Selective Pattern Formation in Reaction-Diffusion Systems

Kenji Kashima (Kyoto Univ.)

Joint work with:
T. Ogawa (Dynamical system theory)
T. Sakurai (Physics)

Selective pattern formation control: Spatial spectrum consensus and Turing instability approach, *Automatica* 2015

Turing Pattern

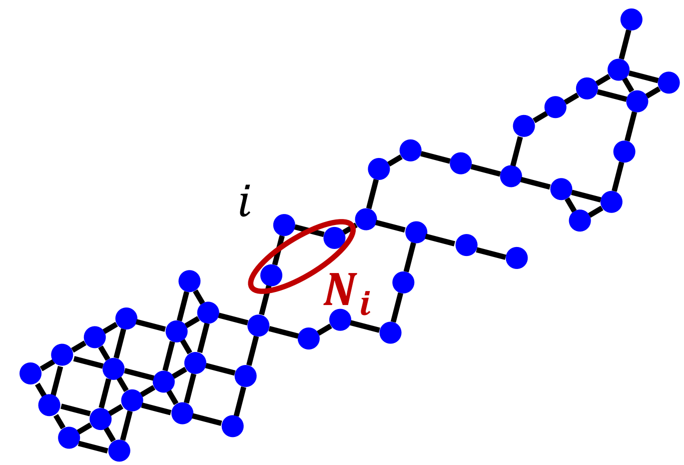
- The Chemical Basis of Morphogenesis (1952)
- A key mechanism for pattern formation
  - Chemical reaction: Belousov-Zhabotinsky
  - Biosystems: sk

- Specially unifo

Outline

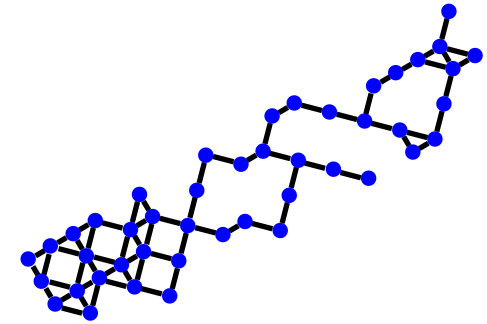
- Preliminary: Diffusively coupled identical (stable) systems
  - Synchronization
  - Turing Instability
- Problem formulation: Nonlinear PDE with unknown target state
- Approach: Spatial spectrum consensus
- Simulation and concluding remarks

Synchronization



- Subsystem dynamics
  - $\dot{x}_i(t) = Ax_i(t) + Du_i(t)$
- Rendezvous, synchronization
  - $\|x_i - x_j\| \to 0, t \to \infty \;\; \forall i, j$
- Diffusive coupling
  - $u_i(t) = \sum_{i \in N_i} \gamma_{ij}(x_j(t) - x_i(t))$
    - $\gamma_{ij} \geq 0 \; (j \neq i)$, connection topology, strength

Closed-loop dynamics



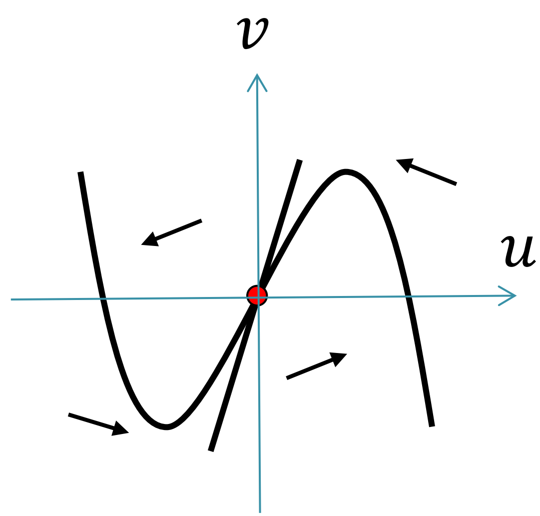
- $\dot{x} = (A \otimes I_n + D \otimes \Gamma)x$
  - $x := [x_1 \quad \cdots \quad x_n]^{\mathrm{T}}$
  - Γ: Graph Laplacian
    - Eigenvalues: $0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots$
- $\mathrm{eig}(A \otimes I_n + D \otimes \Gamma) = \bigcup_{\lambda \in \mathrm{eig}(\Gamma)} \mathrm{eig}(A + \lambda D)$
- Special case: $D = cI, c > 0$
  - Stability/Instability preserving: $\lambda_A + c\lambda_\Gamma$
    - Consensus achieved
- Other cases: Not necessarily true

Formulation by reaction-diffusion

- Reaction (stable) $\frac{dz}{dt} = f(z)$ $z(t)$
  - GAS equil. $z = 0$

- Diffusion $\frac{\partial z}{\partial t} = f(z) + D\Delta z,\ \xi \in \mathrm{P}$ $z(t, \xi)$
  - Diagonal $D \succcurlyeq 0$
  - Is $z_{eq}(\xi) \equiv 0$ stable as well?

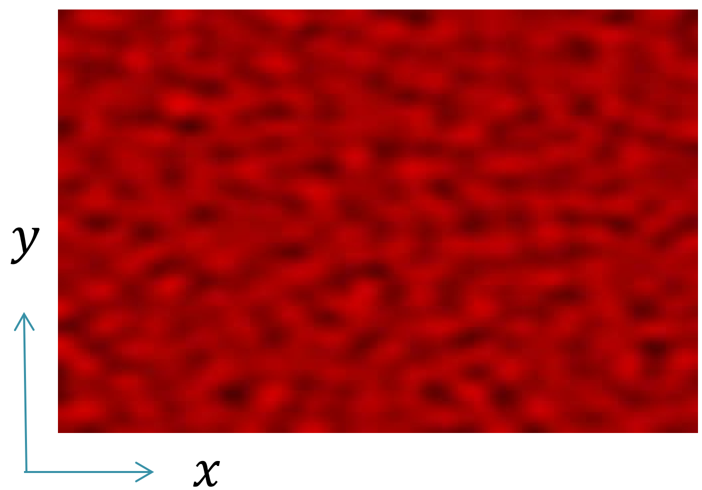
- Example:
  - $\dot{u} = u - v - u^3 + d_u \Delta u$
  - $\dot{v} = 3u - 2v \quad + d_v \Delta v$



Rectangular Domain Example

- $\dot{u} = u - v - u^3 + d_u \Delta u$
- $\dot{v} = 3u - 2v \quad + d_v \Delta v$
- $d_v / d_u \gg 1$

$u(t, x, y)$



Mathematical model

- Reaction-Diffusion
  - $$\begin{cases} \dot{u} = a_{11}u + a_{12}v - u^3 + d_u \Delta u \\ \dot{v} = a_{12}u + a_{22}v \qquad\qquad + d_v \Delta v \end{cases}$$

  - Square domain, periodic boundary conditions
- This toy models is…
  - an activator-inhibitor RD systems with cubic nonlinearities
  - able to captures the essential mathematical structure of spatial symmetry
  - essentially equivalent to
    - phase-field model, the Cahn-Hilliard model,
      the Ginzburg-Landau equation,
      the Swift-Hohenberg equation, and so on

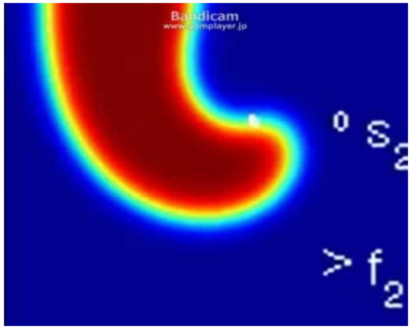
Outline

- Preliminary: Diffusively coupled identical (stable) systems
  - Multi-agent systems
  - Turing Instability
- Problem formulation: Nonlinear PDE with unknown target state
- Approach: Spatial spectrum consensus
- Simulation and concluding remarks

Control of pattern formation

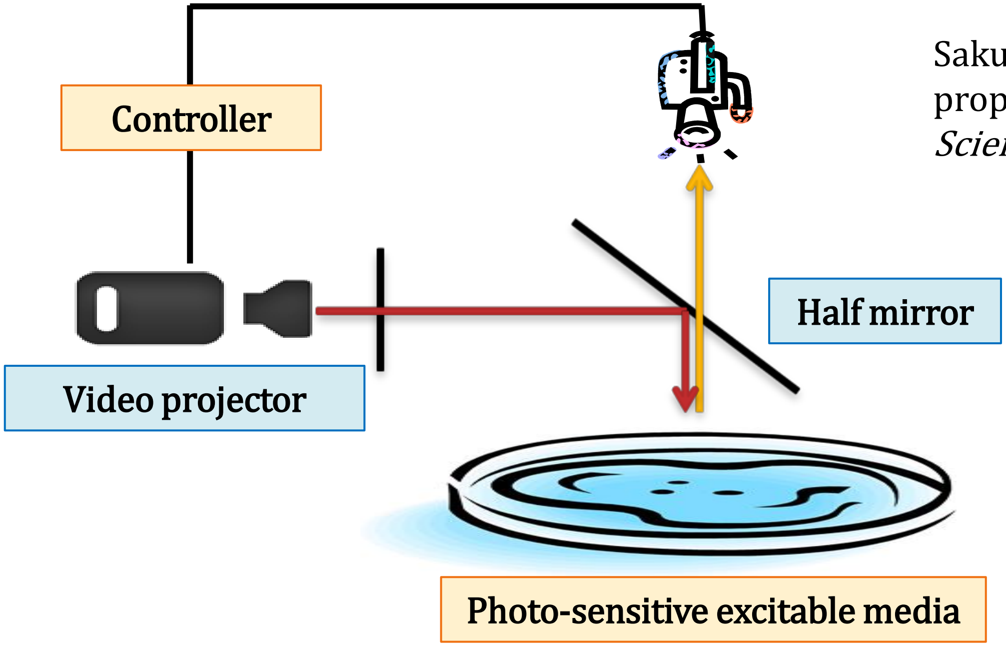
- Cardiac arrhythmias

S. Luther et al., Low-energy control of electrical turbulence in the heart, Nature (2011)



- Chemical reaction

$$\frac{\partial z(t,p)}{\partial t} = f(z(t,p)) + D\Delta z(t,p) + \boldsymbol{w(t,p)}$$

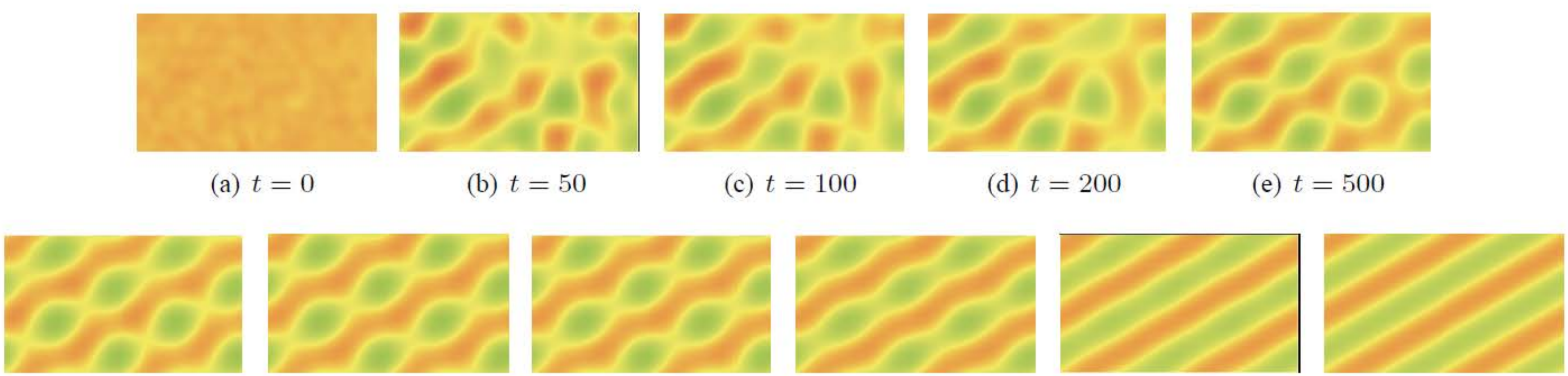


Sakurai et al., Design and control of wave propagation patterns in excitable media, *Science* (2001)

What we attempt to do is …

- not to generate an arbitrarily given pattern by forcing some inputs

- but to assist a (possibly hidden) pattern generation

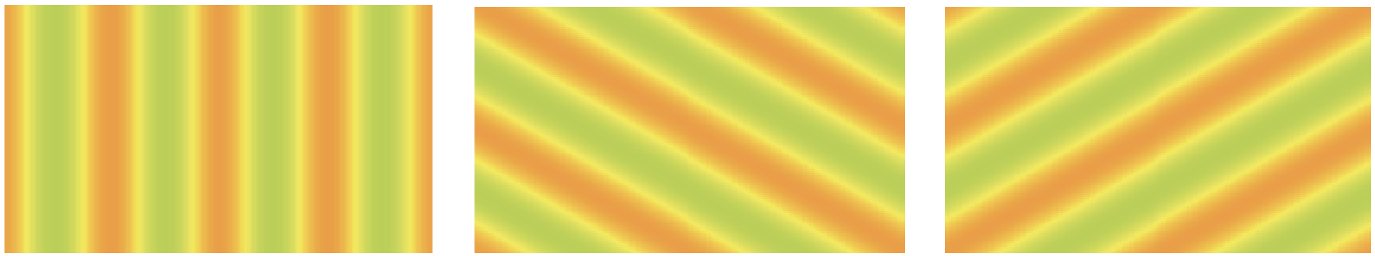


(a) $t = 0$ (b) $t = 50$ (c) $t = 100$ (d) $t = 200$ (e) $t = 500$

(f) $t = 1000$ (g) $t = 5000$ (h) $t = 10000$ (i) $t = 15000$ (j) $t = 20000$ (k) $t = 30000$

Assisted pattern formation

- Example revisited:
  - Initial pattern dependent convergence to one of:



- Can we generate their superposition as well?

Spatial Fourier Transformation

- Basis functions are stripes labeled by $m \in \mathbb{Z}^2$
  - $p_m(x, y) = e^{2\pi j(m_x x + m_y y)}$
  - $u(t, x, y) = \sum_m \hat{u}_m(t) p_m(x, y)$, $\hat{v}_m(t)$
- Reaction-Diffusion
  - $\begin{cases} \dot{u} = a_{11}u + a_{12}v - u^3 + d_u \Delta u \\ \dot{v} = a_{12}u + a_{22}v \qquad\quad + d_v \Delta v \end{cases}$
  - $\begin{cases} \dot{\hat{u}}_m = (a_{11} - s_m d_u)\hat{u}_m + a_{12}\hat{v}_m - \hat{u}_m(couple) \\ \dot{\hat{v}}_m = a_{12}\hat{u}_m + (a_{22} - s_m d_v)\hat{v}_m \end{cases}$
  - $s_m := \left(\frac{2\pi m_x}{L_x}\right)^2 + \left(\frac{2\pi m_y}{L_y}\right)^2$

$(A - s_m D)\begin{bmatrix} \hat{u}_m \\ \hat{v}_m \end{bmatrix}$

Problem Formulation

- Controlled Reaction-Diffusion
  - $\begin{cases} \dot{u} = a_{11}u + a_{12}v - u^3 + d_u \Delta u \\ \dot{v} = a_{12}u + a_{22}v \qquad\quad + d_v \Delta v \end{cases}$
  - $\mathcal{M} := \{m : A - s_m D \text{ is unstable}\}$



(4,0) (2,2) (2, −2)

- Control objective: in large t limit
  - $u(t,x,y) \to \bar{u}(t) \sum_{m \in \mathcal{M}} p_m(x,y)$
  - $\|\bar{u}(t)\| \nrightarrow 0, \infty$
  - $w(t,x,y) \to 0$

  - Remarks
    - Not to maximize $\|\bar{u}(t)\|$
    - Not to regulate $u(t,x,y) \to u^{\circ}(x,y)$

Outline

- Preliminary: Diffusively coupled identical (stable) systems
  - Multi-agent systems
  - Turing Instability
- Problem formulation: nonlinear PDE with unknown target state
- Approach: Spatial spectrum consensus
- Concluding remarks

Assumptions

$$s_m := \left(\frac{2\pi m_x}{L_x}\right)^2 + \left(\frac{2\pi m_y}{L_y}\right)^2$$

- Controlled RD
  - $$\begin{cases} \dot{\hat{u}}_m = (a_{11} - s_m d_u)\hat{u}_m + a_{12}\hat{v}_m - \hat{u}_m(couple) + \hat{w}_m \\ \dot{\hat{v}}_m = a_{12}\hat{u}_m + (a_{22} - s_m d_v)\hat{v}_m \end{cases}$$

    $$(A - s_m D)\begin{bmatrix}\hat{u}_m \\ \hat{v}_m\end{bmatrix}$$

    for $m \in \mathbb{Z}^2$

$(4,0)$ $(2,2)$ $(2,-2)$

  - $\mathcal{M} = \{m: A - s_m D \text{ is unstable}\}$
  - Approximation: $\hat{u}_m, \hat{v}_m \approx 0$ for $m \notin \mathcal{M}$

- Assumption
  - $s_m = \bar{s}$ for all $m \in \mathcal{M}$

Theorem: Spatial spectrum consensus

- Consider the truncated coupled $\mathbb{C}^2$-valued dynamics

$$\begin{cases} \dot{\hat{u}}_m = (a_{11} - \bar{s} d_u)\hat{u}_m + a_{12}\hat{v}_m - \hat{u}_m(couple) + \hat{w}_m \\ \dot{\hat{v}}_m = a_{12}\hat{u}_m + (a_{22} - \bar{s} d_v)\hat{v}_m \end{cases}$$

$$(A - \bar{s}D)\begin{bmatrix} \hat{u}_m \\ \hat{v}_m \end{bmatrix}$$

for $m \in \mathcal{M}$. Then,

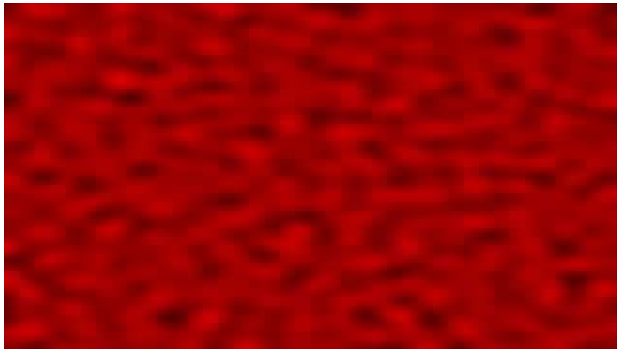
$$u(t,x,y) \to \bar{u}(t) \sum_{m \in \mathcal{M}} \boldsymbol{p_m}(x,y)$$
$$\|\bar{u}(t)\| \nrightarrow 0, \infty$$
$$w(t,x,y) \to 0$$

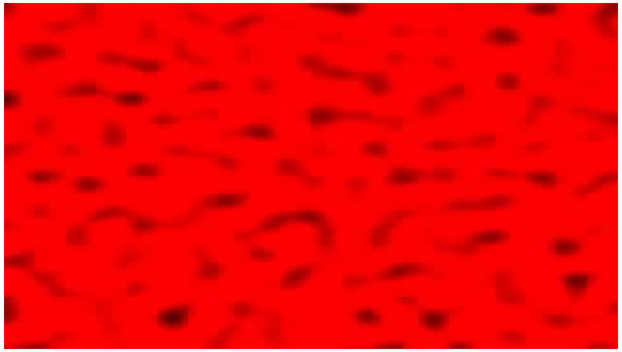
$A - \bar{s}D$ is unstable

Embedding onto PDE

- Controller implementation
  - $w(t,x,y) = \sum_{m\in\mathcal{M}} \widehat{w}_m(t) p_m(x,y)$
  - $\widehat{w}_m = \sigma \sum_{n\in\mathcal{M}} \gamma_{nm}(\hat{u}_m - \hat{u}_n)$

$$\begin{cases} \dot{u} = a_{11}u + a_{12}v - u^3 + d_u \Delta u + w \\ \dot{v} = a_{12}u + a_{22}v \qquad + d_v \Delta v \end{cases}$$

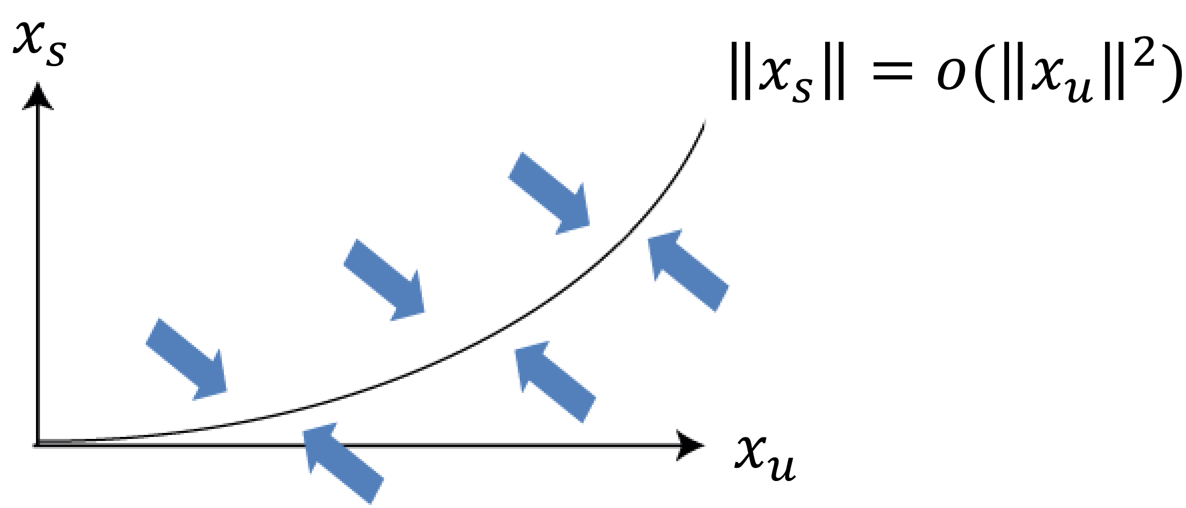




Controlled ($w(t,p) \to 0$)

Center manifold theorem

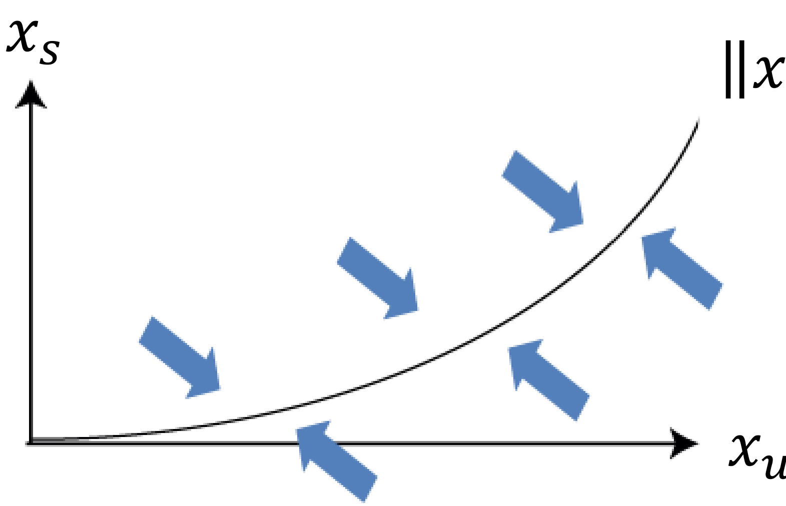
- $\dot{x}_s = f_s{}^q(x_s, x_u), \ \ \dot{x}_u = f_u{}^q(x_s, x_u)$
  - Local stability: x_s stable, x_u unstable for $q > q^*$
- When q is slightly larger than q^*,
  - Existence of an invariant set
  - If the dynamics on the invariant set is stable, then this set is attractive.



Application to pattern formation

- Reaction-Diffusion
  - $\begin{cases} \dot{u} = a_{11}u + a_{12}v - u^3 + d_u \Delta u \\ \dot{v} = a_{12}u + a_{22}v \qquad\quad + d_v \Delta v \end{cases}$ 
  - $\begin{cases} \dot{\hat{u}}_m = (a_{11} - s_m d_u)\hat{u}_m + a_{12}\hat{v}_m - \hat{u}_m(couple) \\ \dot{\hat{v}}_m = a_{12}\hat{u}_m + (a_{22} - s_m d_v)\hat{v}_m \end{cases}$
  - $s_m := \left(\frac{2\pi m_x}{L_x}\right)^2 + \left(\frac{2\pi m_y}{L_y}\right)^2$

$$(A - s_m D)\begin{bmatrix} \hat{u}_m \\ \hat{v}_m \end{bmatrix}$$

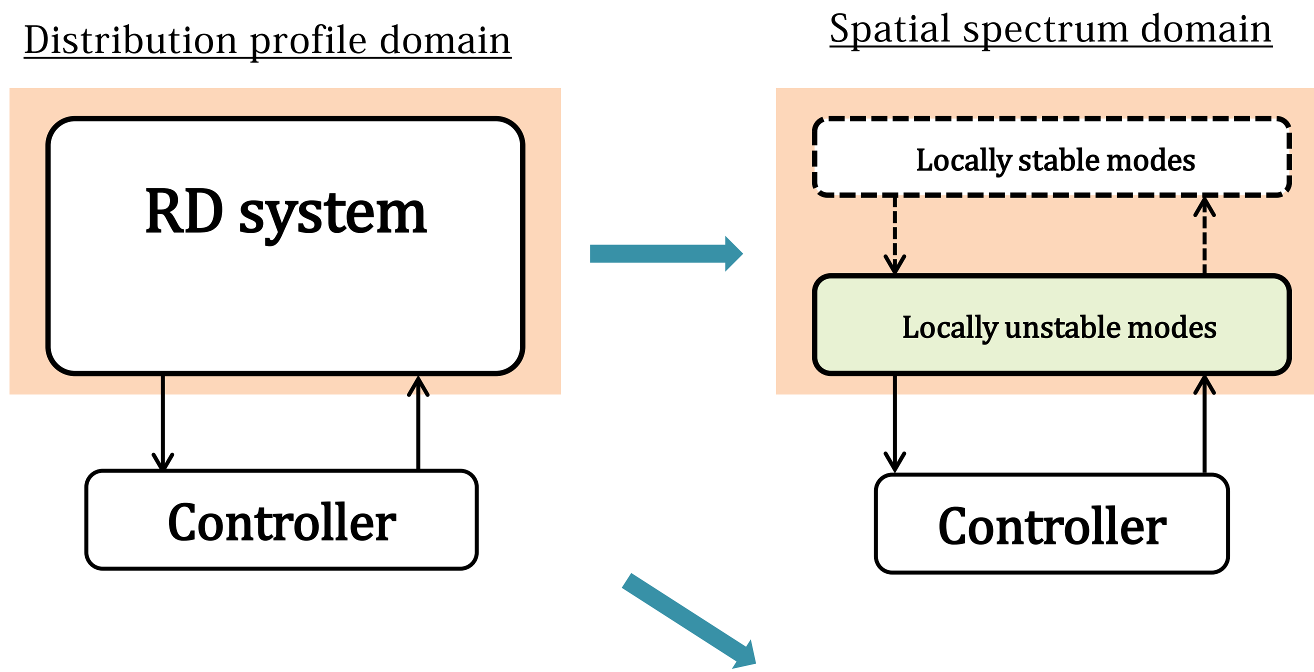


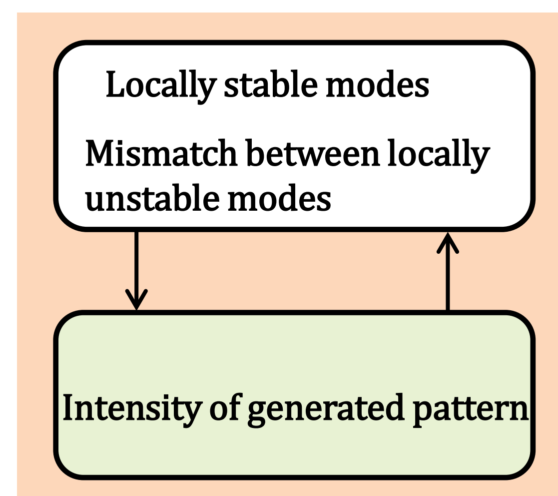
$\|x_s\| = o(\|x_u\|^2)$

$\mathcal{M} := \{m : A - s_m D \text{ is unstable}\}$

$(u_m, v_m)_{m \in \mathcal{M}}$

Spillover





Technical assumptions guaranteed by suitable Lyapunov functionals

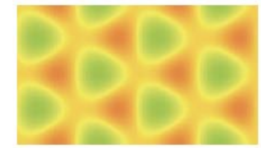
$$\boldsymbol{J}(\boldsymbol{z}(t,x,y)) := \iint_{\Omega} V(\boldsymbol{u}(t,x,y), \boldsymbol{v}(t,x,y))dxdy.$$

$$\dot{\boldsymbol{J}}(\boldsymbol{z}(t,x,y)) \leq -\iint_{\Omega} H(\boldsymbol{u}(t,x,y), \boldsymbol{v}(t,x,y))dxdy.$$

Center manifold for closed-loop PDE

- $$\begin{cases} \dot{u} = a_{11}u + a_{12}v - u^3 + d_u\Delta u + w \\ \dot{v} = a_{12}u + a_{22}v \qquad\qquad + d_v\Delta v \end{cases}$$

Notation:



- Orthogonal decomposition
  - $u(t,x,y) = \bar{u}(t)\sum_{m\in\mathcal{M}} \boldsymbol{p}_{\boldsymbol{m}}(x,y) + u_{res}(t,x,y)$
- Autonomous pattern formation for $d_v > d^*$

- Theorem:

For any $\epsilon > 0$, there exist $\delta, T > 0$ such that

- $\|u(0)\| < \delta, d_v < d^* + \delta \implies$

$$\|u_{res}(t)\| + \|w(t)\| < \epsilon\|\bar{u}(t)\| \text{ for } t > T$$

Residual component Input Intensity of generated pattern

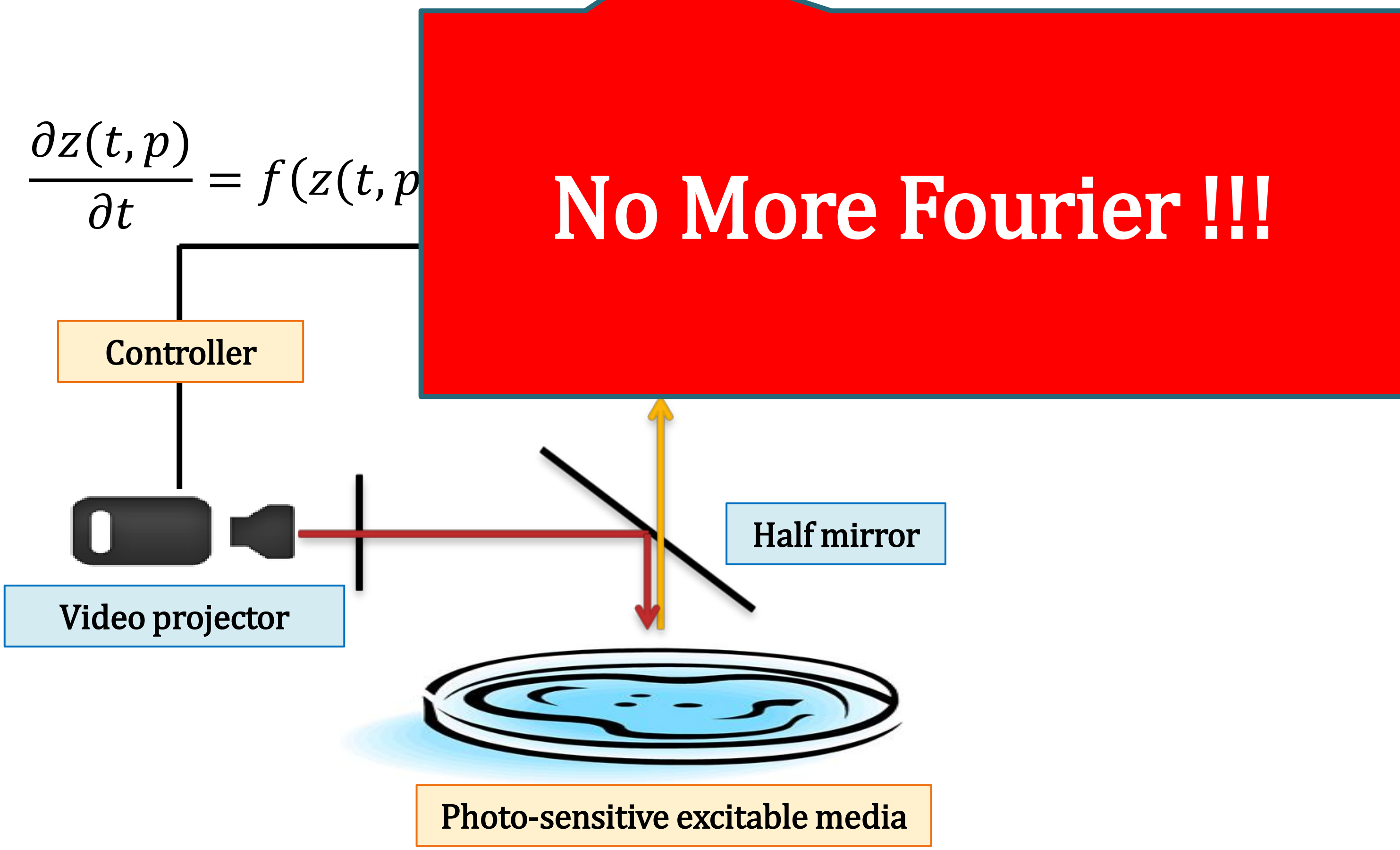
ε_1: Dynamic attractor

- Stabilization of standing waves
  - Autonomous traveling waves

Controlled ($\boldsymbol{w(t,p) \to 0}$)

ε_2: Delay robustness

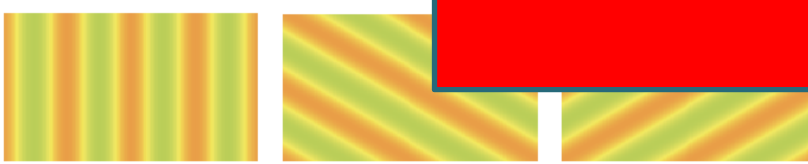
- $w(t,x,y) = \sigma \sum_{m\in\mathcal{M}} \hat{w}_m(t - {\color{red}T})p_m(x,y)$
  $\hat{w}_m = \sigma \sum_n \gamma_{nm}(\hat{u}_m - \hat{u}_n)$
- Pattern formation condition for (σ, T)

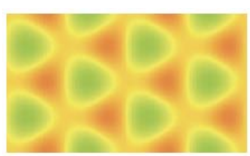


ε_3: Enhancing rotational symmetry

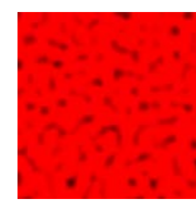
- Utilizing the inherent spatial symmetry
- $w(t,x,y) = \sigma(\mathcal{R}_{3\pi/2}u(t,x,y) - u(t,x,y))$
  - $\hat{w}_m = \sigma(\hat{u}_r$
  - Stability fo
    small- gain

No Distributed Actuation !!!





uncontrolled



Controlled ($w(t,p) \to 0$)

Conclusion

- Formulated and solved feedback stabilization of hidden Turing patterns
- Unavailability of the target profile was managed by synchronization formulation
- Guided self-organization
  - Semi-conductor engineering

- Acknowledgement
  - Toshiyuki Ogawa (Dynamical system theory)
  - Tatsunari Sakurai (Physics)

  - Selective pattern formation control: Spatial spectrum consensus and Turing instability approach, *Automatica* 2015