Optimization of the Principal Eigenvalue for Elliptic Operators

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Outline

Eigenvalue of Differential Operators

- $\Omega \subseteq \mathbb{R}^n$: bounded domain with a C^1 boundary $\partial\Omega$.

  $A(\cdot)$: uniformly elliptic matrix-valued function.

- Well-known: for some positive numbers (**eigenvalues**) λ,

$$\begin{cases} -\nabla \cdot (A(x)\nabla\psi(x)) = \lambda\psi(x), & x \in \Omega, \\ \psi|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

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- Such λ is called an eigenvalue of the differential operator

$$\mathcal{L}_{A(\cdot)}\varphi(\cdot) = -\nabla \cdot \big(A(\cdot)\nabla\varphi(\cdot)\big), \qquad \varphi(\cdot) \in H_0^1(\Omega).$$

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Important of $\lambda_{A(\cdot)}$.

1. $C = \frac{1}{\lambda_{A(\cdot)}}$:

   the sharp constant to the following Poincaré-type inequality:

$$\int_\Omega |y(x)|^2 \, dx \leq C \int_\Omega \langle A(x) \nabla y(x), \nabla y(x) \rangle \, dx, \quad \forall \, y \in H_0^1(\Omega). \tag{2}$$

2. Consider the following parabolic equation:

$$\left\{\begin{array}{ll} y_t(t,x) = \nabla \cdot \big(A(x)\nabla y(t,x)\big), & (t,x) \in [0,\infty) \times \Omega, \\ y(t,x) = 0, & (t,x) \in [0,\infty) \times \partial\Omega, \\ y(0,x) = y_0(x), & x \in \Omega. \end{array}\right. \quad (3)$$

Then

$$\frac{d}{dt}\|y(t,\cdot)\|_2^2 = -2\int_\Omega \langle A(x)\nabla y(t,x), \nabla y(t,x)\rangle \, dx$$
$$\leq -2\lambda_{A(\cdot)}\|y(t,\cdot)\|_2^2, \quad (4)$$

where $\|\cdot\|_2 = \|\cdot\|_{L^2(\Omega)}$.

This implies

$$\|y(t,\cdot)\|_2 \le e^{-\lambda_{A(\cdot)}t}\|y_0(\cdot)\|_2, \qquad \forall\, t \ge 0,\ y_0(\cdot) \in L^2(\Omega). \qquad (5)$$

The equality holds if $y_0(\cdot) = y_{A(\cdot)}$,

$y_{A(\cdot)}$: eigenfunction corresponding to $\lambda_{A(\cdot)}$.

In some sense:

$\lambda_{A(\cdot)}$ is the smallest decay rate

for the evolutionary map $y_0(\cdot) \mapsto y(t,\cdot\,; y_0(\cdot))$

—uniform in the initial state $y_0(\cdot)$.

Our Problems

- $\mathcal{S}^n$: the set of all $(n \times n)$ symmetric matrices

  $M(\lambda, \Lambda) := \{A \in \mathcal{S}^n | \lambda I \leq A \leq \Lambda I\}$ for some $\Lambda \geq \lambda > 0$.

  $\mathcal{M}(\lambda, \Lambda) := L^\infty(\Omega; M(\lambda, \Lambda))$.

  $\mathcal{E}$: a non-empty subset of $\mathcal{M}(\lambda, \Lambda)$.

- Find the value of

$$\bar{\lambda} := \inf_{A(\cdot) \in \mathcal{E}} \lambda_{A(\cdot)}.$$

- Does there exists an $\overline{A}(\cdot) \in \mathcal{E}$ such that $\bar{\lambda} = \lambda_{\overline{A}(\cdot)}$?

  If such an $\overline{A}(\cdot)$ exists, how can we find it?

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The Special Cases We Will Consider

- For simplicity:
  Let $n = 2$. Consider $A, B \in M(\lambda, \Lambda)$,

  $$E = \{A, B\}, \qquad \mathcal{E} = L^\infty(\Omega; E).$$

- A simple observation: If $A \geq B$, then

  $$\bar{\lambda} := \inf_{A(\cdot)\in\mathcal{E}} \lambda_{A(\cdot)} = \lambda_{B\chi_\Omega(\cdot)}.$$

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Optimal control problem

- **System and cost functional:**

$$\begin{cases} -\nabla \cdot (A(x)\nabla y(x)) = v(x), & x \in \Omega, \\ y|_{\partial\Omega} = 0, \end{cases} \tag{6}$$

$$J(y(\cdot), A(\cdot), v(\cdot)) = \int_\Omega v(x)y(x)\,dx, \tag{7}$$

$$\text{sub} : A(\cdot) \in \mathcal{E}, \quad v(\cdot) \in L^2(\Omega), \quad \int_\Omega |y(x)|^2\,dx \geq 1. \tag{8}$$

- **Relation to the original problem.** One can easily see that

$$\bar{\lambda} = \inf_{(6),\,(8)} J(y(\cdot), A(\cdot), v(\cdot)). \tag{9}$$

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H-convergence

Def. $A_\varepsilon(\cdot) \xrightarrow{H} A^*(\cdot)$ (on Ω) iff: $\forall f \in H^{-1}(\Omega)$,

$$\begin{cases} -\nabla\cdot\Big(A_\varepsilon(x)\nabla y_\varepsilon(x)\Big) = f, \quad \text{in } \ \Omega, \\ y_\varepsilon|_{\partial\Omega} = 0, \end{cases} \tag{10}$$

implies

$$\begin{cases} y_\varepsilon(\cdot) \rightharpoonup y(\cdot), & \text{weakly in } \ H_0^1(\Omega), \\ A_\varepsilon(\cdot)\nabla y_\varepsilon(\cdot) \rightharpoonup A^*(\cdot)\nabla y(\cdot), & \text{weakly in } \ L^2(\Omega)^n. \end{cases} \tag{11}$$

Here $A_\varepsilon(\cdot)$ are not necessary symmetric.

Equivalent definition for symmetric cases

Let $A_\varepsilon(\cdot) \in \mathcal{M}(\lambda, \Lambda)$.

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implies

$$y_\varepsilon(\cdot) \rightharpoonup y(\cdot), \quad \text{weakly in } H_0^1(\Omega)$$

and

$$\begin{cases} -\nabla \cdot \Big(A^*(x)\nabla y(x)\Big) = f, & \text{in } \Omega, \\ y|_{\partial\Omega} = 0. \end{cases} \tag{12}$$

Remark

Symmetric case: G-convergence, S. Spagnolo, 1968,

General case: H-convergence, L. Tartar, 1978

Basic Properties of H-convergence

- **Existence.**

  $\mathcal{M}(\lambda, \Lambda) = L^\infty(\Omega; M(\lambda, \Lambda))$: sequentially compact

  —in the sense of H-convergence.

- **Locality.** $\omega \subset \Omega$.

  $A_\varepsilon(\cdot) \xrightarrow{H} A^*(\cdot)$ on Ω $\Longrightarrow A_\varepsilon(\cdot) \xrightarrow{H} A^*(\cdot)$ on ω.

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Basic Properties of H-convergence

- **Uniqueness.**

$$\left.\begin{array}{l} A_\varepsilon(\cdot) \xrightarrow{H} A^*(\cdot) \quad \text{on} \quad \Omega \\ A_\varepsilon(\cdot) \xrightarrow{H} B^*(\cdot) \quad \text{on} \quad \Omega \end{array}\right. \implies A^*(\cdot) = B^*(\cdot) \quad \text{on} \quad \Omega.$$

- **Monotonicity.**

$$\left.\begin{array}{l} A_\varepsilon(\cdot) \xrightarrow{H} A^*(\cdot) \\ B_\varepsilon(\cdot) \xrightarrow{H} B^*(\cdot) \\ A_\varepsilon(\cdot) \geq B_\varepsilon(\cdot) \end{array}\right. \implies A^*(\cdot) \geq B^*(\cdot).$$

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Basic Properties of H-convergence

- ***Upper and under bounds.** Let

$$A_\varepsilon(\cdot) \quad \rightharpoonup \quad \overline{A}(\cdot), \text{ weakly in } L^2(\Omega)^{n\times n},$$

$$A_\varepsilon^{-1}(\cdot) \quad \rightharpoonup \quad \underline{A}^{-1}(\cdot), \text{ weakly in } L^2(\Omega)^{n\times n}.$$

Then

$$\underline{A}(x) \leq A^*(x) \leq \overline{A}(x), \qquad \text{a.e. } x \in \Omega.$$

$\underline{A}(\cdot)$: “Harmonic Mean" $\overline{A}(\cdot)$: “Arithmetic Mean"

Basic Properties of H-convergence

- ***Commutative with congruent transformation.**

$$A_\varepsilon(\cdot) \xrightarrow{H} A^*(\cdot)$$

$$\Updownarrow$$

$$QA_\varepsilon(\cdot)Q^\top \xrightarrow{H} QA^*(\cdot)Q^\top$$

Basic Properties of H-convergence

- **Non-homogeneous boundary conditions.** Let $A_\varepsilon(\cdot) \xrightarrow{H} A^*(\cdot)$, $\varphi \in H^1(\Omega)$, $f \in H^{-1}(\Omega)$,

$$\begin{cases} -\nabla \cdot \Big(A_\varepsilon(x) \nabla y_\varepsilon(x) \Big) = f, \quad \text{in} \;\; \Omega, \\ y_\varepsilon|_{\partial\Omega} = \varphi, \end{cases} \tag{13}$$

$$\begin{cases} -\nabla \cdot \Big(A^*(x) \nabla y^*(x) \Big) = f, \quad \text{in} \;\; \Omega, \\ y^*|_{\partial\Omega} = \varphi. \end{cases} \tag{14}$$

Then

$$y_\varepsilon(\cdot) \rightharpoonup y^*(\cdot), \qquad \text{weakly in } H^1(\Omega).$$

H-closure

- **Def.** $\mathcal{E} \subseteq \mathcal{M}(\lambda, \Lambda)$.
  $\mathrm{cl}_{\mathcal{H}}(\mathcal{E})$—Closure of $\mathcal{E}$ under H-convergence.

- **Pointwisity.** Let $G \subseteq M(\lambda, \Lambda)$. $\mathcal{G} := L^{\infty}(\Omega; G)$,

  $$\mathscr{G} \equiv \mathrm{cl}_{\mathcal{H}}^{C}(G) := \left\{A \in M(\lambda, \Lambda) \middle| A\chi_{\Omega}(\cdot) \in \mathrm{cl}_{\mathcal{H}}(\mathcal{G})\right\}.$$

  Then

  $$\mathrm{cl}_{\mathcal{H}}(\mathcal{G}) = \left\{A^*(\cdot) \in \mathcal{M}(\lambda, \Lambda) \middle| A^*(x) \in \mathrm{cl}_{\mathcal{H}}^{C}(\mathcal{G}), \quad \text{a.e.} \quad \right\}.$$

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H-closure

- More general result. (B. Li and H. Lou)
  Assume that: $\mathcal{Q} \subseteq \mathcal{M}(\lambda, \Lambda)$,
  $F(\cdot) \in L^\infty(\Omega; \mathbb{R}^m)$
  $\omega(\cdot)$: continuous module,

$$\left|A(x) - A(\tilde{x})\right| \leq \omega\Big(\left|F(x) - F(\tilde{x})\right|\Big), \quad \forall\, x, \tilde{x} \in \Omega;\ A(\cdot) \in \mathcal{Q}. \tag{15}$$

  Then

$$\mathrm{cl}_{\mathcal{H}}(\mathcal{Q}) = \Big\{A^*(\cdot) \in \mathcal{M}(\lambda, \Lambda) \Big| A^*(x) \in \mathrm{cl}_{\mathcal{H}}^{C}(Q_x)\Big\},$$

  where $Q_x := \{B(x) | B(\cdot) \in \mathcal{Q}\}$.

Difficulty

Even if $G = \{A, B\}$. $\mathcal{G} := L^\infty(\Omega; G)$,

$\mathscr{G}$ is not completely known!

Special limits: lamination.

Let $G = \{A, B\}$, $\alpha \in (0,1)$, $e \in S^{n-1}$—unit sphere,

$$A_\varepsilon(x) := \begin{cases} A, & \left\{\frac{\langle x,e\rangle}{\varepsilon}\right\} \in [0,\alpha), \\ B, & \left\{\frac{\langle x,e\rangle}{\varepsilon}\right\} \in [\alpha,1). \end{cases}$$

Then, as $\varepsilon \to 0^+$,

$$A_\varepsilon \xrightarrow{H} A^* \equiv \alpha A + (1-\alpha)B - \frac{\alpha(1-\alpha)(A-B)ee^\top(A-B)}{e^\top\Big(\alpha B + (1-\alpha)A\Big)e}. \tag{16}$$

Special limits: lamination.

Equivalently

$$A^* - B = \alpha(A-B) - \frac{\alpha(1-\alpha)(A-B)ee^\top(A-B)}{e^\top\Big(\alpha B + (1-\alpha)A\Big)e}$$

$$= \alpha(A-B)\Big[I - \frac{\alpha(1-\alpha)ee^\top(A-B)}{e^\top\Big(\alpha B + (1-\alpha)A\Big)e}\Big]. \qquad (17)$$

Lamination based on B.

1. Tartar $\quad A \xrightarrow{B} A_1^* \xrightarrow{B} A_2^* \xrightarrow{B} A_3^* \xrightarrow{B} \cdots$

$$\mathscr{G} \equiv \mathrm{cl}_{\mathcal{H}}^{C}(G) \supseteq \Gamma(A, B)$$

$$\equiv \left\{ A^* \middle| (A^* - B)\left[I + (1-\alpha) \sum_{k=1}^{m} \beta_k \frac{e_k e_k^\top (A-B)}{e_k^\top B e_k} \right] = \alpha(A-B), \right.$$

$$\left. \alpha \in [0,1], e_k \in S^{n-1}, \beta_k \in [0,1], \sum_{k=1}^{m} \beta_k = 1 \right\}. \tag{18}$$

2. In fact, we have

$$\Gamma(A,B) = \left\{ B + \alpha(A-B)\left[I + (1-\alpha) B^{-\frac{1}{2}} H B^{-\frac{1}{2}} (A-B) \right]^{-1} \middle| \right.$$

$$\left. \alpha \in [0,1], \quad H \geq 0, \quad \mathrm{tr}\,(H) = 1 \right\} \tag{19}$$

Difficulties

It is possible: $\Gamma(A, B) \neq \Gamma(B, A)$ when $n \geq 3$.

It seems possible: $\mathscr{G} \neq \Gamma(A, B) \bigcup \Gamma(B, A)$.

It is possible: $\Gamma(A, B)$ is not convex.

It is probably: $\mathscr{G}$ is not convex for many cases.

Main Results

- **Recall Our Assumptions.**

  $n = 2, \quad A, B \in M(\lambda, \Lambda), \quad A \ngeq B, A \nleq B.$

- **Optimal Solution Does not Exist in Many Cases.**

Main Results

- **Recall Our Assumptions.**

  $n = 2, \quad A, B \in M(\lambda, \Lambda), \quad A \ngeq B, A \nleq B.$

- **Optimal Solution Does not Exist in Many Cases.**

Existence of optimal relaxed solution.

- **Optimal Relaxed Solution Exists.**

That is,

$$\begin{aligned} \bar{\lambda} &\equiv \inf_{(6),\,(8)} J(y(\cdot), A(\cdot), v(\cdot)) \\ &= J(y^*(\cdot), A^*(\cdot), v^*(\cdot)) \end{aligned} \tag{20}$$

for some

$$A^*(\cdot) \in \mathrm{cl}_{\mathcal{H}}(\mathcal{E}), \quad v^*(\cdot) \in L^2(\Omega), \quad \int_\Omega |y^*(x)|^2\,dx \geq 1. \tag{21}$$

$$\begin{cases} -\nabla \cdot (A^*(x)\nabla y^*(x)) = v^*(x), & x \in \Omega, \\ y^*|_{\partial\Omega} = 0, \end{cases} \tag{22}$$

- **Proof.**

  There is a sequence $(y_k(\cdot), A_k(\cdot)) \in H_0^1(\Omega) \times \mathcal{E}$ such that

$$\|y_k\|_{L^2(\Omega)} = 1,$$

$$-\nabla \cdot (A_k(x)\nabla y_k(x)) = \lambda_k y_k(x),$$

$$\lim_{k \to +\infty} \lambda_k = \bar{\lambda}.$$

  Denote

$$v_k(\cdot) = \lambda_k y_k(\cdot). \tag{23}$$

  Then

$$\lambda_k = J(y_k, A_k(\cdot), v_k(\cdot)). \tag{24}$$

- We can suppose that

$$A_k(\cdot) \xrightarrow{H} A^*(\cdot) \in \mathrm{cl}_{\mathcal{H}}(\mathcal{E}),$$

$$y_k(\cdot) \rightharpoonup y^*(\cdot), \qquad \text{weakly in } H_0^1(\Omega).$$

Then

$$\|y^*\|_{L^2(\Omega)} = 1, \quad -\nabla \cdot (A^*(x)\nabla y^*(x)) = \bar{\lambda} y^*(x),$$

That is,

$$\bar{\lambda} = \lambda_{A^*(\cdot)}. \tag{25}$$

Necessary conditions.

Let $(y^*(\cdot), A^*(\cdot), v^*(\cdot))$ be an optimal relaxed triple.

By Maximum Principle (Lou and Yong, 2009 SICON):

$\exists\ (\overline{\psi}_0, \overline{\psi}(\cdot), \beta) \in \mathbb{R} \times H_0^1(\Omega) \times \mathbb{R}$ such that

non-trivial condition

$$\overline{\psi}_0 \leq 0, \qquad (\overline{\psi}_0, \overline{\psi}(\cdot), \beta) \neq 0, \tag{26}$$

transversality condition

$$\beta(\eta - \|y^*\|^2_{L^2(\Omega)}) \leq 0, \qquad \forall\, \eta \geq 1. \tag{27}$$

state equation

$$-\nabla \cdot \big(A^*(x)\nabla\overline{\psi}(x)\big) = \psi_0 v^*(x) - \beta y^*(x), \qquad \text{in } \ \Omega, \tag{28}$$

Necessary Conditions.

maximum condition

$$\begin{aligned}
& H\big(y^*(x), \overline{\psi}_0, \overline{\psi}(x), \nabla y^*(x), \nabla\overline{\psi}(x), A^*(x), v^*(x)\big) \\
& - H\big(y^*(x), \overline{\psi}_0, \overline{\psi}(x), \nabla y^*(x), \nabla\overline{\psi}(x), \widetilde{A}, v\big) \\
\geq\ & \frac{1}{2}\left|\widetilde{A}^{-\frac{1}{2}}(A^*(x) - \widetilde{A})\nabla y^*(x)\right| \left|\widetilde{A}^{-\frac{1}{2}}(A^*(x) - \widetilde{A})\nabla\overline{\psi}(x)\right| \\
& + \frac{1}{2}\left\langle \widetilde{A}^{-\frac{1}{2}}(A^*(x) - \widetilde{A})\nabla y^*(x), \widetilde{A}^{-\frac{1}{2}}(A^*(x) - \widetilde{A})\nabla\overline{\psi}(x)\right\rangle, \\
& \qquad \forall \widetilde{A} \in \mathscr{G} \equiv \mathrm{cl}^C_{\mathcal{H}}(\{A, B\}), v \in \mathbb{R}, \quad \text{a.e. } x \in \Omega. \quad (29)
\end{aligned}$$

where

$$\begin{aligned}
& H(y, \psi_0, \psi, \xi, \eta, Q, v) = \psi v + \psi_0 v y - \langle Q\xi, \eta\rangle \\
& \qquad (y, \psi_0, \psi, \xi, \eta, Q, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \times \mathbb{R}.
\end{aligned}$$

Simplification.

- We can get that

  $\beta \leq 0$, $\overline{\psi}_0 \neq 0$. Then one can set $\overline{\psi}_0 = -1$.

- $\overline{\psi}(\cdot) = y^*(\cdot) \quad \Longrightarrow \quad v^* = -\frac{\beta}{2} y^*(\cdot)$

  $$\Longrightarrow \quad \beta < 0 \Longrightarrow \quad \|y^*\|_{L^2(\Omega)} = 1$$

  $$\Longrightarrow -\nabla \cdot (A^*(x)\nabla y^*(x)) = \frac{-\beta}{2} y^*(x), \quad \text{in} \;\; \Omega.$$

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Simplification.

- Condition (29) becomes

$$
\begin{aligned}
&\left\langle \widetilde{A}\nabla y^*(x), \nabla\overline{\psi}(x)\right\rangle - \left\langle A^*(x)\nabla y^*(x), \nabla\overline{\psi}(x)\right\rangle \\
\geq \quad & \frac{1}{2}\left|\widetilde{A}^{-\frac{1}{2}}(A^*(x)-\widetilde{A})\nabla y^*(x)\right| \left|\widetilde{A}^{-\frac{1}{2}}(A^*(x)-\widetilde{A})\nabla\overline{\psi}(x)\right| \\
& +\frac{1}{2}\left\langle \widetilde{A}^{-\frac{1}{2}}(A^*(x)-\widetilde{A})\nabla y^*(x), \widetilde{A}^{-\frac{1}{2}}(A^*(x)-\widetilde{A})\nabla\overline{\psi}(x)\right\rangle, \\
& \qquad \forall \widetilde{A} \in \mathscr{G}, \qquad \text{a.e. } \; x \in \Omega. \qquad (30)
\end{aligned}
$$

Simplification.

- Moreover, we have $y^*(\cdot) = \overline{\psi}(\cdot)$. Then (30) becomes

$$\begin{aligned}&\left\langle (\widetilde{A} - A^*(x))\nabla y^*(x), \nabla y^*(x) \right\rangle \\ \geq\ &\left\langle \widetilde{A}^{-1}(\widetilde{A} - A^*(x))\nabla y^*(x), (\widetilde{A} - A^*(x))\nabla y^*(x) \right\rangle, \\ &\forall \widetilde{A} \in \mathscr{G}, \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{31}$$

- That is,

$$\begin{aligned}&\langle A^*(x)\nabla y^*(x), \nabla y^*(x) \rangle \\ \geq\ &\left\langle \widetilde{A}^{-1} A^*(x)\nabla y^*(x), A^*(x)\nabla y^*(x) \right\rangle, \\ &\forall \widetilde{A} \in \mathscr{G}, \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{32}$$

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$$\begin{aligned} &\Big\langle (\widetilde{A} - A^*(x))\nabla y^*(x), \nabla y^*(x) \Big\rangle \\ \geq\ &\Big\langle \widetilde{A}^{-1}(\widetilde{A} - A^*(x))\nabla y^*(x), (\widetilde{A} - A^*(x))\nabla y^*(x) \Big\rangle, \\ &\forall\, \widetilde{A} \in \mathscr{G}, \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{31}$$

- That is,

$$\begin{aligned} &\langle A^*(x)\nabla y^*(x), \nabla y^*(x) \rangle \\ \geq\ &\Big\langle \widetilde{A}^{-1} A^*(x)\nabla y^*(x), A^*(x)\nabla y^*(x) \Big\rangle, \\ &\forall\, \widetilde{A} \in \mathscr{G}, \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{32}$$

Simplification.

- Recall that

$$\underline{A}(x) \le \widetilde{A}(x), \quad \left(\widetilde{A}(x)\right)^{-1} \le \left(\underline{A}(x)\right)^{-1} \tag{33}$$

if

$$A_\varepsilon^{-1}(\cdot) \rightharpoonup \underline{A}^{-1}(\cdot), \quad A_\varepsilon(\cdot) \xrightarrow{H} \widetilde{A}(\cdot). \tag{34}$$

- We see that (32) is equivalent to

$$\begin{aligned} & \langle A^*(x)\nabla y^*(x), \nabla y^*(x)\rangle \\ \ge\ & \left\langle \widetilde{A}^{-1} A^*(x)\nabla y^*(x), A^*(x)\nabla y^*(x)\right\rangle, \\ & \qquad \forall\, \widetilde{A} \in \{A, B\} \quad \text{a.e.} \ \ x \in \Omega. \end{aligned} \tag{35}$$

Evaluate $A^*(\cdot)$.

- For any $\xi \in S^{n-1}$, we need to find $A^* \equiv A^*_\xi \in \mathscr{G}$, s.t.

$$P_\xi : \quad \begin{aligned} \langle A^*\xi, \xi\rangle &\geq \left\langle A^{-1} A^*\xi, A^*\xi \right\rangle, \\ \langle A^*\xi, \xi\rangle &\geq \left\langle B^{-1} A^*\xi, A^*\xi \right\rangle. \end{aligned} \tag{36}$$

- Problems like P_ξ are not easy because $\mathscr{G}$ is not known!

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- **Problems like P_ξ are not easy because $\mathscr{G}$ is not known!**

Evaluate $A^*(\cdot)$.

We have

- A^*_ξ exists.
- There is a non-empty set $E \subset S^{n-1}$, s.t.
  $\forall \xi \in E$, $A^*_\xi \notin \{A, B\}$.

  This means that usually $\bar{\lambda}$ is not attainable in $\mathcal{E}$.
- Sometimes, P_ξ admits more than one solution.

  However, if A^*, B^* are both solutions of P_ξ, then $A^*\xi = B^*\xi$.

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- we guess that P_ξ admits a solution $A_\xi^* \in \Gamma(A,B)\bigcup\Gamma(B,A)$.
- in fact, $\Gamma(A,B)=\Gamma(B,A)$ when $n=2$.
- Since H-convergence and $\mathcal{L}_{A(\cdot)}$ are commutative with congruent transformation,

  we can suppose $A=\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $B=I$ with $0<a<1<b$.

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Any way, after careful discussions, we get that

- $$\begin{cases} A^*_\xi = A, & \text{if } \xi \in E_A, \\ A^*_\xi = B(=I), & \text{if } \xi \in E_B, \\ A^*_\xi \neq A, B, & \text{if } \xi \in E, \end{cases} \tag{37}$$

  where

  $$\begin{cases} E_A = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\middle|\; \dfrac{\sin^2\theta}{\cos^2\theta} \leq \dfrac{a(1-a)}{b(b-1)}, \quad \theta \in [0, 2\pi) \right\}, \\ E_B = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\middle|\; \dfrac{\sin^2\theta}{\cos^2\theta} \geq \dfrac{b(1-a)}{a(b-1)}, \quad \theta \in [0, 2\pi) \right\}, \\ E = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \;\middle|\; \dfrac{a(1-a)}{b(b-1)} < \dfrac{\sin^2\theta}{\cos^2\theta} < \dfrac{b(1-a)}{a(b-1)}, \theta \in [0, 2\pi) \right\}. \end{cases} \tag{38}$$

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- When $\xi \in E$, $A^*_\xi \notin \{A, B\}$.

Evaluate $A^*(\cdot)$.

- If $\xi \in E$. Then

$$A_\xi^* = I + \alpha\Big[(A-I)^{-1} + (1-\alpha)H\Big]^{-1}$$

where

$$H = \begin{pmatrix} s & \beta \\ \beta & 1-s \end{pmatrix}, \quad \beta^2 = s(1-s), \quad s = \frac{a^{-1}-1}{a^{-1}-b^{-1}}. \tag{39}$$

$$\operatorname{sgn}(\beta) = -\operatorname{sgn}(\cos\theta\sin\theta), \tag{40}$$

$$\alpha = \frac{X-Y}{(1-b^{-1})X + (a^{-1}-1)Y}, \tag{41}$$

$$X = \sqrt{\frac{a^{-1}-1}{1-b^{-1}}}, \quad Y = |\tan\theta| \in \left(\frac{aX}{b}, X\right). \tag{42}$$

Evaluate $A^*(\cdot)$.

- Moreover, for $\xi = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, we have

$$A^*_\xi \xi = \begin{cases} \begin{pmatrix} 1-s & \varepsilon\sqrt{s(1-s)} \\ \varepsilon\sqrt{s(1-s)} & s \end{pmatrix} \xi, & \xi \in E, \\ \xi, & \xi \in E_B, \\ A\xi, & \xi \in E_A, \end{cases} \quad (43)$$

were $\varepsilon = \operatorname{sgn}(\cos\theta \sin\theta)$

- $\xi \mapsto A^*_\xi \equiv A^*_{\frac{\xi}{|\xi|}}$ is continuous on $\mathbb{R}^n \setminus \{0\}$.

- $\xi \mapsto A^*_\xi \xi$ is Lipschitz continuous on $\mathbb{R}^n$. Yet, it is not differentiable.

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Further works.

Properties of $\bar{\lambda}$ and the corresponding eigenfunction.

Multi-materials: $\overline{\overline{U}} \geq 3$.

High dimensional cases: $n \geq 3$.

Maximize the first eigenvalue.

....

Thanks for Attention!