Null Controllability for One-dimensional Linear Degenerate Wave Equations

Hang Gao
(Joint with Muming Zhang)

School of Mathematics and Statistics
Northeast Normal University
hangg@nenu.edu.cn

July 2, 2015

Table of Contents

1.1 Problem

We discuss null controllability problem for the system governed by the degenerate wave equation:

$$
\begin{cases}
y_{tt} - (a(x)y_x)_x = 0, & x \in \Omega = (0, m),\ t \in (0, T), \\
y|_{\partial\Omega\times(0,T)} = \theta(t), & t \in (0, T), \\
y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & x \in \Omega,
\end{cases}
\tag{1}
$$

where $a(x) = x^{\alpha}$, $\alpha > 0$, $x \in \Omega$. $\theta(\cdot)$ is the control variable, $y(\cdot)$ is the state variable.

Introduction

(1) If for some constant $a_0 > 0$, $a(x) \geq a_0$, equation (1) is called a uniformly hyperbolic equation.

(2) If $a(x) \geq 0$, equation (1) is called a degenerate hyperbolic equation.

- Difference
  For degenerate wave equations, we can't obtain the estimate for y_x.

Classification:

Due to $a(x) = x^{\alpha}$, $\alpha > 0$, $x \in \Omega$, $x = 0$ is a degenerate point. If there exists $\delta > 0$ such that $\int_0^{\delta} \frac{1}{a(x)} dx < +\infty$, then $x = 0$ is the weakly degenerate point, otherwise, $x = 0$ is the strongly degenerate point.

Therefore,

- $\alpha \in (0, 1)$, (1) is a weakly degenerate system.
- $\alpha \geq 1$, (1) is a strongly degenerate system.

Goal:

We would like to find a function $\theta(\cdot)$ and $T^* > 0$, such that $y(x, T; \theta) = y_t(x, T; \theta) = 0$, where $(y_0, y_1) \in L^2(\Omega) \times H_a^*(\Omega)$, and $T > T^*$.

1.2 The relative works

When $a(x) = 1$, we have classical results.

- J.-L. Lions, SIAM Rev. 30 (1988), no. 1, 1-68: author introduced the Hilbert Uniqueness Method, and proved the boundary controllability for linear wave equation, when $T > T^* = 2diam\Omega$.
- E. Zuazua, Pitman Res. Notes Math. Ser., 220, Longman Sci. Tech., Harlow, 1991: author proved the semilinear wave equation is exactly controllable when the nonlinearity is globally Lipschitz.
- X. Zhang, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 456 (2000), no. 1997, 1101-1115: author proved the controllability of the semilinear wave equation by means of the global Carleman estimate.
- ...

When $a(x) \geq a_0 > 0$, we have the following results.

- P. Yao, SIAM J. Control Optim. 37 (1999), no. 5, 1568-1599: author proved the boundary controllability for the wave equation with variable coefficients by the Riemannian geometry method. And author gived a counterexample, if there is a closed geodesic that is contained in $\widehat{\Omega}(\subset \mathbb{R}^n)$, then the system has no exact controllability where the control is exerted on the whole boundary.
- L. Cui; X. Liu; H. Gao, J. Math. Anal. Appl. 402 (2013), no. 2, 612-625: the authors proved the controllability for the wave equation with constant coefficient in the non-cylindrical domain, this equation can be transformed into a wave equation with variable coefficients in a cylindrical domain.
- ...

When $a(x)=x^\alpha,\ \alpha>0,\ x\in(0,1)$, as far as we know, the only work we know is

- M. Gueye, SIAM J. Control Optim. 52 (2014), no. 4, 2037-2054.

The author consider the following system:

$$\begin{cases} y_{tt}-(x^\alpha y_x)_x=0,\ \alpha\in(0,1), & (x,t)\in(0,1)\times(0,T),\\ y(0,t)=\theta(t),\ y(1,t)=0, & t\in(0,T),\\ y(x,0)=y_0(x),\ y_t(x,0)=y_1(x), & x\in(0,1), \end{cases} \tag{2}$$

and proved that, for $(y_0,y_1)\in A\times B$ and $T>T^*=\frac{4}{2-\alpha}$, there exists a control $\theta\in L^2(0,T)$ such that the solution of (2) satisfies $y(x,T;\theta)=y_t(x,T;\theta)=0$, where $A\times B\subsetneqq L^2(0,1)\times H_a^*(0,1)$.

1.3 Our problem

We investigate null controllability of the strongly degenerate wave equation.

For strongly degenerate wave equation, remark that there is no trace for the solution on degenerate(left) boundary. Therefore, we can only let the control acts on nondegenerate(right) boundary.

Introduction

Let $Q = (0, m) \times (0, T)$, for $\alpha \in (0, 1)$, we consider the following system:

$$\begin{cases} y_{tt} - (a(x)y_x)_x = 0, & (x, t) \in Q, \\ y(0, t) = 0, \ \ y(m, t) = \theta(t), & t \in (0, T), \\ y(x, 0) = y_0(x), \ \ y_t(x, 0) = y_1(x), & x \in \Omega, \end{cases} \tag{3}$$

For $\alpha \geq 1$, we consider the following system:

$$\begin{cases} y_{tt} - (a(x)y_x)_x = 0, & (x, t) \in Q, \\ y(m, t) = \theta(t), & t \in (0, T), \\ y(x, 0) = y_0(x), \ \ y_t(x, 0) = y_1(x), & x \in \Omega, \end{cases} \tag{4}$$

2.1 Well-posedness of the systems (3) and (4)

At first, we give a set $H_a^1(\Omega)$, as the following:

$$\text{For } 0<\alpha<1,\ H_a^1(\Omega):=\big\{y\in L^2(\Omega)\big|y \text{ absolutely continuous in } [0,m], \sqrt{a}y_x\in L^2(\Omega) \text{ and } y(0)=y(m)=0\big\}.$$

$$\text{For } \alpha\geq 1,\ H_a^1(\Omega):=\big\{y\in L^2(\Omega)\big|y \text{ locally absolutely continuous in } (0,m], \sqrt{a}y_x\in L^2(\Omega) \text{ and } y(m)=0\big\}.$$

$H_a^1(\Omega)$ is a Hilbert space with the inner product

$$(y,v)_{H_a^1(\Omega)}=\int_\Omega yv+ay_xv_xdx,\ \forall\ y,v\in H_a^1(\Omega).$$

$H_a^*(\Omega)$ denotes the conjugate space of $H_a^1(\Omega)$.

Our main results

Further, in order to investigate the well-posedness of the systems (3) and (4), we first study the well-posedness for the following adjoint degenerate wave equation:

$$\begin{cases} v_{tt} - (a(x)v_x)_x = f, & (x,t) \in Q, \\ v(0,t) = 0, \quad \alpha \in (0,1) & t \in (0,T), \\ v(m,t) = 0, & t \in (0,T), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{5}$$

where $f \in L^1(0,T;L^2(\Omega)), (v_0, v_1) \in H^1_a(\Omega) \times L^2(\Omega)$. Notice that, on the degenerate boundary, only if $\alpha \in (0,1)$, we let $v(0,t) = 0$.

Definition 2.1 A function $v \in C([0,T]; H_a^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ is said to be a weak solution of the system (5), if for any function $\varphi \in C^\infty(\overline{Q})$, $\varphi(0,t) = \varphi(m,t) = 0$ and $\varphi(\cdot, T) = 0$ in Ω, it holds that

$$\int_Q (-v_t\varphi_t + av_x\varphi_x)dxdt - \int_\Omega v_1(x)\varphi(x,0)dx = \int_Q f\varphi dxdt,$$

$$v(x,0) = v_0(x).$$

- $v \in C([0,T]; H_a^1(\Omega))$ means that $v(0,t) = v(m,t) = 0(\alpha \in (0,1))$, and $v(m,t) = 0(\alpha \geq 1)$. However, actually, on degenerate boundary, the condition $(av_x)(0,t) = 0,\ \ t \in (0,T)$ is contained within the integral identify. Namely, by the definition of weak solution for the system (5), it shows that $(av_x)(0,t) = 0,\ \ t \in (0,T)$ naturally holds in the case $\alpha \geq 1$.

  So, hereon, we denote that $k_1 v(0,t) + k_2(av_x)(0,t) = 0, t \in (0,T)$, where $k_2 = 0$ when $\alpha \in (0,1)$ and $k_1 = 0$ when $\alpha \geq 1$.

Our main results

Theorem 2.1 For any $f \in L^1(0, T; L^2(\Omega))$ and $(v_0, v_1) \in H_a^1(\Omega) \times L^2(\Omega)$, the system (5) has a unique solution $v \in C([0, T]; H_a^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$. Moreover,

$$
\begin{aligned}
& |v|_{L^\infty(0,T;L^2(\Omega))} + |v_t|_{L^\infty(0,T;L^2(\Omega))} + \left|a|v_x|^2\right|_{L^\infty(0,T;L^1(\Omega))} \\
& \leq C\left(|f|_{L^1(0,T;L^2(\Omega))} + |v_0|_{H_a^1(\Omega)} + |v_1|_{L^2(\Omega)}\right),
\end{aligned}
$$

where $C = C(T, \Omega, |a|_{L^\infty(\Omega)})$.

Next, we give the definition of the solution of (3) and (4) in the sense of transposition.

Definition 2.1 A function $y \in C([0,T]; L^2(\Omega))$ is called the solution by transposition of (3) (or (4)), if for any $f \in L^1(0,T; L^2(\Omega))$, it holds that

$$\int_Q yfdxdt = \langle y_1, v(0)\rangle_{H_a^*(\Omega), H_a^1(\Omega)} - \int_\Omega y_0 v_t(0)dx - m^\alpha \int_0^T \theta(t) v_x(m,t)dt.$$

where v satisfies

$$\begin{cases} v_{tt} - (a(x)v_x)_x = f, & (x,t) \in Q, \\ k_1 v(0,t) + k_2(av_x)(0,t) = 0, & t \in (0,T), \\ v(m,t) = 0, & t \in (0,T), \\ v(x,T) = 0, \ \ v_t(x,T) = 0, & x \in \Omega, \end{cases}$$

Our main results

At last, we have the following well-posedness result for systems (3) and (4).

Theorem 2.2 For any $(y_0, y_1) \in L^2(\Omega) \times H_a^*(\Omega)$ and $\theta \in L^2(0, T)$, the system (3) (or (4)) admits a unique solution $y \in \mathscr{K} = C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H_a^*(\Omega))$ in the sense of transposition. Moreover,

$$|y|_{\mathscr{K}} \leq C\big(|y_0|_{L^2(\Omega)} + |y_1|_{H_a^*(\Omega)} + |\theta|_{L^2(0,T)}\big).$$

2.2 Controllability of the systems (3) and (4)

Theorem 2.3 Let $\alpha \in (0,2)$ and $T^* := \frac{4}{2-\alpha} m^{\frac{2-\alpha}{2}}$.

(1) If $T > T^*$, then, system (3)(or (4)) is null controllable, i.e. for any $(y_0, y_1) \in L^2(\Omega) \times H_a^*(\Omega)$, one can find a control $\theta \in L^2(0, T)$ such that the solution y of (3)(or (4)) satisfies $y(x, T; \theta) = y_t(x, T; \theta) = 0$.

(2) If $T < T^*$, then, system (3)(or (4)) is not null controllable.

Theorem 2.4 Let $\alpha \geq 2$, then, system (4) is not null controllable.

2.3 Remarks

- Theorem 2.3 shows that, if $\alpha \in (0, 2)$, all initial datas in $L^2 \times H_a^*$ can be driven to zero, when control acts on nondegenerate boundary.
- The reason for the different results:
  Degenerate wave equation can be viewed as describing the vibration of the elastic string. In the degenerate point, the propagation speed of waves is zero. When control acts on this point, it has little effect on the whole waveform. Hence, in this case, only the part of initial datas in general space $L^2 \times H_a^*$ can be driven to zero.

  On the contrary, when control acts on nondegenerate boundary, it has a great influence on the whole waveform, thus, all initial datas in $L^2 \times H_a^*$ can be driven to zero.

- Clearly, $T^* \to 2m$ as $\alpha \to 0$, It is well known that the controllability time $T = 2m$ is sharp for the classical wave equation with constant coefficients. But we do not know whether the time $T^* = \frac{4}{2-\alpha} m^{\frac{2-\alpha}{2}}$ is sharp for the degenerate wave equation.
- When $T = T^*$, whether the system (3)(or (4)) is null controllable is a open question, it remains to be done.

we consider the following adjoint degenerate wave equation:

$$\begin{cases} v_{tt}-(a(x)v_x)_x=0, & (x,t)\in Q, \\ k_1 v(0,t)+k_2(av_x)(0,t)=0, & t\in(0,T), \\ v(m,t)=0, & t\in(0,T), \\ v(x,0)=v_0(x),\ v_t(x,0)=v_1(x), & x\in\Omega, \end{cases} \tag{6}$$

where $(v_0,v_1)\in H_a^1(\Omega)\times L^2(\Omega)$.

By HUM, it is easy to see that, the system (3)(or (4)) is null controllable in time T with a control θ, if and only if there exists a constant $C > 0$, such that any solution v of (6) satisfies

$$|v_0|^2_{H^1_a(\Omega)} + |v_1|^2_{L^2(\Omega)} \leq C \int_0^T v_x^2(m,t)dt, \tag{7}$$

$\forall\ (v_0, v_1) \in H^1_a(\Omega) \times L^2(\Omega)$.

Lemma 3.1 Let $(\omega_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers that satisfying the gap condition

$$\omega_{k+1}-\omega_k \geq \delta, \quad \forall\ n\in\mathbb{Z}.$$

Then, for any $T>\frac{2\pi}{\delta}$, there exists a positive constant $C(T,\delta)$ such that

$$\int_0^T \Big|\sum_{k\in\mathbb{Z}} a_k e^{i\omega_k t}\Big|^2 dt \asymp \sum_{k\in\mathbb{Z}} \big|a_k\big|^2, \tag{8}$$

for all sequence of complex numbers $(a_k)_{k\in\mathbb{Z}}\in l^2(\mathbb{Z})$, where $M\asymp N \Leftrightarrow \exists C_1,\ C_2>0$ s.t. $C_1M\leq N\leq C_2M$.

- In fact, (8) holds if $T > 2\pi D^+$ and it is false if $T < 2\pi D^+$, where D^+ is the upper density of the sequence $(\omega_k)_{k\in\mathbb{Z}}$ defined by $D^+ = \lim\limits_{r\to\infty} \frac{n^+(r)}{r}$, with $n^+(r)$ denoting the largest number of terms of the sequence $(\omega_k)_{k\in\mathbb{Z}}$ contained in an interval of length r. (see V. Komornik, P. Loreti, Fourier Series in Control Theory, Springer Monographs in Mathematics. Springer-Verlag, New York, 2005.)

Proof of Theorem 2.3: Method: Fourier expansion.

If $\alpha \in (0,1)$, we consider the boundary value problem

$$\begin{cases} -(x^{\alpha}y'(x))' = \lambda y(x), \quad x \in (0,m), \quad \lambda \in \mathbb{R}, \\ y(0) = y(m) = 0. \end{cases} \tag{9}$$

Then, problem (9) has a solution

$$y(x,\lambda_n) = x^{\frac{1}{2}(1-\alpha)} J_{\nu}(j_{\nu,n}(\frac{x}{m})^{\kappa}), \quad x \in (0,m), \quad n \geq 1. \tag{10}$$

$$\lambda_n = (\frac{\kappa j_{\nu,n}}{m^{\kappa}})^2, \ n \geq 1.$$

The key steps of proof

Where

$$J_\nu(x) = \sum_{k\geq 0} \frac{(-1)^k}{k! \cdot \Gamma(k+\nu+1)} (\frac{x}{2})^{2k+\nu}, \; x \geq 0,$$

are the first type of Bessel functions, $\Gamma(\cdot)$ is the Gamma function, $j_{\nu,n}$ $(n \geq 1)$ are the positive zeros of the Bessel function J_ν, $\nu := \frac{1-\alpha}{2-\alpha}$, $\kappa := \frac{2-\alpha}{2}$.

The eigenfunctions are normalized into

$$\Phi_n(x) = \frac{(2\kappa)^{\frac{1}{2}}}{m^\kappa |J'_\nu(j_{\nu,n})|} x^{\frac{1-\alpha}{2}} J_\nu(j_{\nu,n}(\frac{x}{m})^\kappa), \; x \in (0,m), \; n \geq 1.$$

Thus, for any $(v_0, v_1) \in H_a^1(\Omega) \times L^2(\Omega)$, set

$$v_0(x) = \sum_{n \in \mathbb{N}^*} v_0^n \Phi_n(x), \quad v_1(x) = \sum_{n \in \mathbb{N}^*} v_1^n \Phi_n(x).$$

Then the corresponding solution v of (6) is as follows:

$$v(x,t) = \sum_{n \in \mathbb{N}^*} v_n(t) \Phi_n(x), \quad v_n(t) = b_n e^{i\omega_n t} + b_{-n} e^{-i\omega_n t},$$

where $\omega_n = \frac{\kappa j_{\nu,n}}{m^\kappa} = \sqrt{\lambda_n}$ and

$$b_n = \frac{1}{2}(v_0^n - i\frac{v_1^n}{\omega_n}), \quad b_{-n} = \frac{1}{2}(v_0^n + i\frac{v_1^n}{\omega_n}).$$

The key steps of proof

By computation,

$$v_x(m,t) = m^{-\frac{3}{2}}\kappa\sqrt{2\kappa}\sum_{n\in\mathbb{N}^*} j_{\nu,n}\frac{J_\nu'(j_{\nu,n})}{|J_\nu'(j_{\nu,n})|}(b_n e^{i\omega_n t} + b_{-n}e^{-i\omega_n t}).$$

By **Lemma 3.1**, we have

$$\int_0^T v_x^2(m,t)dt \asymp \sum_{n\in\mathbb{N}^*}(j_{\nu,n})^2(|v_0^n|^2 + \frac{|v_1^n|^2}{(\kappa j_{\nu,n})^2})$$

$$\asymp \sum_{n\in\mathbb{N}^*}(\lambda_n|v_0^n|^2 + |v_1^n|^2) \asymp (|v_0|^2_{H_a^1(\Omega)} + |v_1|^2_{L^2(\Omega)}). \tag{11}$$

The key steps of proof

In fact, (11) holds if $T > 2\pi D^+$ and it is false if $T < 2\pi D^+$. Moreover, by the definition of $n^+(r)$, we have

$$\frac{r}{n^+(r)} \approx \frac{1}{n^+(r)} \sum_{n=1}^{n^+(r)-1} \frac{\kappa}{m^\kappa} \int_{j_{\nu,n}}^{j_{\nu,n+1}} dx, \text{ as } r \to \infty.$$

By $(j_{\nu,n+1} - j_{\nu,n})_n$ converges to π as $n \to +\infty$, and the definition of D^+, we have

$$D^+ = \frac{m^\kappa}{\kappa\pi}.$$

Thus, $2\pi D^+ = \frac{2}{\kappa} m^\kappa = \frac{4}{2-\alpha} m^{\frac{2-\alpha}{2}}$.

The key steps of proof

If $\alpha \in [1,2)$, we consider the boundary value problem

$$\begin{cases} -(x^{\alpha}y'(x))' = \lambda y(x), \quad x \in (0,m), \quad \lambda \in \mathbb{R}, \\ (x^{\alpha}y')(0) = y(m) = 0. \end{cases} \tag{12}$$

Then, the corresponding solution y of (12) is

$$y(x,\lambda) = x^{\frac{1}{2}(1-\alpha)} J_{\widehat{\nu}}(\kappa^{-1}\sqrt{\lambda}x^{\kappa}), \ x \in (0,m), \ \lambda \in \mathbb{R}, \tag{13}$$

where

$$\widehat{\nu} := \frac{\alpha-1}{2-\alpha}, \ \kappa := \frac{2-\alpha}{2}.$$

The rest of proof runs as before.

The key steps of proof

Proof Theorem 2.4: If $\alpha > 2$, by means of the following classical change of variables (see R. Courant, D. Hilbert, Methods of mathematical physics):

$$X := \int_x^m \frac{1}{\sqrt{a}} \quad \text{and} \quad U(X,t) := a(x)^{\frac{1}{4}} y(x,t),$$

equation (4) is transformed into the nondegenerate wave equation on the half-line:

$$\begin{cases} U_{tt}(X,t) - U_{XX}(X,t) + M(X)U(X,t) = 0, & (X,t) \in \mathbb{R}^+ \times (0,T), \\ U(0,t) = \theta(t), & t \in (0,T), \end{cases} \tag{14}$$

where $M(X) = \frac{\alpha(3\alpha-4)}{(4m^{\frac{2-\alpha}{2}} - 2(2-\alpha)X)^2}$.

The key steps of proof

By the duality, the null controllability of (14) is equivalent to the existence of a constant $C > 0$ such that

$$|\varphi(x,0)|^2_{H^1_0(\mathbb{R}^+)} + |\varphi_t(x,0)|^2_{L^2(\mathbb{R}^+)} \le C \int_0^T \varphi_x(0,t)^2 dt \tag{15}$$

holds for any solution of the following adjoint system:

$$\begin{cases} \varphi_{tt} - \varphi_{xx} + M(x)\varphi = 0, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ \varphi(0,t) = 0, & t \in (0,T), \\ \varphi(x,T) = \varphi_0(x),\ \varphi_t(x,T) = \varphi_1(x), & x \in \mathbb{R}^+. \end{cases} \tag{16}$$

The key steps of proof

Take $\widehat{\varphi}_0, \widehat{\varphi}_1 \in \mathcal{D}(\mathbb{R}^+)$ and $\varphi_0^k(x) = \widehat{\varphi}_0(x-k)$, $\varphi_1^k(x) = \widehat{\varphi}_1(x-k)$ with $k > 0$ large enough. Let φ^k be the solution of (16) with initial data $(\varphi_0^k, \varphi_1^k)$. It is easy to check that

$$\frac{|\varphi^k(x,0)|^2_{H_0^1(\mathbb{R}^+)} + |\varphi_t^k(x,0)|^2_{L^2(\mathbb{R}^+)}}{\int_0^T \varphi_x^k(0,t)^2 dt} \to \infty, \quad \text{as } k \to \infty.$$

Thus, system (14) is not null controllable. i.e. (4) is not null controllable.

- $\alpha = 2$, we can prove the same result in the similar way.

Further works

- Internal controllability of the 1-D degenerate wave equations.
- Global Carleman estimate for the 1-D degenerate wave equations.
- Controllability of coupled 1-D degenerate wave equations.
- Controllability of the high dimensional degenerate wave equations.

Thank you!