Control and Inverse Problems for Networks of Vibrating Strings with Attached Masses

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Beijing, July 2, 2015

The talk is based in part on joint work with
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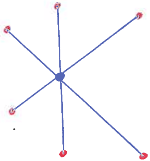
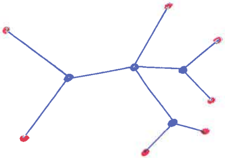
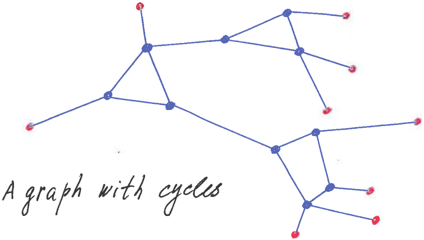


A star graph



A tree



One Interval

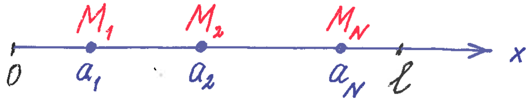
$$\rho(x)\,\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad \rho \in \text{ piecewise } \ \mathrm{C}^2[0,\ell],$$

$$t \in (0,T),\ x \in \Omega := (0,a_1) \cup (a_1,a_2) \cup \ldots \cup (a_N,\ell),$$

$$u(a_j^-,t) = u(a_j^+,t), \ \ M_j\, u_{tt}(a_j,t) = u_x(a_j^+,t) - u_x(a_j^-,t),$$

$$u(x,0) = u_t(x,0) = 0,$$

$$u(0,t) = f(t),\ u(\ell,t) = 0.$$



IBVP

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + q(x)u = 0, \quad t \in (0,T),\ x \in \Omega,\ q \in C[0,\ell].$$

$$u(x,0) = u_t(x,0) = 0,$$

$$u(a_j^-,t) = u(a_j^+,t), \quad M_j\, u_{tt}(a_j,t) = u_x(a_j^+,t) - u_x(a_j^-,t),$$

$$u(0,t) = f(t),\ u(\ell,t) = 0.$$

Controllability

Theorem 1. Let $T > 2\ell$ for $N \geq 2$ and $T \geq 2\ell$ for $N = 1$;

$$y \in W_0 := L^2(0, a_1) \times H^1(a_1, a_2) \times \cdots \times H^N(a_N, \ell),$$

$$z \in W_{-1} := H^{-1}(0, a_1) \times L^2(a_1, a_2) \times \cdots \times H^{N-1}(a_N, \ell),$$

$$y(a_j^-) = y(a_j^+),\ j \geq 2,$$

$$y'(a_j^-) = y'(a_j^+) - M_j[y_{xx}(a_j^+) - q(a)y(a_j^+)],\ j \geq 3, \quad y(\ell^-) = 0,$$

$$z(a_j^-) = z(a_j^+),\ j \geq 3,$$

$$z'(a_j^-) = z'(a_j^+) - M_j[z_{xx}(a_j^+) - q(a)z(a^+)],\ j \geq 4, \quad z(\ell^-) = 0.$$

There exists

$$f \in L^2(0, T),\ \|f\|^2_{L^2(0,T)} \asymp \|y\|^2_{W_0} + \|z\|^2_{W_{-1}},$$

such that

$$u^f(x, T) = y(x),\ u^f_t(x, T) = z(x),\ x \in \Omega.$$

For $N = 1$ and constant $\rho(x)$: S. Hansen and E. Zuazua (1995).

Spectrum

$$-\varphi''(x,\lambda)+q(x)\varphi(x,\lambda)=\lambda^2\varphi(x,\lambda),$$

$$x\in\Omega:=(0,a_1)\cup(a_1,a_2)\cup\ldots\cup(a_N,\ell),$$

$$\varphi(0,\lambda)=\varphi(\ell,\lambda)=0,\ \ \varphi(a_j^-,\lambda)=\varphi(a_j^+,\lambda),$$

$$-M_j\,\lambda^2\varphi(a_j,\lambda)=\varphi'(a_j^+,\lambda)-\varphi'(a_j^-,\lambda),$$

The (simple) eigenfrequences λ_n of this problem are zeros of the generating function

$$F(\lambda):=\Phi(\ell,\lambda),\ \ \Phi(0,\lambda)=0,\ \Phi'(0,\lambda)=1.$$

They can be presented as a union of $N+1$ separated series:

$$\{\lambda_n\}_{n=1}^{\infty}=\{\lambda_m^0\}_{m=1}^{\infty}\cup\left[\bigcup_{j=1}^{N}\{\lambda_m^j\}_{m=0}^{\infty}\right],\ \lambda_m^j=\frac{\pi m}{a_{j+1}-a_j}+O\left(\frac{1}{m}\right).$$

We present the solution of the IBVP in the form of the series

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x),\;\; u_t(x,t) = \sum_{n=1}^{\infty} \dot{b}_n(t)\,\phi_n(x).$$

Coefficients $b_n(t)$ can be found as

$$b_n(t) = \phi_n'(0)\int_0^t f(\tau)\,\frac{\sin\lambda_n(t-\tau)}{\lambda_n}\,d\tau,$$

$$\dot{b}_n(t) = \phi_n'(0)\int_0^t f(\tau)\cos\lambda_n(t-\tau)\,d\tau.$$

These equalities can be conveniently written in the form

$$c_n^{\pm}(t) = \int_0^t f(\tau)\,e^{\pm i\lambda_n\tau}\,d\tau,\;\; c_n^{\pm}(t) := \left(\mp\frac{i\lambda_n}{\phi_n'(0)}\,b_n(t) + \frac{\dot{b}_n(t)}{\phi_n'(0)}\right)\,e^{i\lambda_n t}.$$

Fourier Method

We use the spectral representation to create a scale of spaces $\mathcal{H}_p$ (for $p > 0,\ \mathcal{H}_p = D(A^{p/2})$:

$$\mathcal{H}_p = \{\varphi(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x) \ : \ ||\varphi||_p^2 = \sum_{n=1}^{\infty} |\alpha_n|^2 |n|^{2p} < \infty\},\ p \in \mathbb{R}.$$

So, for example,

$$\|u(\cdot,t)\|^2_{L^2(0,\ell)} \asymp \sum_{n=1}^{\infty} |b_n(t)|^2,\ \|u_t(\cdot,t)\|^2_{H^{-1}(0,\ell)} \asymp \sum_{n=1}^{\infty} |\dot{b}_n(t)|^2.$$

For a regular string, $|\lambda_n| \asymp n,\ |\phi_n'(0)| \asymp n$, and so

$$\|u(\cdot,t)\|^2_{L^2(0,\ell)} + \|u_t(\cdot,t)\|^2_{H^{-1}(0,\ell)} \asymp \sum_{n=1}^{\infty} |c_n^{\pm}(t)|^2.$$

The Moment Problem

Exact controllability of our system in $L^2(0,\ell) \times H^{-1}(0,\ell)$ in the time interval $[0,T]$ is equivalent to solvability of the following moment problem in $L^2(0,T)$ for all $\{c_n^\pm\} \in \ell^2$:

$$c_n^\pm = \int_0^T f(t)\, e^{\pm i\lambda_n t}\, d\tau,\ n \in \mathbb{N}.$$

The latter is in turn equivalent to the fact that the family $\{\exp(\pm i\lambda_n t)\}$ is a Riesz sequence in $L^2(0,T)$.
This is true (for a regular string) iff $T \geq 2\ell$.
The sharp regularity result is also follows from the Riesz sequence property.

Exponential Divided Differences (EDD)

For strings with attached masses:
(i) it is not clear how express non-symmetric spaces W_0 and W_1 in terms of the coefficients α_n;
(ii) the relation $|\phi_n'(0)| \asymp n$ is not generally true;
(iii) for no T the family $\{\exp(\pm i\lambda_n t)\}$ can be a Riesz sequence in $L^2(0,T)$ because the sequence $\{\lambda_n\}$ is not separated: $\inf_{n\neq k}|\lambda_n - \lambda_k| = 0.$

Assume $\{\mu_j\}$ is a non-repeating sequence. The exponential divided difference (EDD) of order zero for $\{e^{i\mu_n t}\}$ is $[e^{i\mu_1 t}](t) := e^{i\mu_1 t}$. The EDD of order $n-1$ is given by

$$[e^{i\mu_1 t},\dots,e^{i\mu_n t}] = \frac{[e^{i\mu_1 t},\dots,e^{i\mu_{n-1} t}] - [e^{i\mu_2 t},\dots,e^{i\mu_n t}]}{\mu_1-\mu_n},\ \mu_1\neq\mu_n.$$

One then easily derives the formulas

$$[e^{i\mu_1 t},e^{i\mu_2 t}] = \frac{e^{i\mu_1 t}-e^{i\mu_2 t}}{\mu_1-\mu_2},\quad [e^{i\mu_1 t},\dots,e^{i\mu_n t}] = \sum_{k=1}^{n}\frac{e^{i\mu_k t}}{\prod_{j\neq k}(\mu_k-\mu_j)}.$$

The Riesz basis theory of exponential divided differences was developed in (Avdonin and Ivanov, 2001; Avdonin and Moran, 2001).

The generating function of the sequence $\{\pm\lambda_n\}$ is an entire function

$$F(\lambda) = \prod_{n\in\mathbb{N}} \left(1 - \frac{\lambda^2}{\lambda_n^2}\right)$$

of the exponential type of ℓ in $\mathbb{C}_\pm$. For some $h \in \mathbb{R}$,

$$|F(x+ih)| \asymp |x+ih|^{N-1}.$$

Therefore, the corresponding family of EDD $\mathcal{E} = \bigcup_{p\in\pm\mathbb{N}} \mathcal{E}_p$ forms a Riesz basis in the closure of its linear span in $L^2(0,T)$ for $T > 2\ell$.

Controllability

The original moment problem

$$c_n^{\pm} = \int_0^T f(t)\, e^{\pm i\lambda_n t}\, d\tau,\ n \in \mathbb{N},$$

can be rewritten in the form

$$\hat{c}_n^{\pm} = \int_0^T f(t)\, e_n^{\pm}(t)\, d\tau,\ n \in \mathbb{N},$$

where $\mathcal{E} = \{e_n^{\pm}(t)\}$ is the corresponding family of EDD. Similarly, the relations

$$\alpha_n = \int_0^T f(t)\ \sin(\lambda_n t)\, d\tau,\ n \in \mathbb{N},$$

can be rewritten in the form

$$\beta_n = \int_0^T f(t)\, s_n(t)\, d\tau,\ n \in \mathbb{N},$$

Shape Controllability

Since $\mathcal{E}$ forms a Riesz sequence in $L^2(0,T)$ for $T > 2\ell$, the family $S = \{s_n\}$ of the corresponding DD of sine functions forms a a Riesz sequence in $L^2(0,T)$ for $T > \ell$.
Now we rewrite the relation $y(x,T) = \sum \alpha_n \phi_n(x)$ in the form $y(x,T) = \sum \beta_n \psi_n(x)$, where ψ_n are the corresponding linear combinations of ϕ_n.
Theorem 2 (Shape controllability). Let $T > \ell,\ y \in W_0$. There exists

$$f \in L^2(0,T),\ \|f\|^2_{L^2(0,T)} \asymp \|y\|^2_{W_0},$$

such that

$$u^f(x,T) = y(x),\ x \in \Omega.$$

This theorem implies that $\{\psi_n\}$ forms a Riesz basis in W_0. From that we can derive the full controllability (Theorem 1).

Inverse problem

Response operator

$$R^T : L^2(0,T) \mapsto L^2(0,T), \ \ Dom(R^T) = \{f \in H^1(0,T), f(0) = 0\},$$

$$(R^T f)(t) = u_x^f(0,t), \ t \in (0,T).$$

Theorem 2. Let $T > 2l$. Given R^T, one can find $\rho(x), \ a_j, \ M_j, \ l.$
Step 1. From R^T, $T > 2a_1,$ we find $\rho(x), \ x \in [0, a_1]; \ a_1, \ M_1.$
Step 2. Leaf-peeling method. Let $T > 2l$. From R^T, we find the response operator of the reduced system defined on the interval $x > a_1$.

$$-M_j \lambda^2 \varphi(a_j, \lambda) = \varphi'(a_j^+, \lambda) - \varphi'(a_j^-, \lambda),$$

$$\tilde{m}(\lambda) = \frac{\varphi'(a_j^-, \lambda)}{\varphi(a_j, \lambda)} - M_j \lambda^2.$$

Our approach is based on the boundary control (BC) method.