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# Exact controllability for a coupled system of wave equations with Neumann / Robin boundary controls

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§**1. Introduction.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ satisfying the usual geometric control condition. Consider the following coupled system of wave equations with Neumann boundary controls :

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in} \quad \Omega, \\ U = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu U = DV & \text{on} \quad \Gamma_1, \\ t = 0 : \qquad U = U_0, \quad U' = U_1, & \end{cases}$$

where

$$U = \big(u_1, \cdots, u_N\big)^T, \qquad V = \big(v_1, \cdots, v_N\big)^T$$

and the coupling matrix $A, D \in \mathbb{M}^{N\times N}(\mathbb{R})$ are both with constant elements.

**The exact controllability as the rank of control matrix $D = N$ ;**

**The non-exact controllability as rank $D < N$, in the case of lack of controls.**

Let us denote

$$\mathcal{H}_0 = (L^2(\Omega))^N, \qquad \mathcal{H}_1 = (H^1_{\Gamma_0}(\Omega))^N$$

**Theorem 1.1.** There exist positive constants $T > 0$ and $C > 0$ independent of initial data, such that the following observability inequality

$$\|(\Phi_0, \Phi_1)\|^2_{\mathcal{H}_0 \times \mathcal{H}_{-1}} \leq C \int_0^T \int_{\Gamma_1} |\Phi|^2 d\Gamma dt$$

holds for all solutions $\Phi$ to the corresponding adjoint problem :

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in} \quad \Omega, \\ \Phi = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu \Phi = 0 & \text{on} \quad \Gamma_1, \\ t = 0 : \quad \Phi = \Phi_0, \quad \Phi' = \Phi_1, \end{cases}$$

where

$$\Phi = (\phi_1, \cdots, \phi_N)^T.$$

In the case of lack of boundary controls, we have the non exact controllability :

**Theorem 1.2.** Assume that $D$ is not invertible. Then problem

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in} \quad \Omega, \\ U = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu U = DV & \text{on} \quad \Gamma_1, \\ t = 0 : \qquad U = U_0, \quad U' = U_1, \end{cases}$$

is not exactly controllable for all initial data $(U_1, U_0) \in \mathcal{H}_0 \times \mathcal{H}_1$.

Direct methods such as multiplier methods or Carlman's estimates :

Th. Duyckaerts, X. Zhang, E. Zuazua, *On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials*, Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008), 1-41.

X. Zhang, E. Zuazua, *A sharp observability inequality for Kirchhoff plate systems with potentials*, Comput. Appl. Math. 25 (2006), 353-372.

I. Lasiecka, R. Triggiani, X. Zhang, *Nonconservative wave equations with unobserved Neumann B.C. : global uniqueness and observability in one shot*, Differential geometric methods in the control of partial differential equations, Contemp. Math., 268, Amer. Math. Soc., Providence, RI, 2000, 227-324.
*One only wave equation with Neumann boundary the whole condition on* $\Gamma_1$ *and* $\Gamma_0$.

L. Miller, *Controllability cost of conservative systems : resolvent condition and transmutation,* J. Func. Anal. 218 (2005), 425-444

It turns out that the methods of compact perturbation are particularly simple and efficient for dealing with some systems with lower order terms.

N. Cîndea, M.Tucsnak, *Internal exact observability of a perturbed Euler-Bernoulli equation*, Ann. Acad. Rom. Sci. Ser. Math. Appl. 2 (2010), 205-221.

V. Komornik, P. Loreti, *Observability of compactly perturbed systems*, J. Math. Anal. Appl., 243 (2000), 409-428.

M. Mehrenberger, *Observability of coupled systems*, Acta Math. Hungar., 103 (2004), 321-348.

As an application, in Li Tatsien-B.R. [2013], we have considered :

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in} \quad \Omega, \\ \Phi = 0 & \text{on} \quad \Gamma, \\ t = 0 : \quad \Phi = \Phi_0, \quad \Phi' = \Phi_1. \end{cases}$$

Following Mehrenberger's result, in order to establish observability inequalities (direct and inverse),

$$c\int_0^T \int_{\Gamma_1} |\partial_\nu \Phi|^2 d\Gamma dt \leq \|(\Phi_0, \Phi_1)\|^2_{\mathcal{H}_1\times\mathcal{H}_0} \leq C\int_0^T \int_{\Gamma_1} |\partial_\nu \Phi|^2 d\Gamma dt$$

it is sufficient to check :

(i) The direct and inverse inequalities hold for decoupled problem with $A = 0$.

(ii) The system of eigenvectors forms a Riesz basis in the usual energy space ($A$ is constant matrix).

The results remains true for matrix $A$ with variable elements by Carlman's estimate in Duyckaerts- Zhang-Zuazua (2008). But the compact perturbation method is really simple.

**Basic Lemma.** Let $\mathcal{F}$ be a Hilbert space endowed with the $p$-norm. Assume that

$$\mathcal{F} = \mathcal{N} \bigoplus \mathcal{L},$$

where $\oplus$ denotes the direct sum and $\mathcal{L}$ is a finite co-dimensional closed subspace in $\mathcal{F}$. Assume that $q$ is a weaker norm in $\mathcal{F}$ such that the projection from $\mathcal{F}$ into $\mathcal{N}$ is continuous with respect to the $q$-norm.
Assume furthermore that

$$(i) \qquad q(y) \leq p(y), \qquad \forall y \in \mathcal{L}.$$

Then there exists a positive constant $C > 0$ such that

$$(ii) \qquad q(z) \leq Cp(z), \qquad \forall z \in \mathcal{F}.$$

**Remark.** Since $\mathcal{L}$ is not necessarily closed for the weaker $q$-norm, so the continuity of the projection from $\mathcal{F}$ onto $\mathcal{N}$ with respect to the $q$-norm is not a trivial question. But it often occurs that the subspaces $\mathcal{N}$ and $\mathcal{L}$ are mutually orthogonal with respet to the $q$-inner product, in these situations, the continuity of the projection from $\mathcal{F}$ into $\mathcal{N}$ is evident, much easier than Riesz basis property!

We next consider the controllability with Robin boundary controls :

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in} \quad \Omega, \\ U = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu U + BU = DV & \text{on} \quad \Gamma_1, \\ t = 0 : \qquad U = U_0, \quad U' = U_1, \end{cases}$$

where $B$ is a symmetric matrix.

At first glance, Robin problem is more complicated because of the second coupling term $BU$ on the boundary $\Gamma_1$.
The new observation consists in transforming the Neumann boundary condition into

$$\partial_\nu U = DV \Longleftrightarrow \partial_\nu U + BU = D(V + D^{-1}BU) = D\widetilde{V}.$$

So, Neumann problem would yield a solution to Robin problem, provided that

$$U|_{\Gamma_1} \in L^2(0, T; L^2(\Gamma_1))^N.$$

This is the strategy that we will follow.

**Remark.** Even for one single equation ($N = 1$), in order to get the corresponding observability, the positivity on the scalar coefficient $B$ is technically necessary (Lions [1988], Komornik [1994]).

**§2. Proof of observability.** Let

$$\Phi = (\phi_1, \cdots, \phi_N)^T.$$

We consider the following homogenous adjoint problem

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in} \quad \Omega, \\ \Phi = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu \Phi = 0 & \text{on} \quad \Gamma_1, \\ t = 0: \quad \Phi = \Phi_0, \quad \Phi' = \Phi_1. \end{cases} \tag{2.1}$$

Let

$$\mathcal{H}_0 = (L^2(\Omega))^N.$$

We define the linear unbounded operator $-\Delta$ in $\mathcal{H}_0$ by

$$D(-\Delta) = \{\Phi \in H^2(\Omega)^N : \quad \Phi|_{\Gamma_0} = 0, \quad \partial\Phi_\nu|_{\Gamma_1} = 0\}.$$

Then define the power operator $(-\Delta)^{s/2}$ for any given $s \in \mathbb{R}$. In particular, we have

$$\mathcal{H}_1 = D(\sqrt{-\Delta}) = \{\Phi \in H^1(\Omega)^N : \quad \Phi = 0 \quad \text{on} \quad \Gamma_0\}.$$

Now let $e_m$ be the normalized eigenfunction defined by

$$\begin{cases} -\Delta e_m = \mu_m^2 e_m & \text{in} \quad \Omega, \\ e_m = 0 & \text{on} \quad \Gamma_0. \\ \partial_\nu e_m = 0 & \text{on} \quad \Gamma_1, \end{cases}$$

where the sequence of positive terms $\{\mu_m\}_{m\geq 1}$ is increasing such that $\mu_m \to +\infty$. For each $m \geq 1$, we define the subspace $Z_m$ by

$$Z_m = \{\alpha e_m : \quad \alpha \in \mathbb{R}^N\}.$$

For any integers $m \neq n$ and any vectors $\alpha, \beta \in \mathbb{R}^N$, we have

$$(\alpha e_m, \beta e_n)_{\mathcal{H}_s} = (\alpha, \beta)\big((-\Delta)^{s/2} e_m, (-\Delta)^{s/2} e_n\big)_{L^2(\Omega)} = (\alpha, \beta)\mu_m^s \mu_n^s \delta_{mn}.$$

Then the subspaces $Z_m (m \geq 1)$ are mutually orthogonal in the Hilbert space $\mathcal{H}_s$. In particular, we have

$$\|\Phi\|_{\mathcal{H}_s} = \frac{1}{\mu_m}\|\Phi\|_{\mathcal{H}_{s+1}}, \quad \forall \Phi \in Z_m. \tag{2.2}$$

Since $A$ is constant matrix, the subspaces $Z_m (m \geq 1)$ are invariant for $A^T$. Then for the initial data $(\Phi_0, \Phi_1) \in \bigoplus_{m \geq m_0} (Z_m \times Z_m)$, the corresponding solution $\Phi$ belongs to $\bigoplus_{m \geq m_0} Z_m$.

Moreover, setting the energy as

$$E(t) = \frac{1}{2} \int_{\Omega} (|\Phi'|^2 + |\nabla \Phi|^2) dx,$$

we have the following energy estimates :

$$e^{\frac{-\sigma t}{\mu m_0}} E(0) \leq E(t) \leq e^{\frac{\sigma t}{\mu m_0}} E(0), \quad t \geq 0$$

for all $(\Phi_0, \Phi_1) \in \bigoplus_{m \geq m_0} (Z_m \times Z_m)$.

This is almost the conservation of energy for $m_0$ large enough. Noting that $A$ is not symmetric positively definite, the adjoint system is not conservative.

**Proposition 2.1.** There exist an integer $m_0 \geq 1$ and positive constants $T > 0$ and $C > 0$ independent of initial data, such that the following observability inequality

$$\|(\Phi_0, \Phi_1)\|^2_{\mathcal{H}_1 \times \mathcal{H}_0} \leq C \int_0^T \int_{\Gamma_1} |\Phi'|^2 d\Gamma dt$$

holds for all solution $\Phi$ of adjoint problem (2.1) with $(\Phi_0, \Phi_1) \in \bigoplus_{m \geq m_0} (Z_m \times Z_m)$.

*Proof.* Multiplying the $k$-th equation by

$$\phi_k'' - \Delta\phi_k + \sum_{p=1}^{N} a_{pk}\phi_p = 0 \qquad \text{in} \quad \Omega$$

by the classic multiplier :

$$2m \cdot \nabla\phi_k + (N-1)\phi_k \quad \text{with} \quad m = x - x_0$$

and using Cauchy-Schwartz's inequality :

$$\Big|\int_0^T \int_\Omega \phi_p m \cdot \nabla\phi_k dxdt\Big| \leq c\int_0^T \|\Phi\|_{\mathcal{H}_0}\|\Phi\|_{\mathcal{H}_1} \leq \frac{c}{\mu_{m_0}}\int_0^T \|\Phi\|_{\mathcal{H}_1}^2 dt.$$

we get easily the following estimate :

$$\int_0^T E(t)dt \leq c\int_0^T \int_{\Gamma_1} (m,\nu)|\Phi'|^2 d\Gamma dt + \frac{c}{\mu_{m_0}}\int_0^T E(t)dt + cE(0).$$

**Proposition 2.2.** There exist an integer $m_0 \geq 1$ and positive constants $T > 0$ and $C > 0$ independent of initial data, such that the following observability inequality :

$$\|(\Phi_0, \Phi_1)\|^2_{\mathcal{H}_0 \times \mathcal{H}_{-1}} \leq C \int_0^T \int_{\Gamma_1} |\Phi|^2 d\Gamma dt$$

holds for all solutions $\Phi$ of adjoint problem (2.1) with $(\Phi_0, \Phi_1) \in \bigoplus_{m \geq m_0} (Z_m \times Z_m)$.

Now we give the proof of Theorem 1.1.

First for any $(\Phi_0, \Phi_1) \in \bigoplus_{m\geq 1}(Z_m \times Z_m)$, define

$$p(\Phi_0, \Phi_1) = \sqrt{\int_0^T \int_{\Gamma_1} |\Phi|^2 d\Gamma dt}$$

where $\Phi$ is the solution to the adjoint problem (2.1).

By Holmgren's uniqueness theorem, $p(\cdot)$ defines a norm in $\bigoplus_{m\geq 1}(Z_m \times Z_m)$ for $T > 0$ is large enough.
Then, we denote by $\mathcal{F}$ the completion of $\bigoplus_{m\geq 1}(Z_m \times Z_m)$ with respect to the $p$-norm.
We write

$$\mathcal{F} = \mathcal{N} \bigoplus \mathcal{L}$$

with

$$\mathcal{N} = \bigoplus_{1\leq m<m_0} (Z_m \times Z_m), \qquad \mathcal{L} = \overline{\Big\{ \bigoplus_{m\geq m_0} (Z_m \times Z_m)\Big\}}^{p}.$$

Clearly, $\mathcal{N}$ is a finite-dimensional subspace and $\mathcal{L}$ is a closed subspace in $\mathcal{F}$.

We next introduce a weaker $q$-norm in $\mathcal{F}$ :

$$q(\Phi_0, \Phi_1) = \|(\Phi_0, \Phi_1)\|_{\mathcal{H}_0 \times \mathcal{H}_{-1}}.$$

Noting that the subspaces $(Z_m \times Z_m)$ are mutually orthogonal in $\mathcal{H}_0 \times \mathcal{H}_{-1}$ for all $m \geq 1$, so we have the orthogonality :

$$\mathcal{N} \perp \mathcal{L}$$

with respect to the $q$-inner product. In particular, the othogonal projection from $\mathcal{F}$ into $\mathcal{N}$ is continuous with respect to the $q$-norm.

On the other hand, Proposition 2.2 means that

$$q(\Phi_0, \Phi_1) \leq p(\Phi_0, \Phi_1)$$

for all $(\Phi_0, \Phi_1) \in \bigoplus_{m \geq m_0} (Z_m \times Z_m)$, therefore in $\mathcal{L}$.

Then, applying Lemma, we get

$$q(\Phi_0, \Phi_1) \leq cp(\Phi_0, \Phi_1)$$

for all $(\Phi_0, \Phi_1) \in \mathcal{F}$.

§**3. Non exact boundary controllability.** Let $D$ be a boundary control matrix of order $N$ and denote

$$U = (u_1, \cdots, u_N)^T, \qquad V = (v_1, \cdots, v_N)^T.$$

We consider the following inhomogeneous problem :

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in} \quad \Omega, \\ U = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu U = DV & \text{on} \quad \Gamma_1, \\ t = 0: \quad U = U_0, \quad U' = U_1. \end{cases} \tag{3.1}$$

**Theorem 3.1.** Assume that $D$ is invertible. Then there exists a positive constant $T > 0$ such that Neumann problem (3.1) is exactly controllable at the moment $T$ for any given initial data $(U_1, U_0) \in \mathcal{H}_0 \times \mathcal{H}_1$. That precisely means that there exist a control $V$

$$\|V\|_{L^2(0,T;L^2(\Gamma_1)^N} \leq C\|(U_1, U_0)\|_{\mathcal{H}_0 \times \mathcal{H}_1}$$

such that the correspondind solution $U$ satidfies the final null conditions :

$$t = T: \qquad U = U' = 0.$$

**Proposition 3.1.** For any given $V \in L^2(0,T;L^2(\Gamma_1))^N$ and any given $(U_1, U_0) \in \mathcal{H}_0 \times \mathcal{H}_1$, the Neumann problem (3.1) admits a unique weak solution $U$. Moreover, the linear map

$$(U_1, U_0, V) \to (U', U) \in C^0([0,T]; \mathcal{H}_{-s} \times \mathcal{H}_{1-s}), \qquad s > 1/2$$

is continuous with respect to the corresponding topologies.

**Remark** A weaker regularity (classic one) such as

$$(U', U) \in C^0([0,T]; \mathcal{H}_{-1} \times \mathcal{H}_0)$$

is sufficient to make sense to the value $(U'(0), U(0))$, therefore, sufficient for proving the exact boundary controllability. At this stage, it is not necessary to pay much attention to the regularity of the weak solution with respect to the space variable. However, in order to establish the non exact boundary controllability, we need more strong regularity for the argument of compact perturbation. For example for $s = 3/4$ :

$$(U', U) \in C^0([0,T]; \mathcal{H}_{-3/4} \times \mathcal{H}_{1/4}).$$

**Theorem 3.2.** Assume that $D$ is not invertible. Then problem (3.1) is not exactly controllable for all initial data $(U_1, U_0) \in \mathcal{H}_0 \times \mathcal{H}_1$.

*Proof.* We choose a special initial data as

$$\theta \in \mathcal{D}(\Omega), \qquad U_0 = \theta e, \qquad U_1 = 0.$$

If problem (3.1) is exactly controllable, we can find a control $V$ such that

$$\|V\|_{L^2(]0,T[;L^2(\Gamma_1))^N} \leq C\|\theta\|_{H^1(\Omega)}.$$

By the improved smoothness in Proposition 3.1 we have

$$\|U\|_{L^2(0,T;\mathcal{H}_{1-s}(\Omega))} \leq C\|\theta\|_{H^1(\Omega)}, \quad \forall s > 1/2.$$

Since $D$ is not invertible, there exists a vector $e \in \mathbb{R}^N$ such that $D^T e = 0$. Taking the inner product of $e$ with (3.1) and noting $\phi = (e, U)$, we get

$$
(3.2) \qquad \begin{cases} \phi'' - \Delta\phi = -(e, AU) & \text{in} \quad \Omega, \\ \phi = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu \phi = 0 & \text{on} \quad \Gamma_1, \\ t = 0: \quad \phi = \theta, \quad \phi' = 0, \\ t = T: \quad \phi = 0, \quad \phi' = 0. \end{cases}
$$

By the well-posedness of the backward problem (3.2), we get

$$
\|\theta\|_{H^{2-s}(\Omega)} \le C\|U\|_{L^2(0,T;\mathcal{H}_{1-s}(\Omega))} \le C'\|\theta\|_{H^1(\Omega)}
$$

Taking $s = 3/4, \quad 2 - s = 5/4$, we get a contracdiction :

$$
\|\theta\|_{H^{5/4}(\Omega)} \le C'\|\theta\|_{H^1(\Omega)}, \qquad \theta \in \mathcal{D}(\Omega).
$$

**Remark.** In the case of Dirichlet boundary condition, thanks to the direct inequality, the weak solution has the same smoothness as the initial data :

$$(U', U) \in C^0([0, T]; \mathcal{H}_{-1} \times \mathcal{H}_0), \qquad (U_1, U_0) \in \mathcal{H}_{-1} \times \mathcal{H}_0.$$

This regularity yields the non exact boundary controllability in the case of less boundary controls (Li Tatsien-B.R. [2014]).

But for Neumann boundary conditions, the direct inequality is much weaker than the inverse inequality (in fact the trace embedding), we can only get

$$(U', U) \in C^0([0, T]; \mathcal{H}_{-s} \times \mathcal{H}_{1-s}), \qquad (U_1, U_0) \in \mathcal{H}_0 \times \mathcal{H}_1.$$

Even though this regularity is not sharp, it is already sufficient for the proof of the non exact controllability of Neumann problem.

Of course, the non exact boundary controllability is valid only in the framework that all the components of the initial data have the same regularity. In fact, if the components of the initial data are allowed to have different levels of finite energy, then we can realize the exact boundary controllability by means of only one boundary control for a system of two wave equations (Alabau [2003], Liu Zh.-B.R. [2009]), or more generally, for a cascade system of $N$ wave equations (Alabau [2013]) and Rosier for a cascade system of conservative equations (2011).

§**4. Exact boundary controllability for Robin problem.** Let $B$ be a symmetric matrix of order $N$ with constant elements. We will establish the exact boundary controllability for the following Robin boundary problem :

$$
\begin{cases} U'' - \Delta U + AU = 0 & \text{in} \quad \Omega, \\ U = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu U + BU = DV & \text{on} \quad \Gamma_1, \\ t = 0 : \quad U = U_0, \quad U' = U_1. \end{cases} \tag{4.1}
$$

**Theorem 4.1.** Assume that the control matrix $D$ is invertible. Then there exists $T > 0$ such that Robin boundary problem (4.1) is exactly controllable at the moment $T$ for any initial data $(U_1, U_0) \in \mathcal{H}_0 \times \mathcal{H}_1$.

*Proof.* By Theorem 3.1, Neumann problem is exactly controllable. Let $V \in L^2(0,T;L^2(\Gamma_1))^N$ be a boundary control which realizes the exact controllability of Neumann problme (3.1). We write

$$\partial_\nu U + BU = DV + BU = D(V + D^{-1}BU).$$

On the other hand, we rewrite problem (3.1) as

$$\begin{cases} u_k'' - \Delta u_k = f_k & \text{in} \quad \Omega, \\ u_k = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu u_k = g_k & \text{on} \quad \Gamma_1, \end{cases}$$

where

$$f_k = -\sum_{l=1}^N a_{kl} u_l \in L^2(0,T;L^2(\Omega)), \qquad g_k = -\sum_{l=1}^N d_{kl} v_l \in L^2(0,T;L^2(\Gamma_1)).$$

Then following the regularity of Neumann problem in Lasiecka-Triggiani [1989, 1990], we have

$$\|u_k\|_{H^{1/5-\epsilon}(]0,T[\times\Gamma_1)} \leq C\|(U_1, U_0)\|_{\mathcal{H}_0\times\mathcal{H}_1}. \tag{4.2}$$

This implies that the new control $V + D^{-1}BU \in L^2(0,T;L^2(\Gamma_1))^N$ realizes well the exact controllability of the Robin problem (4.1).

As a consequence of the exact controllability, we have

**Corollary 4.1.** There exist positive constants $T > 0$ and $C > 0$ independent of initial data, such that the following observability inequality :

$$\|(\Phi_0, \Phi_1)\|^2_{\mathcal{H}_0 \times \mathcal{H}_{-1}} \leq C \int_0^T \int_{\Gamma_1} |\Phi|^2 d\Gamma dt \tag{4.3}$$

holds for the solution $\Phi$ of the adjoint problem :

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in} \quad \Omega, \\ \Phi = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu \Phi + B\Phi = 0 & \text{on} \quad \Gamma_1, \\ t = 0 : \quad \Phi = \Phi_0, \quad \Phi' = \Phi_1 \end{cases} \tag{4.4}$$

**Remark** Theorem 4.1 and Corollary 4.1 are valid without any assumption on the positivity of the matrix $B$, which was technically necessary for the usual multiplier method even in the case of a single wave equation. On the other hand, the observability inequality (4.3) has not been proved yet by a direct method such as the multiplier method or Carleman estimates.

We give the well-posedness result :

**Proposition 4.1.** For any given $V \in L^2(0,T;L^2(\Gamma_1))^N$ and any given $(U_1,U_0) \in \mathcal{H}_0 \times \mathcal{H}_1$, the inhomogeneous problem (4.1) admits a unique weak solution $U$ such that

$$(U_1,U_0,V) \to (U',U) \in C^0([0,T];\mathcal{H}_{-s} \times \mathcal{H}_{1-s}), \qquad s > 1/2$$

Moreover the linear map

$$(U_1,U_0,V) \to (U',U).$$

is continuous with respect to the corresponding topologies.

**Theorem 4.2.** Assume that

$$\text{rank}(D, BD, \cdots, B^{N-1}D) < N. \tag{4.5}$$

Then the Robin problem (4.1) is not exactly controllable for any given initial data $(U_1,U_0) \in \mathcal{H}_0 \times \mathcal{H}_1$.

*Proof.* Condition (4.5) is equivalent to Hautus's criterion :

$$\mathrm{rank}[D, B - \lambda I] < N.$$

There exists a vector $e \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$ such that

$$B^T e = \lambda e, \qquad D^T e = 0.$$

We choose a special initial data as

$$\theta \in \mathcal{D}(\Omega), \qquad U_0 = \theta e, \qquad U_1 = 0.$$

Taking the inner product of $e$ with (4.1) and noting $\phi = (e, U)$, we get

$$\begin{cases} \phi'' - \Delta\phi = -(e, AU) & \text{in} \quad \Omega, \\ \phi = 0 & \text{on} \quad \Gamma_0, \\ \partial_\nu \phi + \lambda\phi = 0 & \text{on} \quad \Gamma_1, \\ t = 0: \quad \phi = \theta, \quad \phi' = 0, & \\ t = T: \quad \phi = 0, \quad \phi' = 0. & \end{cases}$$

Assume that problem (4.1) is exactly controllable with the boundary control $V$ :

$$\|V\|_{L^2(]0,T[;L^2(\Gamma_1))^N} \leq C\|\theta\|_{H^1(\Omega)}.$$

By Proposition 4.1, we have

$$\|U\|_{L^2(]0,T[;\mathcal{H}_{1-s})} \leq C\|\theta\|_{H^1(\Omega)}, \qquad \forall s > 1/2.$$

By the well-posedness, we get

$$\|\theta\|_{H^{2-s}(\Omega)} \leq C\|U\|_{L^2(0,T;\mathcal{H}_{1-s}(\Omega))} \leq C'\|\theta\|_{H^1(\Omega)}.$$

Taking $s = 3/4, \quad 2 - s = 5/4$, we get a contracdiction :

$$\|\theta\|_{H^{5/4}(\Omega)} \leq C'\|\theta\|_{H^1(\Omega)}, \qquad \theta \in \mathcal{D}(\Omega).$$

In one space-dimension, we can show the non exact boundary controllability of Robin's problem for all non invertible matrices $D$.

**Theorem 4.2.** Assume that $D$ is not invertible. Then, the one space-dimensional Robin problem (4.1) is not exactly controllable in $\mathcal{H}_0 \times \mathcal{H}_1$.

*Proof.* We choose a special initial data as

$$\theta \in \mathcal{D}(\Omega), \qquad U_0 = \theta e, \qquad U_1 = 0.$$

Let $e \in \mathbb{R}^N$ be a vector such that $D^T e = 0$. Taking the inner product of $e$ with (4.6) and denoting $\phi = (e, U)$, we get

$$\begin{cases} \phi'' - \phi_{xx} = f & 0 < x < 1, \\ x = 0: \quad \phi = 0, & \\ x = 1: \quad \phi_x = g, & \\ t = 0: \quad \phi = \theta, \quad \phi' = 0 & \\ t = T: \quad \phi = 0, \quad \phi' = 0 & \end{cases} \tag{4.7}$$

where

$$f = -(e, AU), \quad g = -(e, BU) \neq -\lambda\phi.$$

Assume that problem (4.1) is exactly controllable. The linear map

$$(a) \qquad \theta \to U, \quad \text{the solution to the forward problem (4.1)}$$

is continuous from $H_0^1(0,1)$ into $L^2(0,T;\mathcal{H}_{1-s}) \cap H^1(0,T;\mathcal{H}_{-s})$.

Next using the compact embedding (Lions 1968) :

$$L^2(0,T;\mathcal{H}_{1-s}) \cap H^1(0,T;\mathcal{H}_{-s}) \subset L^2(0,T;L^2(\Omega))$$

and the sharp regularity of Lasiecka-Triggiani (4.2), it turns out that the linear map

$$(b) \qquad U \to (f,g) = \big\{ -(e,AU), \quad -(e,BU)\big\}$$

is compact from the space $L^2(0,T;\mathcal{H}_{1-s}) \cap H^1(0,T;\mathcal{H}_{-s})$ into the space $L^2(0,T;L^2(0,1)) \times L^2(0,T)$.

But the linear map

$$(c) \qquad (f,g) \to \phi, \quad \text{the solution to the backward problem (4.7)}$$

is continuous from $L^2(0,T;L^2(0,1)) \times L^2(0,T)$ into $C^0([0,T];H_0^1(0,1))$.

Noting that $\phi(0) = \theta$, then the identity operator, obtained by the composition of (a), (b) and (c), would be compact in the space $H_0^1(0,1)$.

**THANK YOU FOR YOUR ATTENTION**