Optimal Damping Coefficient of Some Elastic Systems

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Outline

- Introduction
- Fastest decay rate
- Least total energy
- further study

Consider a first order linear evolution equation on Hilbert space $\mathcal{H}$

$$\frac{dz}{dt}(t) = \mathcal{A}z(t), \qquad z(0) = z_0.$$

where $\mathcal{A}$ generates a C_0 semigroup, $e^{\mathcal{A}t}$, of contractions on $\mathcal{H}$.

- The growth rate of $e^{\mathcal{A}t}$ is defined by

$$\omega_0(\mathcal{A}) = \inf\{\omega \mid \|z(t)\| \leq Me^{\omega t}\}.$$

- We say $e^{\mathcal{A}t}$ satisfies the spectrum-determined growth property if

$$\omega_0(\mathcal{A}) = \sup\{\Re\lambda \mid \lambda \in \sigma(\mathcal{A})\}.$$

Example 1: Consider a wave equation with viscous damping

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) - 2au_t(x,t), & x \in (0,1), \quad t > 0 \\ u(0,t) = u(1,t) = 0, & t \geq 0 \\ u(x,0) = u_0(x), & u_t(x,0) = u_1(x) \end{cases}$$

where $a > 0$, the damping coefficient, is a constant.
Let

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1)$$

with norm $\|z_a\|^2 = \|(u,v)\|_{\mathcal{H}} = \|u_x\|^2 + \|v\|^2$. Then

$$\begin{cases} \frac{dz_a}{dt} = \mathcal{A}_a z_a \\ z_a(0) = z_0 \end{cases}$$

where

$$\mathcal{A}_a = \begin{bmatrix} 0 & I \\ D^2 & -2aI \end{bmatrix}.$$

- $\mathcal{A}_a$ generates a C_0 semigroup of contraction $e^{\mathcal{A}_a t}$ on $\mathcal{H}$
- $e^{\mathcal{A}_a t}$ is exponentially stable, i.e., there exists $M > 0, \omega_a < 0$ such that

$$\|z_a\|_{\mathcal{H}} \leq M\|z_0\|_{\mathcal{H}} e^{\omega_a t}, \qquad t > 0.$$

- spectrum determined growth property (Cox and Zuazua, 1993), i.e.,

$$\omega_0(\mathcal{A}_a) = \min \omega_a = \sup_{\lambda \in \sigma(\mathcal{A}_a)} Re\lambda.$$

It is easy to find the eigenvalues

$$\lambda_n = -a \pm \sqrt{a^2 - n^2\pi^2}, \qquad n = 1, 2, \cdots .$$

$$\omega_0(\mathcal{A}_a) = \sup_n \mathrm{Re}\lambda_n = \begin{cases} -a & a \leq \pi \\ -a + \sqrt{a^2 - \pi^2} & a > \pi \end{cases}$$

Optimal Damping Coefficient

Theorem

$$\min_a \omega_0(\mathcal{A}) = -\pi \qquad \textit{when} \quad a = \pi.$$

Example 2: Consider a wave equation with Kelvin-Voigt damping

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + 2au_{xxt}(x,t), \qquad x \in (0,1), \quad t > 0 \\ u(0,t) = u(1,t) = 0, \qquad t \geq 0 \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \end{cases}$$

where $a > 0$, the damping coefficient, is a constant.
Let

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1)$$

with norm $\|z_a\|^2 = \|(u,v)\|_{\mathcal{H}} = \|u_x\|^2 + \|v\|^2$. Then

$$\begin{cases} \frac{dz_a}{dt} = \mathcal{A}_a z_a \\ z_a(0) = z_0 \end{cases}$$

where

$$\mathcal{A}_a = \begin{bmatrix} 0 & I \\ D^2 & 2aD^2 \end{bmatrix}.$$

- $\mathcal{A}_a$ generates a C_0 semigroup of contraction $e^{\mathcal{A}_a t}$ on $\mathcal{H}$
- $e^{\mathcal{A}_a t}$ is exponentially stable, i.e., there exists $M > 0, \omega_a < 0$ such that

$$\|z_a\|_{\mathcal{H}} \leq M\|z_0\|_{\mathcal{H}} e^{\omega_a t}, \qquad t > 0.$$

- spectrum determined growth property, i.e.,

$$\omega_0(\mathcal{A}_a) = \min \omega_a = \sup_{\lambda \in \sigma(\mathcal{A}_a)} \mathrm{Re}\lambda.$$

It is easy to find the eigenvalues

$$\lambda_n = -an^2\pi^2 \pm \sqrt{a^2 n^4 \pi^4 - n^2\pi^2}, \qquad n = 1, 2, \cdots .$$

$$\omega_0(\mathcal{A}_a) = \sup_n \mathrm{Re}\lambda_n = \begin{cases} -a\pi^2 & a \leq \frac{1}{\sqrt{2}\pi} \\ -\frac{1}{2a} & a > \frac{1}{\sqrt{2}\pi} \end{cases}$$

Optimal Damping Coefficient

Theorem

$$\min_a \omega_0(\mathcal{A}) = -\frac{\pi}{\sqrt{2}} \qquad \textit{when} \quad a = \frac{1}{\sqrt{2}\pi}.$$

Remark: There are sevearl disadvantage of using the growth rate of the semigroup as the objective function

- $\omega_0(\mathcal{A})$ is very difficult to compute, but the spectral abscissa is relatively easy to compute. Therefore, we need the spectrum determined growth property.
- The spectrum abssisa provide a uniform decay rate for all modes by using the decay rate of one mode. Therefore, it does not provide a complete picture.
- When the $\omega_0(\mathcal{A}) = 0$, this only implies that the semigroup is bounded, no decay information. For example, when the system has a sequence of eigenvalues approching the imaginary axis asymptotically from the left, we still expect the existence of optimal damping in certain sense.

We propose another objective function which measures the total system energy over the infinite time horizon.

$$\min_{a} \left(\max_{\|z_0\|=1} \int_0^{\infty} E(t)dt \right)$$

such that

$$\frac{dz}{dt} = \mathcal{A}_a z, \qquad z(0) = z_0.$$

where $E(t) = \|z(t)\|^2 = \|e^{\mathcal{A}_a t} z_0\|^2$

- This object function does not have the disadvantages of the previous objective function. If $e^{\mathcal{A}t}$ is polynomially stable of order great than 1, then the integral is finite.

We now consider a second order evolution equation on a Hilbert space H:

$$\begin{cases} u_{tt} + Au + Bu_t = 0, \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases}$$

where

- A is a self-adjoint, positive and densely defined operator on a Hilbert space H. B is a positive and densely defined operator on H.
- There exist constants $0 < c < C < \infty$, $C_0 > 0$ and $\alpha \neq 0$ such that

$$c\langle A^{\alpha} u, u\rangle \leq \Re\langle Bu, u\rangle \leq C\langle A^{\alpha} u, u\rangle,$$

and

$$\left|\Im\langle Bu, u\rangle\right| \leq C_0 \left|\Re\langle Bu, u\rangle\right|, \quad \forall\, u \in D(B).$$

The C_0 semigroup on $D(A^{1/2}) \times H$ generated by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A & -B \end{bmatrix}$$

is (G. Chen and D. Russell, F. Huang, S. Taylor, S. Chen and R. Triggianni)

- analytic for $\frac{1}{2} \leq \alpha \leq 1$,
- of Gevrey class of order $\delta = \frac{1}{2\alpha}$ for $0 < \alpha < \frac{1}{2}$,
- exponentially stable.

Theorem (Z. Liu and Q. Zhang, 2015)

$e^{\mathcal{A}t}$ is polynomially stable of order $\frac{1}{2|\alpha|}$ for $\alpha < 0$. Moreover, If eigenvalues of A are

$$0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \to \infty$$

as $n \to \infty$ this rate is the best polynomial rate for the system.

For example, the Raleigh beam medel with viscous damping

$$\begin{cases} u_{tt} - \gamma u_{xxtt} = -u_{xxxx} - a u_t, & x \in (0, L), \quad t > 0 \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0. \end{cases}$$

- If $\gamma > 0$, the associated semigroup is polynomially stable of order $|\alpha| = 1$.
- If $\gamma = 0$, the associated semigroup is exponentially stable.

Take

$$B = -aA^{-\alpha}, \qquad -1 < \alpha \leq 1.$$

where $a > 0$ is the damping coeffient.
Define a functional

$$J(a, z_0) = \int_0^\infty \|e^{\mathcal{A}_a t} z_0\|^2 dt = \int_0^\infty E(t, a, z_0) dt.$$

Find

$$\min_{a} \max_{\|z_0\|=1} J(a, z_0)$$

subject to

$$\begin{cases} \dfrac{dz_a}{dt} = \mathcal{A}_a z_a \\ z_a(0) = z_0 \end{cases}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & A^{1/2} \\ -A^{1/2} & -aA^{\alpha} \end{bmatrix}$$

Theorem

The optimal damping coefficient for the above problem is

$$a^* = \sqrt{2(\sqrt{5} - 1)\mu_0^{\frac{1}{2} - \alpha}}$$

where $\mu_0 = \min \sigma(A)$.

Sketch of Proof:

- $J(a, z_0) = \langle \mathcal{E}_a z_0, z_0 \rangle$ where

$$\mathcal{E}_a = \int_0^\infty e^{\mathcal{A}_a^* t} e^{\mathcal{A}_a t} dt$$

  which is the solution of the Lyapunov Equation

$$\mathcal{E}_a \mathcal{A}_a + \mathcal{A}_a \mathcal{E}_a = -I.$$

- Solve

$$\begin{bmatrix} E_1 & E_2 \\ E_2 & E_3 \end{bmatrix} \begin{bmatrix} 0 & A^{1/2} \\ -A^{1/2} & -aA^\alpha \end{bmatrix} + \begin{bmatrix} 0 & -A^{1/2} \\ A^{1/2} & -aA^\alpha \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ E_2 & E_3 \end{bmatrix}$$

$$= \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}.$$

- $$\mathcal{E}_a = \begin{bmatrix} \frac{a}{2}A^{-1+\alpha} + \frac{1}{a}A^{-\alpha} & \frac{1}{2}A^{-1/2} \\ \frac{1}{2}A^{-1/2} & \frac{1}{a}A^{-\alpha} \end{bmatrix}.$$

- Since
$$J(a) \equiv \max_{\|z_0\|=1} J(a, z_0) = \max \sigma(\mathcal{E}_a),$$
we calculate the eigenvalues of $\mathcal{E}_a$ from its characteristic equation
$$\lambda^2 - (\frac{a}{2}\mu^{-1+\alpha} + \frac{2}{a}\mu^{-\alpha})\lambda + (\frac{1}{a^2}\mu^{-2\alpha} + 4\mu^{-1}) = 0,$$
where μ is eigenvalue of A. We obtain
$$\lambda = \frac{a}{4}\mu^{-1+\alpha} + \frac{1}{a}\mu^{-\alpha} + \frac{1}{4}\sqrt{a^2\mu^{-2+2\alpha} + 4\mu^{-1}}.$$

- Thus,
$$J(a) = \frac{a}{4}\mu_0^{-1+\alpha} + \frac{1}{a}\mu_0^{-\alpha} + \frac{1}{4}\sqrt{a^2\mu_0^{-2+2\alpha} + 4\mu_0^{-1}}.$$

where μ_0 is the smallest eigenvalue of A.

- It is now straight forward to show that

$$a^* = \sqrt{2(\sqrt{5}-1)}\mu_0^{\frac{1}{2}-\alpha}.$$

- It is interesting to see that the optimal damping coefficient does not depend on α for structural damping.

Consider an abstract thermoelastic system

$$\begin{cases} u_{tt} = -Au + \gamma A^{1/2}\theta \\ \theta_t = -\gamma A^{1/2} u_t - aA\theta \end{cases}$$

on $H \times H$.
Find

$$\min_{a} \max_{\|z_0\|=1} J(a, z_0)$$

subject to

$$\begin{cases} \dfrac{dz_a}{dt} = \mathcal{A}_a z_a \quad \text{on} \quad H \times H \times H \\ z_a(0) = z_0 \end{cases}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & A^{1/2} & 0 \\ -A^{1/2} & 0 & bA^{1/2} \\ 0 & -bA^{1/2} & -aA \end{bmatrix}$$

Take $\gamma = 1$.

It is known that the associated semigroup is exponentially stable. Solve the Lyapunov equation

$$\mathcal{E}_a \mathcal{A}_a + \mathcal{A}_a \mathcal{E}_a = -I.$$

We have

$$\mathcal{E}_a = \begin{bmatrix} aI + \frac{3}{2a}A^{-1} & \frac{1}{2}A^{-1/2} & -\frac{1}{2a}A^{-1} \\ \frac{1}{2}A^{-1/2} & aI + \frac{2}{a}A^{-1} & A^{-1/2} \\ -\frac{1}{2a}A^{-1} & A^{-1/2} & \frac{3}{2a}A^{-1} \end{bmatrix}$$

The characteristic equation of $\mathcal{E}$ is

$$\lambda^3 + \left(-\frac{5}{a\mu} - 2a\right)\lambda^2 + \left(\frac{8}{a^2\mu^2} + a^2 + \frac{21}{4\mu}\right)\lambda - \frac{4}{a^3\mu^3} - \frac{21}{8a\mu^2} - \frac{a}{2\mu} = 0,$$

where μ is eigenvalue of A.

After a long computation by the Cardano formula, the three positive real roots are

$$\lambda_0 = \frac{2a^2\mu + 5 + \sqrt{(a^2\mu + 4)(4a^2\mu + 1)}\cos\left(\frac{\theta}{3}\right)}{3a\mu},$$

$$\lambda_+ = \frac{4a^2\mu + 10 - \sqrt{(a^2\mu + 4)(4a^2\mu + 1)}\left(\sqrt{3}\sin\left(\frac{\theta}{3}\right) + \cos\left(\frac{\theta}{3}\right)\right)}{6a\mu},$$

$$\lambda_- = \frac{4a^2\mu + 10 + \sqrt{(a^2\mu + 4)(4a^2\mu + 1)}\left(\sqrt{3}\sin\left(\frac{\theta}{3}\right) - \cos\left(\frac{\theta}{3}\right)\right)}{6a\mu}.$$

where

$$\theta = \arccos\left(-\frac{(4a^2\mu + 1)\left(4a^4\mu^2 + 11a^2\mu + 16\right)}{2\left(4a^4\mu^2 + 17a^2\mu + 4\right)^{3/2}}\right).$$

It can be proved that

$$\lambda_0 > \lambda_- > \lambda_+.$$

Thus,

$$J(a) = \frac{2a^2\mu_0 + 5 + \sqrt{(a^2\mu_0 + 4)(4a^2\mu_0 + 1)}\cos\left(\frac{\theta}{3}\right)}{3a\mu_0},$$

We can prove that $J(a)$ is a convex function of a. The unique minimum point a^* exists. Unfortunately, we are not able to obtain an explicit expression of a^*.

Remark:

- a, γ represent the heat conductivity and thermal strain, respectively. The above result gives the best conductivity when the thermal strain is fixed.
- We can fix a and find an optimal value for γ, or find the optimal pair (α^*, γ^*).

We consider the following Abstract Coupled System

$$\begin{array}{l} u_{tt} = -Au + \gamma A^{\alpha}\theta, \qquad \text{on} \quad H \\ \theta_t = -\gamma A^{\alpha} u_t - kA^{\beta}\theta, \qquad \text{on} \quad H \\ u(0) = u_0, \quad u_t(0) = v_0, \quad \theta(0) = \theta_0 \end{array}$$

The corresponding semigroup generator is

$$\mathcal{A}_{\alpha,\beta} = \begin{pmatrix} 0 & I & 0 \\ -A & 0 & \gamma A^{\alpha} \\ 0 & -\gamma A^{\alpha} & -kA^{\beta} \end{pmatrix}$$

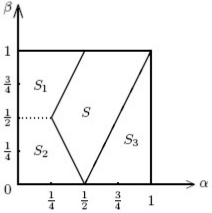


Fig.2

$$S_1 = \left\{ (\alpha, \beta) |\ 0 < \beta - 2\alpha,\ \alpha \geq 0,\ \frac{1}{2} \leq \beta \leq 1 \right\},$$

$$S_2 = \left\{ (\alpha, \beta) |\ \beta + 2\alpha < 1,\ \alpha \geq 0,\ 0 \leq \beta \leq \frac{1}{2} \right\},$$

$$S_3 = \{ (\alpha, \beta) |\ 2\alpha - \beta > 1,\ \alpha \leq 1,\ 0 \leq \beta \}.$$

Theorem (J. Hao and Z. Liu)

The semigroup $e^{\mathcal{A}_{\alpha,\beta}t}$

- *is polynomially stable of optimal order* $j_1 = \frac{1}{2(\beta-2\alpha)}$ *in the region* S_1*;*
- *is polynomially stable of optimal order* $j_2 = \frac{1}{2-2(\beta+2\alpha)}$ *in the region* S_2*;*
- *is instable in the region* S_3*.*

Abstract system of two second order evolution equations

$$\begin{cases} u_{tt} = -A^{\gamma} u + bA^{\alpha} y_t, \\ y_{tt} = -aAy - bA^{\alpha} u_t - A^{\beta} y_t, \\ u(0) = u_0, \quad u_t(0) = v_0, \quad y(0) = y_0, \quad y_t(0) = w_0. \end{cases}$$

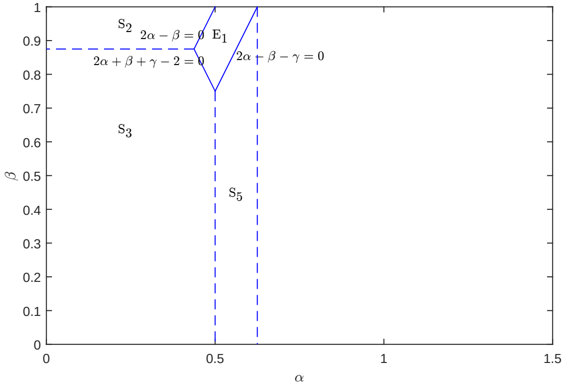
$$(\alpha, \beta, \gamma) \in E = [0, \frac{\gamma+1}{2}] \times [0,1] \times [0,2]$$

Let

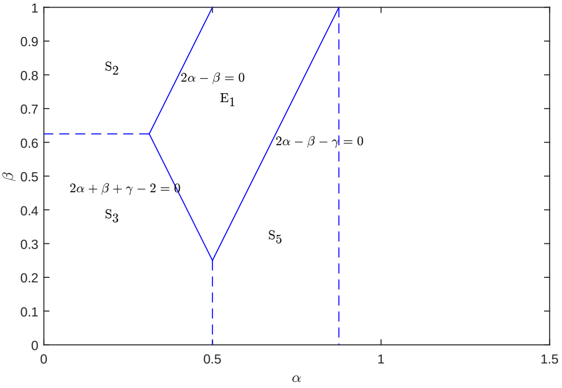
$$\mathcal{A}_{\alpha,\beta,\gamma} = \begin{pmatrix} 0 & I & 0 & 0 \\ -aA^{\gamma} & 0 & 0 & bA^{\alpha} \\ 0 & 0 & 0 & I \\ 0 & -bA^{\alpha} & -A & -kA^{\beta} \end{pmatrix}.$$

We partition E into six subregions $S_1, \cdots, S_6$.

$$\begin{cases} S_{1,1} = \Big\{(\alpha,\beta,\gamma) \in E \mid 2\alpha-\beta \geq 0, 2\alpha-\beta-\gamma \leq 0, 2\alpha+\beta+\gamma \geq 2, 0 < \gamma \leq 1\Big\}, \\ S_{1,2} = \Big\{(\alpha,\beta,\gamma) \in E \mid 2\alpha-\beta \geq 0, 2\alpha-\beta-\gamma \leq 0, 2\alpha+\beta-\gamma \geq 0, 1 \leq \gamma \leq 2\Big\}, \\ S_{2,1} = \Big\{(\alpha,\beta,\gamma) \in E \mid 2\alpha-\beta < 0, 1-\frac{\gamma}{2} \leq \beta, 0 < \gamma \leq 1, \Big\}, \\ S_{2,2} = \Big\{(\alpha,\beta,\gamma) \in E \mid 2\alpha-\beta < 0, \frac{\gamma}{2} \leq \beta, 1 < \gamma \leq 2\Big\}, \\ S_{5,1} = \Big\{(\alpha,\beta,\gamma) \in E \mid 2\alpha-\beta-\gamma > 0, \alpha > \frac{1}{2}, 0 < \gamma \leq 1\Big\}, \\ S_{5,2} = \Big\{(\alpha,\beta,\gamma) \in E \mid 2\alpha-\beta-\gamma > 0, \alpha > \frac{\gamma}{2}, 1 < \gamma \leq 2\Big\}, \\ S_1 = S_{1,1} \cup S_{1,2}, \\ S_2 = S_{2,1} \cup S_{2,2}, \\ S_3 = \Big\{(\alpha,\beta,\gamma) \in E \mid 2\alpha+\beta+\gamma < 2, \beta < 1-\frac{\gamma}{2}, \alpha \leq \frac{1}{2}, 0 < \gamma \leq 1\Big\}, \\ S_4 = \Big\{(\alpha,\beta,\gamma) \in E \mid 2\alpha+\beta-\gamma < 0, \beta < \frac{\gamma}{2}, \alpha \leq \frac{\gamma}{2}, , 1 < \gamma \leq 2\Big\}, \\ S_5 = S_{5,1} \cup S_{5,2}, \\ S_6 = \Big\{(\alpha,\beta,\gamma) \in E \mid \gamma = 0, \alpha \neq \frac{1}{2}, \beta \neq 1\Big\}, S_8 = (\frac{1}{2}, 1, 0) \end{cases}$$



S2
2α − β = 0
E1
2α + β + γ − 2 = 0
2α − β − γ = 0
S3
S5
β
α
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5


S$_2$
$2\alpha - \beta = 0$
E$_1$
$2\alpha - \beta - \gamma = 0$
$2\alpha + \beta + \gamma - 2 = 0$
S$_3$
S$_5$
β
α
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1
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1
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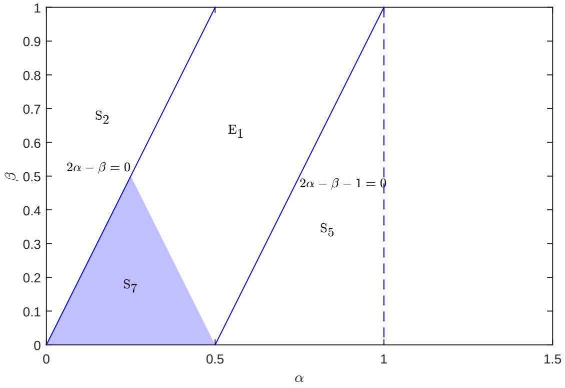

S_2
E_1
$2\alpha - \beta = 0$
$2\alpha - \beta - 1 = 0$
S_5
S_7
β
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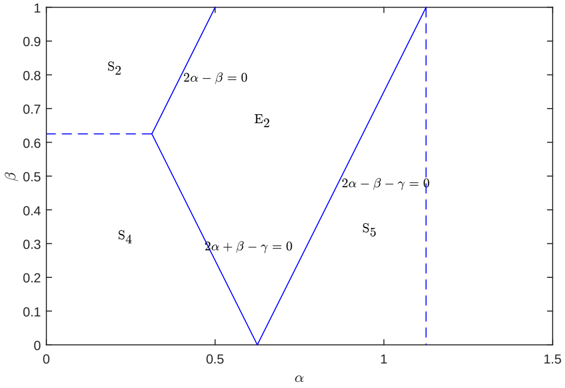
1
1.5



1
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β
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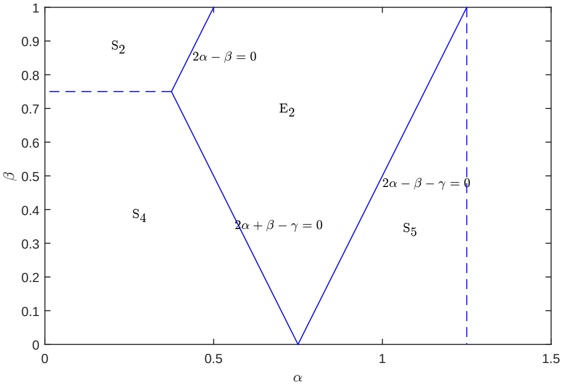
S_2
$2\alpha - \beta = 0$
E_2
$2\alpha - \beta - \gamma = 0$
S_4
$2\alpha + \beta - \gamma = 0$
S_5



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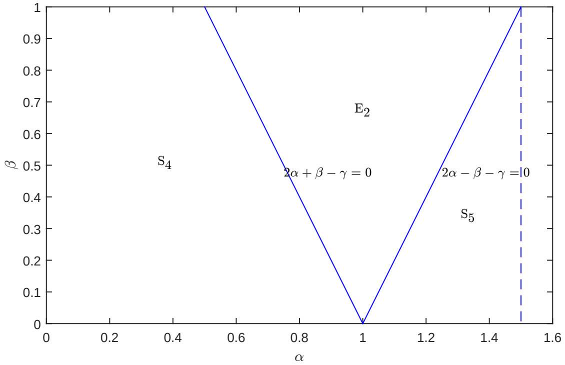
S_2
$2\alpha - \beta = 0$
E_2
S_4
$2\alpha + \beta - \gamma = 0$
$2\alpha - \beta - \gamma = 0$
S_5



1
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β
S_4
E_2
$2\alpha + \beta - \gamma = 0$
$2\alpha - \beta - \gamma = 0$
S_5
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0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
α

Theorem (Hao, Liu, Yong, 2015)

The semigroup $e^{\mathcal{A}_{\alpha,\beta,\gamma}t}$ has the following stability properties:

1. *In $S_1 \cup S_7 \cup S_8$, it is exponentially stable;*
2. *In S_2, it is polynomially stable of order $\frac{\gamma}{2(\beta-2\alpha)}$;*
3. *In S_3, it is polynomially stable of order $\frac{\gamma}{2(2-2\alpha-\beta-\gamma)}$;*
4. *In S_4, it is polynomially stable of order $\frac{\gamma}{2(\gamma-2\alpha-\beta)}$;*
5. *In S_5, it is polynomially stable of order $\frac{\gamma+1-2\alpha}{2(2\alpha-\beta-\gamma)}$;*
6. *In S_6, it is not asymptotically stable.*

Thanks for Your Attention !