Selective Pattern Formation in Reaction-Diffusion Systems

Kenji Kashima (Kyoto Univ.)

Joint work with:
T. Ogawa (Dynamical system theory)
T. Sakurai (Physics)

Selective pattern formation control: Spatial spectrum consensus and Turing instability approach, *Automatica* 2015
Turing Pattern

- The Chemical Basis of Morphogenesis (1952)

- A key mechanism for pattern formation
  - Chemical reaction: Belousov-Zhabotinsky
  - Biosystems: skin

- Specially uniform reaction - diffusion equation
Outline

- Preliminary: Diffusively coupled identical (stable) systems
  - Synchronization
  - Turing Instability
- Problem formulation: Nonlinear PDE with unknown target state
- Approach: Spatial spectrum consensus
- Simulation and concluding remarks
Synchronization

- **Subsystem dynamics**
  - \( \dot{x}_i(t) = Ax_i(t) + Du_i(t) \)

- **Rendezvous, synchronization**
  - \( \|x_i - x_j\| \to 0, \ t \to \infty \ \forall i, j \)

- **Diffusive coupling**
  - \( u_i(t) = \sum_{i \in N_i} \gamma_{ij} (x_j(t) - x_i(t)) \)
    - \( \gamma_{ij} \geq 0 \ (j \neq i) \), connection topology, strength
Closed-loop dynamics

- \( \dot{x} = (A \otimes I_n + D \otimes \Gamma)x \)
  - \( x := [x_1 \cdots x_n]^T \)
  - \( \Gamma: \text{Graph Laplacian} \)
    - Eigenvalues: \( 0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \)
- \( \text{eig}(A \otimes I_n + D \otimes \Gamma) = \bigcup_{\lambda \in \text{eig}(\Gamma)} \text{eig}(A + \lambda D) \)

- Special case: \( D = cI, c > 0 \)
  - Stability/Instability preserving: \( \lambda_A + c\lambda_\Gamma \)
    - Consensus achieved
- Other cases: Not necessarily true
Formulation by reaction-diffusion

- **Reaction** (stable) \( \frac{dz}{dt} = f(z) \)
  - GAS equil. \( z = 0 \)

- **Diffusion** \( \frac{\partial z}{\partial t} = f(z) + D \Delta z, \; \xi \in \mathbb{P} \)
  - Diagonal \( D \geq 0 \)
  - Is \( z_{eq}(\xi) \equiv 0 \) stable as well?

- **Example:**
  - \( \dot{u} = u - v - u^3 + d_u \Delta u \)
  - \( \dot{v} = 3u - 2v + d_v \Delta v \)
Rectangular Domain Example

\[
\begin{align*}
\dot{u} &= u - v - u^3 + d_u \Delta u \\
\dot{v} &= 3u - 2v + d_v \Delta v \\
d_v / d_u &\gg 1
\end{align*}
\]

\[u(t, x, y)\]
Mathematical model

- **Reaction-Diffusion**
  \[
  \begin{cases}
    \dot{u} = a_{11}u + a_{12}v - u^3 + d_u \Delta u \\
    \dot{v} = a_{12}u + a_{22}v + d_v \Delta v
  \end{cases}
  \]
  - Square domain, periodic boundary conditions

- **This toy models is...**
  - an activator-inhibitor RD systems with cubic nonlinearities
  - able to captures the essential mathematical structure of spatial symmetry
  - essentially equivalent to
    - phase-field model, the Cahn-Hilliard model,
    - the Ginzburg-Landau equation,
    - the Swift-Hohenberg equation, and so on
Outline

- Preliminary: Diffusively coupled identical (stable) systems
  - Multi-agent systems
  - Turing Instability
- Problem formulation: Nonlinear PDE with unknown target state
- Approach: Spatial spectrum consensus
- Simulation and concluding remarks
Control of pattern formation

- Cardiac arrhythmias
  S. Luther et al., Low-energy control of electrical turbulence in the heart, Nature (2011)

- Chemical reaction
  \[
  \frac{\partial z(t,p)}{\partial t} = f(z(t,p)) + D \Delta z(t,p) + w(t,p)
  \]

What we attempt to do is ...

- not to generate an arbitrarily given pattern by forcing some inputs

- but to assist a (possibly hidden) pattern generation
Assisted pattern formation

- Example revisited:
  - Initial pattern dependent convergence to one of:

- Can we generate their superposition as well?
Spatial Fourier Transformation

• Basis functions are stripes labeled by $m \in \mathbb{Z}^2$
  - $p_m(x, y) = e^{2\pi j (m_x x + m_y y)}$
  - $u(t, x, y) = \sum_m \hat{u}_m(t)p_m(x, y), \quad \hat{v}_m(t)$

• Reaction-Diffusion
  $$\begin{cases}
  \dot{u} = a_{11} u + a_{12} v - u^3 + d_u \Delta u \\
  \dot{v} = a_{12} u + a_{22} v + d_v \Delta v
  \end{cases}$$

$$\begin{cases}
  \hat{u}_m = (a_{11} - s_m d_u) \hat{u}_m + a_{12} \hat{v}_m - \hat{u}_m(\text{couple}) \\
  \hat{v}_m = a_{12} \hat{u}_m + (a_{22} - s_m d_v) \hat{v}_m
  \end{cases}$$

- $s_m := \left( \frac{2\pi m_x}{L_x} \right)^2 + \left( \frac{2\pi m_y}{L_y} \right)^2$

$$\begin{pmatrix}
  \hat{u}_m \\
  \hat{v}_m
  \end{pmatrix} = \begin{pmatrix}
  A - s_m D
  \end{pmatrix}^{-1} \begin{pmatrix}
  \hat{u}_m \\
  \hat{v}_m
  \end{pmatrix}$$
Problem Formulation

- Controlled Reaction-Diffusion
  \[
  \begin{align*}
    \dot{u} &= a_{11}u + a_{12}v - u^3 + d_u \Delta u \\
    \dot{v} &= a_{12}u + a_{22}v + d_v \Delta v
  \end{align*}
  \]
  \[\mathcal{M} := \{m: A - s_m D \text{ is unstable}\}\]

- Control objective: in large $t$ limit
  \[
  u(t, x, y) \to \bar{u}(t) \sum_{m \in \mathcal{M}} p_m(x, y)
  \]
  \[
  \|\bar{u}(t)\| \nrightarrow 0, \infty
  \]
  \[
  w(t, x, y) \to 0
  \]

- Remarks
  - Not to maximize $\|\bar{u}(t)\|$
  - Not to regulate $u(t, x, y) \to u^\circ(x, y)$
Outline

• Preliminary: Diffusively coupled identical (stable) systems
  ◦ Multi-agent systems
  ◦ Turing Instability

• Problem formulation: nonlinear PDE with unknown target state

• Approach: Spatial spectrum consensus

• Concluding remarks
Assumptions

- Controlled RD
  \[
  \begin{align*}
  \dot{\hat{u}}_m &= (a_{11} - s_m d_u)\hat{u}_m + a_{12}\hat{v}_m \\
  \dot{\hat{v}}_m &= a_{12}\hat{u}_m + (a_{22} - s_m d_v)\hat{v}_m
  \end{align*}
  \]
  for \( m \in \mathbb{Z}^2 \)

- \( \mathcal{M} = \{ m: A - s_m D \text{ is unstable} \} \)

- Approximation: \( \hat{u}_m, \hat{v}_m \approx 0 \) for \( m \notin \mathcal{M} \)

- Assumption
  \( s_m = \bar{s} \) for all \( m \in \mathcal{M} \)

- \( s_m := \left( \frac{2\pi m_x}{L_x} \right)^2 + \left( \frac{2\pi m_y}{L_y} \right)^2 \)

\[(4,0) \ (2,2) \ (2,-2)\]
Theorem: Spatial spectrum consensus

- Consider the truncated coupled $\mathbb{C}^2$-valued dynamics

$$\begin{cases}
\dot{\hat{u}}_m = (a_{11} - \overline{s}d_u)\hat{u}_m - \hat{u}_m(\text{couple}) + \hat{w}_m \\
\dot{\hat{v}}_m = a_{12}\hat{u}_m + (a_{22} - \overline{s}D)\hat{v}_m
\end{cases}$$

for $m \in \mathcal{M}$. Then,

- $u(t, x, y) \to \bar{u}(t) \sum_{m \in \mathcal{M}} p_m(x, y)$
- $\|\bar{u}(t)\| \to 0, \infty$
- $w(t, x, y) \to 0$

$A - \overline{s}D$ is unstable
Embedding onto PDE

- Controller implementation
  - $\mathbf{w}(t, x, y) = \sum_{m \in \mathcal{M}} \mathbf{\hat{w}}_m(t) p_m(x, y)$
  - $\mathbf{\hat{w}}_m = \sigma \sum_{n \in \mathcal{M}} \gamma_{nm} (\mathbf{\hat{u}}_m - \mathbf{\hat{u}}_n)$

\[
\begin{aligned}
\dot{u} &= a_{11} u + a_{12} v - u^3 + d_u \Delta u + w \\
\dot{v} &= a_{12} u + a_{22} v + d_v \Delta v
\end{aligned}
\]

Controlled ($\mathbf{w}(t, p) \to 0$)
Center manifold theorem

- \( \dot{x}_s = f_s^q(x_s, x_u) \), \( \dot{x}_u = f_u^q(x_s, x_u) \)
  - Local stability: \( x_s \) stable, \( x_u \) unstable for \( q > q^* \)
- When \( q \) is slightly larger than \( q^* \),
  - Existence of an invariant set
  - If the dynamics on the invariant set is stable, then this set is attractive.

\[ \|x_s\| = o(\|x_u\|^2) \]
Application to pattern formation

- **Reaction-Diffusion**
  \[
  \begin{aligned}
  \dot{u} &= a_{11} u + a_{12} v - u^3 + d_u \Delta u \\
  \dot{v} &= a_{12} u + a_{22} v + d_v \Delta v \\
  \hat{u}_m &= (a_{11} - s_m d_u) \hat{u}_m + a_{12} \hat{v}_m \\
  \hat{v}_m &= a_{12} \hat{u}_m + (a_{22} - s_m d_v) \hat{v}_m \\
  s_m &:= \left( \frac{2\pi m_x}{L_x} \right)^2 + \left( \frac{2\pi m_y}{L_y} \right)^2
  \end{aligned}
  \]

- \[\mathcal{M} := \{ m : A - s_m D \text{ is unstable} \} \]
Spillover

Distribution profile domain

RD system

Controller

Spatial spectrum domain

Locally stable modes

Locally unstable modes

Controller

Locally stable modes

Mismatch between locally unstable modes

Intensity of generated pattern

Technical assumptions guaranteed by suitable Lyapunov functionals

\[
J(z(t, x, y)) := \iint_{\Omega} V(u(t, x, y), v(t, x, y)) \, dx \, dy.
\]

\[
\dot{J}(z(t, x, y)) \leq - \iint_{\Omega} H(u(t, x, y), v(t, x, y)) \, dx \, dy.
\]
Center manifold for closed-loop PDE

\[
\begin{align*}
\dot{u} &= a_{11}u + a_{12}v - u^3 + d_u \Delta u + w \\
\dot{v} &= a_{12}u + a_{22}v + d_v \Delta v
\end{align*}
\]

Notation:

- **Orthogonal decomposition**
  \[ u(t, x, y) = \bar{u}(t) \sum_{m \in M} p_m(x, y) + u_{res}(t, x, y) \]
- **Autonomous pattern formation for** \( d_v > d^* \)

**Theorem:**

For any \( \varepsilon > 0 \), there exist \( \delta, T > 0 \) such that

\[ \|u(0)\| < \delta, d_v < d^* + \delta \implies \|u_{res}(t)\| + \|w(t)\| < \varepsilon \|\bar{u}(t)\| \text{ for } t > T \]

- **Residual component**
- **Input**
- **Intensity of generated pattern**
\( \varepsilon_1: \) Dynamic attractor

- Stabilization of standing waves
  - Autonomous traveling waves

Controlled \((w(t, p) \to 0)\)
$\varepsilon_2$: Delay robustness

- $w(t, x, y) = \sigma \sum_{m \in M} \hat{w}_m (t - T) p_m (x, y)$
- $\hat{w}_m = \sigma \sum_n \gamma_{nm} (\hat{u}_m - \hat{u}_n)$
- Pattern formation condition for $(\sigma, T)$

$$\frac{\partial z(t, p)}{\partial t} = f(z(t, p))$$
ε₃: Enhancing rotational symmetry

- Utilizing the inherent spatial symmetry

\[ w(t, x, y) = \sigma(R_{3\pi/2}u(t, x, y) - u(t, x, y)) \]

- Stability for residual modes guaranteed by the small-gain theorem

\[ \hat{w}_m = \sigma(\hat{u}_m) \]

No Distributed Actuation !!!!
Conclusion

- Formulated and solved feedback stabilization of hidden Turing patterns
- Unavailability of the target profile was managed by synchronization formulation
- Guided self-organization
  - Semi-conductor engineering

Acknowledgement

- Toshiyuki Ogawa (Dynamical system theory)
- Tatsunari Sakurai (Physics)

  - Selective pattern formation control: Spatial spectrum consensus and Turing instability approach, *Automatica* 2015