha+1}.$$

Dynamical steady-state operator

With aid of the dynamical steady state operator H_{ss} the extended state decomposes into two parts

$$x_e(t) = T_{A_e}(t)[x_e(0) - H_{\mathsf{ss}} v(0)] + H_{\mathsf{ss}} v(t),$$

$$e(t) = C_e T_{A_e}(t)[x_e(0) - H_{\mathsf{ss}} v(0)] + (C_e H_{\mathsf{ss}} + D_e) v(t).$$

For stable A_e, as $t \to \infty$

$$\begin{aligned} x_e(t) - H_{\mathsf{ss}} v(t) &\to 0 \\ e(t) - (C_e H_{\mathsf{ss}} + D_e) v(t) &\to 0. \end{aligned}$$

Hence regulation = stabilization + tracking.

Robust Output Regulation Problem (RORP)

RORP is divided into two parts: Finding a controller that

(i) is (robustly) stabilizing,

(ii) robustly satisfies the regulation constraint
$C_e H_{\mathsf{ss}} + D_e = 0.$

Let us first discuss about the regulation constraint (ii).

G-conditions and regulation

H_{ss} satisfies the Sylvester equation

$$H_{\mathsf{ss}} S - A_e H_{\mathsf{ss}} = B_e \text{ on } W_{\alpha+1}.$$

Decompose $H_{\mathsf{ss}} : W_\alpha \to X \times Z$ as

$$H_{\mathsf{ss}} = \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix}.$$

Then the Sylvester equation decomposes as

$$\Pi S = A\Pi + BK\Gamma + E,$$
$$\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 (C_e H_{\mathsf{ss}} + D_e).$$

Because the controller satisfies $\mathcal{G}$-conditions, then $C_e H_{\mathsf{ss}} + D_e = 0$.

Robust Regulation Constraint

Denote by $green$ color free variables, by $blue$ dependent variables and by $black$ constant variables.

Now assume the system parameters (A, B, C, D, E, F) are perturbed to (A, B, C, D, E, F) so that the perturbed Sylvester equation $H_{\mathsf{ss}} S - A_e H_{\mathsf{ss}} = B_e$ has the solution H_{ss}. Then

$$\Pi S = A\Pi + BK\Gamma + E,$$

$$\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 (C_e H_{\mathsf{ss}} + D_e).$$

$\mathcal{G}$-conditions imply

$$0 = C_e H_{\mathsf{ss}} + D_e,$$

indicating robustness of regulation.

Robust Controller

Theorem

(Hämäläinen, Paunonen, Pohjolainen (2010))
Assume that

- A_e *generates a (robustly) stable* C_0*-semigroup*
- *there exists an operator* $H_{ss} \in \mathcal{L}(W_\alpha, X_e)$ *satisfying* $H_{ss}S - A_e H_{ss} = B_e$ *on* $W_{\alpha+1}$,
- *the controller satisfies the* $\mathcal{G}$*-conditions*

then the controller solves the RORP.

$\mathcal{G}$-conditions and p-copy internal model

Theorem (Paunonen (2010))

Let $\sigma(S) \cap \sigma(A_e) = \emptyset$. The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions iff the restriction $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k - \mathcal{G}_1)} \to Y$ is invertible.

Theorem (Paunonen (2010))

Let $\sigma(S) \cap \sigma(A_e) = \emptyset$. The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions iff

(1) $i\omega_k \in \sigma_p(\mathcal{G}_1)$

(2) *The geometric multiplicity of $i\omega_k$ as an eigenvalue of $\mathcal{G}_1$ is p.*

Stability of the closed loop system

- Due to IMP $\mathcal{G}_1$contains a p-copy of S i.e $\sigma(S) \subset \sigma(\mathcal{G}_1)$.
- This makes stabilizing A_e difficult.
- Exponential stability ($\| T_{A_e} \| \leq M\mathrm{e}^{-\omega t}$) with good robustness properties is out of reach when S is infinite-dimensional.
- Strong stability ($T_{A_e}(t)x_0 \to 0$) available but with weak robustness properties.
- Polynomial stability: weaker than exponential stability, better robustness properties than strong stability.

Polynomial stability

Lemma (Borichev, Tomilov (2010))

Assume A_e *generates a uniformly bounded semigroup and* $i\mathbb{R} \subset \rho(A_e)$*. For a fixed* $\alpha > 0$ *the following are equivalent.*

- $\|T_{A_e}(t){A_e}^{-1}\| \leq \frac{M}{t^{1/\alpha}} \qquad \forall t > 0$
- $\|R(i\omega, A_e)\| = O(|\omega|^{\alpha})$.

Polynomial stability is robust and some results are available by Paunonen (2012),(2013).

Robust Output Regulation Problem with Polynomial Stability

Theorem (Paunonen Pohjolainen (2013))

Assume $(\mathcal{G}_1, \mathcal{G}_2, K)$

- *Satisfies the* $\mathcal{G}$*-conditions.*
- *Stabilizes the closed loop system polynomially for some* α.

Then the controller solves the RORP so that $H_{ss} \in \mathcal{L}(W_\alpha, X_e)$.

Note that the α is the same for reference signal space and degree of polynomial stability.

When can we expect polynomial stability ?

(1) For finite-dimensional systems and infinite-dimensional exosystems.

(2) For wave equation with boundary control (Xu, Sallet (1996)).

(3) For heat equation equation with collocated control and measurement (Paunonen (2013)).

(4) Heat equation with non-collocated control and measurement and invertible D.

(5) Not for heat equation with non-collocated control and measurement, if $D = 0$.

Do we always need is full internal model ?

A partial solution is given by (Paunonen, Pohjolainen (2013)) when S is finite-dimensional $E = 0$,and the closed loop system exponentially stable.

Theorem

A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ solving the output regulation problem is robust with given perturbations (A, B, C, D) if and only if the equations

$$P(i\omega_k)K z_k = F_r \phi_k$$
$$(\mathcal{G}_1 - i\omega_k I) z_k = 0.$$

have a solution for z_k. If $P(i\omega_k)K = P(i\omega_k)K$ then one vector z_k suffices !

Example: A MIMO Wave Equation

Set-point regulation ($r(t) \equiv y_r \in \mathbb{C}^p, \beta > 0$) for

$$\frac{\partial^2 w}{\partial t^2}(z,t) - \beta \frac{\partial w}{\partial t}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) + Bu(t)$$

$$y(t) = Cw(\cdot,t).$$

Exosystem has $S = i\omega_0 I = 0$, and for perturbations in β we have $P(0) = P(0)$. Thus a single copy of the exosystem, instead p, is

sufficient for robustness. Only 1-dimensional servo-compensator (I-controller) is sufficient for small perturbations in β.

Structure of the Robust Controller

Theorem (Hämäläinen, Pohjolainen (2013))

Assume that $\sigma_p(S) \cap \sigma(A_e) = \emptyset$ *and the operators* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *satisfy the* $\mathcal{G}$*-conditions. The controller* $\mathcal{G}_1$ *can be represented as*

$$\mathcal{G}_1 = \begin{bmatrix} G_1 & 0 \\ R_1 & R_2 \end{bmatrix},$$

where $\sigma_p(S) \subset \sigma_p(G_1)$*, each eigenvalue* $\mathrm{i}\omega_k \in \sigma_p(G_1)$ *has multiplicity* p*.*

Structure of the Robust Controller

Let

$$Z_1 = \overline{\sum_{k=-\infty}^{\infty} \mathcal{N}(-\mathrm{i}\omega_k I - \mathcal{G}_1{}^*)}.$$

and select Z_2 so that $Z = Z_1 \dot{+} Z_2$ Then we have

$$\mathcal{G}_1 = \begin{bmatrix} G_1 & 0 \\ R_1 & R_2 \end{bmatrix}, \mathcal{G}_2 = \begin{bmatrix} G_2 \\ R_3 \end{bmatrix}, \quad K = [K_1 \;\; K_2].$$

(G_1, G_2) define a servo-compensator (Davison) on Z_1

$$\dot{z}_1 = G_1 z_1 + G_2 e,$$

with an internal model. The rest defines a stabilizing compensator (Davison) on Z_2

$$\dot{z}_2 = R_1 z_1 + R_2 z_2 + R_3 e$$
$$u = K_1 z_1 + K_2 z_2.$$

Structure of the Robust Controller

Altogether, the robust controller

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ R_1 & R_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} G_2 \\ R_3 \end{bmatrix} e.$$

$$u = K_1 z_1 + K_2 z_2.$$

consists of a servo-compensator (G_1, G_2), and a stabilizing compensator (R_1, R_2, R_3, K) to stabilize the combination of the system and servo-compensator.

Parametrization of all robustly regulating controllers in the time-domain (a'la Davison) !

$\mathbf{P}$-stability in the frequency domain

- Due to internal model principle $C(s)$ contains infinite number of poles on the imaginary axis.
- H^∞ stability out of reach if $P(i\omega_k) \to 0$ as $k \to \pm\infty$ (Hara et al. 1988, Ylinen et. al 2006).

Definition

$f(s)$ is $\mathbf{P}$-stable if

i. f is analytic in an open set containing $\overline{\mathbb{C}^+}$,
ii. f is bounded in $\{s \in \mathbb{C} : \mathrm{Re}(s) > \beta\}$ for all $\beta > 0$,
iii. $\exists M, \alpha > 0 : |f(i\omega)| \leq M(1 + |\omega|)^\alpha \; \forall \omega \in \mathbb{R}$.

$\mathbf{P}$-stability in the frequency domain

- $\mathbf{P}$-stability relaxes boundedness assumption on the imaginary axis, which makes stabilization possible.
  (Laakkonen, Pohjolainen 2015).
- Time domain interpretation (Paunonen, Laakkonen 2015):
  $y \in L^2$ for every $u \in W_\alpha$.

Frequency-Domain Problem Formulation

- Plants are matrices of fractions over the ring $\mathbf{P}$ of stable elements. A plant is stable if it is a matrix over $\mathbf{P}$.
- C stabilizes P if

$$\begin{bmatrix} (I+PC)^{-1} & -(I+PC)^{-1}P \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix}$$

  is stable.
- Reference signal: $r = \sum_{k\in\mathbb{Z}} \frac{f_k}{s-i\omega_k}\mathbf{a}_k$, where (f_k) is square summable and $\|\mathbf{a}_k\| = 1$.

Frequency-Domain Problem Formulation

Definition

Controller C solves the robust regulation problem if

i. C robustly stabilizes P,

ii. C is regulating for every plant P' it stabilizes, i.e., $(I + P'C)^{-1}r$ is stable for every reference signal y_r.

Frequency-Domain Internal Model Principle

Define

$$\theta(s) = \left(1 - \sum_{k\in\mathbb{Z}} \frac{f_k}{s - i\omega_k}\right)^{-1} \in \mathbf{P}.$$

Theorem (Laakkonen, Pohjolainen 2015)

A stabilizing controller with a right coprime factorization $(N(s), D(s))$ solves the robust regulation problem if and only if $\frac{1}{\theta(s)}D(s)$ is stable.

Conclusions

- A review of robust controllers for infinite dimensional systems with infinite-dimensional exosystems was given.
- Smoothness of the exosystem and polynomial stability were used to extend existing theory.
- Various conditions for the robustness of the controller were given in the time and frequency domain.
- Conditions for the existence of the dynamic steady state operator between proper spaces were given.
- Stabilization of the closed loop system was not discussed, although strong stability and polynomial stability have been obtained (under some assumptions).

Conclusions

- Robust controller was shown to consist of servo-compensator (internal model) and stabilizing compensator a'la Davison (Wonham).
- For structured perturbations partial internal model theory was initiated.
- Many results can be extended for exosystems with Jordan block structure (Paunonen, Pohjolainen (2010), (2012)).

Further work

- Proper stability concept with robustness.
- Robustness of strong/weak stability.
- Robustness of the solution of Sylvester equation and the other conditions.
- Finding a way to define order of controllers ("low order is better than high order").

- Thanks !
- home page: http://math.tut.fi/sysgroup/index.html