Control and stabilization of $2 \times 2$ hyperbolic systems on graphs

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Outline

1. Introduction
2. General problems on graphs
3. Well Posedness
4. Reduced problems
5. Link with a wave equation
6. Strong stability
7. Uniform stability results
8. Exact controllability results
The $2 \times 2$ hyperbolic system

\[
\begin{cases}
    p_t + L p_x + G p = 0 & \text{in } (0, \ell) \times (0, +\infty), \\
    q_t + M p_x + K q = 0 & \text{in } (0, \ell) \times (0, +\infty), 
\end{cases}
\]

is a model for dissipative wave equation on the real interval $(0, \ell)$. In electrical engineering:

$p$ represents the voltage $V$, $q$ the electrical current $I$ at $(\ell - x, t)$. In biology, models of arterial blood flow:

$p$ is the pressure, $q$ the flow rate at $(x, t)$. 
Stability results

The exponential stability of (1) with the boundary conditions

\[ q(0, t) = 0, \quad p(\ell, t) = \gamma q(\ell, t), \quad \forall t > 0, \]

with \( \gamma > 0 \), can be proved by using Theorem 1 of [Diagne-Bastin-Coron 12]. Using the method of characteristic we can even show that it can be stabilized in a finite time if \( G = K = 0 \) and \( L = M = \gamma = 1 \), see [Perrollaz-Rosier 14].

Lyapunov exponential stability of 1-D linear hyperbolic systems of balance laws.


V. Perrollaz and L. Rosier.
Finite-time stabilization of \( 2 \times 2 \) hyperbolic systems on tree-shaped networks.

Motivation again

Recent applications, like electrical circuits, arterial networks, networks of open channels, traffic flows on networks, involve such a system set on networks. In that case some transmission conditions, that translate some physical preservations, have to be imposed at the junctions.

**Main goal:** Consider $2 \times 2$ hyperbolic systems on a network and introduce a general class of transmission conditions, including reasonable ones (like the Kirchhoff law or the one introduced in [Zong-Xu14]) so that with dissipative boundary conditions at the exterior vertices, we obtain an exponentially stable system.
Main questions

- Well-posedness of the problem.
- Strong stability of the solution.
- Uniform stability.
- Exact controllability.
All graphs considered here are non empty, finite and simple. Let $G$ be a connected topological graph imbedded in $\mathbb{R}^m$, $m \in \mathbb{N}^*$, with $n$ vertices $V = \{v_i : 1 \leq i \leq n\}$ and $N$ edges $E = \{e_j : 1 \leq j \leq N\}$. Each edge $e_j$ is a Jordan curve in $\mathbb{R}^m$ and is assumed to be parametrized by its arc length parameter $x_j$, such that the parametrization

$$\pi_j : [0, l_j] \rightarrow e_j : x_j \mapsto \pi_j(x_j)$$

is twice differentiable, i.e., $\pi_j \in C^2([0, l_j], \mathbb{R}^m)$ for all $1 \leq j \leq N$. We now define the $C^2$-network $\Gamma$ associated with $G$ as the union

$$\Gamma = E \cup V.$$

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**G. Lumer.**

Connecting of local operators and evolution equations on networks.

The valency of each vertex $v$ is denoted by $\gamma(v)$. 

$V_{\text{ext}} = \{ v \in V : \gamma(v) = 1 \}$ the set of boundary (or exterior) vertices, 

$V_{\text{int}} = V \setminus V_{\text{ext}}$, the set of interior vertices.

For each vertex $v$, $J_v = \{ j \in \{1, \ldots, N\} : v \in e_j \}$ the set of edges adjacent to $v$ and let $N_v$ be the cardinal of $J_v$. Note that if $v \in V_{\text{ext}}$ then $N_v$ is a singleton that we write $\{j_v\}$. For each vertex $v$ and $j \in N_v$

normal vector in $e_j$ at $v$ : $\nu_j(v) = \begin{cases} 
1 & \text{if } \pi_j(l_j) = v, \\
-1 & \text{if } \pi_j(0) = v, 
\end{cases}$

For a function $u : \Gamma \to \mathbb{C}$, we set $u_j = u \circ \pi_j : [0, l_j] \to \mathbb{C}$, its “restriction” to the edge $e_j$. Finally, differentiations are carried out on each edge $e_j$ with respect to the arc length parameter $x_j$.

For any $v \in V$, we finally introduce the mappings

$T_v : \mathbb{C}^{N_v} \to \mathbb{C}^{N_v} : y = (y_j)_{j \in J_v} \to T_v y = (y_j \nu_j(v))_{j \in J_v}$.

“trace” mapping: $\gamma_v : PC(\Gamma) \to \mathbb{C}^{N_v} : u \to \gamma_v u = (u_j(v))_{j \in J_v}$.
The problem

Fix a $C^2$-network $\Gamma$. For each edge $e_j$, we also fix $L_j > 0$, $M_j > 0$ and $G_j \geq 0$, $K_j \geq 0$. For each $v \in V_{\text{int}}$, we fix a subspace $Z_v$ of $\mathbb{C}^{N_v}$ and denote by $Z_v^\perp$ its orthogonal complement in $\mathbb{C}^{N_v}$ with respect to the euclidean inner product. We finally fix a decomposition of $V_{\text{ext}} = V_{\text{ext}}^{\text{Dir}} \cup V_{\text{ext}}^{\text{Diss}}$ with two disjoint subsets $V_{\text{ext}}^{\text{Dir}}$ and $V_{\text{ext}}^{\text{Diss}}$ and $\alpha \geq 0$.

$$
\begin{align*}
\partial_t p_j + L_j \partial_x q_j + G_j p_j &= 0 \text{ in } Q_j = (0, l_j) \times (0, \infty), \forall j = 1, \ldots, N, \\
\partial_t q_j + M_j \partial_x p_j + K_j q_j &= 0 \text{ in } Q_j, \forall j = 1, \ldots, N, \\
\gamma_v p(\cdot, t) &\in Z_v, \forall v \in V_{\text{int}}, t > 0, \text{ (transmission cond.)} \\
T_v \gamma_v q(\cdot, t) &\in Z_v^\perp, \forall v \in V_{\text{int}}, t > 0, \text{ (transmission cond.)} \\
q_{jv}(v, t) &= 0, \forall v \in V_{\text{ext}}^{\text{Dir}}, t > 0, \text{ (Dirichlet bc)} \\
p_{jv}(v, t) &= \alpha \nu_{jv}(v) q_{jv}(v, t), \forall v \in V_{\text{ext}}^{\text{Diss}}, t > 0, \text{ (dissipative bc)} \\
p(\cdot, 0) &= p_0, q(\cdot, 0) = q_0 \text{ in } \Gamma.
\end{align*}$$
Particular cases

1. **Kirchoff conditions**: [Sherwin 03] for all $v \in V_{\text{int}}$

$$\sum_{j \in J_v} \nu_j(v)q_j(v, t) = 0, \text{ (mass conservation)},$$

$$p_j(v, t) = p_k(v, t), \forall j, k \in J_v, \text{ (pressures agree at the junction)}.$$  

(3) \hspace{1cm} Z_v = \text{Span} (1, \ldots, 1)^T.

2. **Flow rates partition model**: [Zong-Xu 14] for all $v \in V_{\text{int}}$, fix one edge $j_v \in J_v$ and take

$$\sum_{j \in J_v} \alpha_j \nu_j(v)p_j(v, t) = 0, \text{ (no loss of energy)},$$

$$q_j(v, t) = \alpha_j q_{j_v}(v, t), \forall j \in J_v, \text{ (flow rates partition)}.$$  

with $\alpha_{j_v} = 1$ and for some mass partition coefficients $\alpha_j \in (0, 1)$ such that $\sum_{j \in J_v \setminus \{j_v\}} \alpha_j = 1$

(4) \hspace{1cm} Z_v^\perp = \text{Span} (\alpha_j)_{j \in J_v}.
The energy space

Introduce the Hilbert space \( H = L^2(\Gamma)^2 = \prod_{j=1}^N L^2(0, l_j) \) with the inner product

\[
((p, q), (r, s))_H = \sum_{j=1}^N \int_0^{l_j} \left( L_j^{-1} p_j(x_j) \bar{r}_j(x_j) + M_j^{-1} q_j(x_j) \bar{s}_j(x_j) \right) \, dx_j,
\]

Define the operator \( A_\alpha \):

\[
D(A_\alpha) = \{ (p, q) \in PH^1(\Gamma)^2 \text{ satisfying (5) to (8) hereafter} \},
\]

\[\gamma_v p \in Z_v, \forall v \in V_{\text{int}},\]
\[T_v \gamma_v q \in Z_v^\perp, \forall v \in V_{\text{int}},\]
\[q_{jv}(v) = 0, \forall v \in V_{\text{ext}}^{\text{Dir}},\]
\[p_{jv}(v) = \alpha v_{jv}(v) q_{jv}(v, t), \forall v \in V_{\text{ext}}^{\text{Diss}}.\]

\[
A_\alpha(p, q) = -(Lq' + Gp, Mp' + Kq), \quad \forall (p, q) \in D(A_\alpha).
\]

\[
Lq' := (L_j q'_j)_{j=1}^N, \quad Mp' := (M_j p'_j)_{j=1}^N, \quad Gp := (G_j p_j)_{j=1}^N, \quad Kq := (K_j q_j)_{j=1}^N.
\]
Well Posedness

Cauchy problem/Maximal dissipativity

Problem (2) can be written as a Cauchy problem:

\[ U_t = A_\alpha U, \quad U(0) = U_0, \]

where \( U = (p, q) \). Existence is obtained using semigroup theory:

**Theorem**

*Under the above assumptions, the operator \( A_\alpha \) generates a \( C_0 \)-semigroup of contractions on \( H \).*

\( A_\alpha \) is m-dissipative, i.e., \( \lambda I - A_\alpha \) is surjective for some \( \lambda > 0 \) and

\[ \Re(A_\alpha U, U)_H = -\alpha \sum_{v \in V^{\text{Diss}}_{\text{ext}}} |\gamma_v q|^2 \]

\[ -\sum_{j=1}^{N} \int_0^{l_j} (L_j^{-1} G_j |p_j|^2 + M_j^{-1} K_j |q_j|^2) \, dx_j \leq 0. \]

Using Lumer-Phillips’ thm, \( A_\alpha \) generates a \( C_0 \)-sg of contraction.
Lemma (Pruss, Huang)

Let \((e^{tA})_{t \geq 0}\) be a \(C_0\) semigroup on a Hilbert space \(H\) satisfying

\[
\|e^{tA}\| \leq M, \forall t \geq 0,
\]

for some \(M > 0\). Then it is exponentially stable, i.e., satisfies

\[
\|e^{tA}U_0\| \leq C e^{-\omega t} \|U_0\|_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,
\]

for some positive constants \(C\) and \(\omega\) if and only if

\[
\rho(A) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R},
\]

(11)

\[
\sup_{\beta \in \mathbb{R}} \|(i\beta - A)^{-1}\|_{\mathcal{L}(H)} < \infty,
\]

(12)

where \(\rho(A)\) denotes the resolvent set of the operator \(A\).
Let $A$ be the generator of an exponentially stable $C_0$ semigroup of contraction on a Hilbert space $H$ with a compact resolvent and let $B$ be a bounded operator on $H$ such that $-B$ is a non negative selfadjoint operator. Suppose further that the $C_0$ semigroup $(e^{t(A+B)})_{t \geq 0}$ generated by $A + B$ on $H$ satisfies

$$\|e^{t(A+B)}\| \leq M, \forall t \geq 0,$$

for some $M > 0$. Then $(e^{t(A+B)})_{t \geq 0}$ is exponentially stable.
Proof

A standard perturbation result yields that $A + B$ generates a $C_0$ semigroup. Due to the assumption (13), according to the previous Lemma, it suffices to show that $A + B$ satisfies (11) and (12).

(11): As $A + B$ has also a compact resolvent, it suffices to check that $A + B$ has no eigenvalue on $i\mathbb{R}$: Let $\xi \in \mathbb{R}, u \in D(A)$ be s. t.

$$(i\xi - A - B)u = 0.$$ 

Then taking the inner product in $H$ with $u$ and taking the real part:

$$\Re(Au, u)_H + (Bu, u)_H = 0.$$ 

As $A$ is dissipative, we deduce that $(Bu, u)_H = 0$, or equivalently

$$\|(-B)^{1/2}u\|^2_H = 0 \Rightarrow Bu = 0.$$ 

Hence $(i\xi - A)u = 0$ and by (11) applied to $A$, we deduce that $u = 0$. Hence $A + B$ satisfies (11).

The second property is proved similarly by using the fact that $A$ satisfies (12).
Let $A^{(0)}_{\alpha}$ be the operator $A_{\alpha}$ defined before with $G_j = K_j = 0$ for all $j = 1, \ldots, N$. If $A^{(0)}_{\alpha}$ generates an exponentially stable $C_0$ semigroup on $H$, then $A_{\alpha}$ generates an exponentially stable $C_0$ semigroup on $H$. It suffice to notice that

$$A_{\alpha} = A^{(0)}_{\alpha} + B,$$

where $B$ is defined by

$$B(p, q) = -((G_j p_j)_{j=1}^N, (K_j q_j)_{j=1}^N).$$

From now on we then assume that $G_j = K_j = 0$ for all $j = 1, \ldots, N$. 
Theorem (Thm 2)

Let \((p, q) \in C([0, \infty); D(A_{\alpha}^2)) \cap C^1([0, \infty); D(A_{\alpha})) \cap C^2([0, \infty); H)\) be a sol. of (9) with \(K_j = G_j = 0\). Then \(\forall t > 0\), \(q(\cdot, t)\) satisfies the wave eq.

\[
\partial_t^2 q_j - M_j L_j q_j'' = 0 \text{ in } Q_j, \forall j = 1, \ldots, N,
\]

the boundary/transmission conditions

\[
T \gamma_v q(\cdot, t) \in Z_v^\perp, \forall v \in V_{\text{int}}, \ t > 0,
\]

\[
\gamma_v (Lq')(v, t) \in Z_v, \forall v \in V_{\text{int}}, \ t > 0,
\]

\[
q_{jv}(v, t) = 0, \forall v \in V_{\text{ext}}^{\text{Dir}}, t > 0,
\]

\[
L_{jv} \partial_v q_{jv}(v, t) = -\alpha \partial_t q_{jv}(v, t), \forall v \in V_{\text{ext}}^{\text{Diss}}, t > 0,
\]

and the initial conditions

\[
q(\cdot, 0) = q_0, \partial_t q(\cdot, 0) = -Lp_0'.
\]
Proof

The regularity on \((p, q)\) allows to deduce that \((p, q)\) satisfies (2) strongly, implying that

\[
\begin{align*}
\partial_t p_j + L_j \partial_x q_j &= 0 \text{ in } Q_j, \forall j = 1, \ldots, N, \\
\partial_t q_j + M_j \partial_x p_j &= 0 \text{ in } Q_j, \forall j = 1, \ldots, N.
\end{align*}
\]

(19)

Furthermore we can derive the first identity in space and the second one in time and obtain

\[
\begin{align*}
\partial_t p'_j + L_j q''_j &= 0 \text{ in } Q_j, \\
\partial_t^2 q_j + M_j \partial_t p'_j &= 0 \text{ in } Q_j,
\end{align*}
\]

eliminating \(\partial_t p'_j\) we arrive at (14).
An associated operator

system (14)-(18) \iff

\[ U_t = A_\alpha U, \quad U(0) = U_0, \]

with \( U = (q, q_t) \) and

\[ D(A_\alpha) = \{(q, r) \in PH^2(\Gamma) \times PH^1(\Gamma) : (q, r) \text{ satisfies (20) to (23)}\}, \]

(20) \hspace{1cm} T_v \gamma_v q \in Z_v^\perp, \forall v \in V_{\text{int}},

(21) \hspace{1cm} \gamma_v(Lq')(v) \in Z_v, \forall v \in V_{\text{int}},

(22) \hspace{1cm} \partial_v L_{j_v} q_{j_v}(v) = -\alpha r_{j_v}(v), \forall v \in V_{\text{ext}}^{\text{Diss}},

(23) \hspace{1cm} q_{j_v}(v) = 0, \forall v \in V_{\text{ext}}^{\text{Dir}}.
A remark

Note that the transmission condition (15) can be equivalently written as

(24) \[ \gamma_v q \in T_v Z_v^\perp, \forall v \in V_{\text{int}}. \]

Hence if for all \( v \in V_{\text{int}} \), we chose \( T_v Z_v^\perp = \text{Span} (1, 1, \ldots, 1)^\top \), system (14)-(18) is nothing else than the wave equation on \( \Gamma \) with the so-called Kirchoff condition at the interior node and a standard damping condition at the node of \( V_{\text{Diss}}^{\text{ext}} \). Accordingly owing to [Schmidt 99], if \( \Gamma \) has some cycles or if two exterior nodes are uncontrolled, we cannot expect to have an exponential decay from a boundary dissipation.

Therefore in the remainder of the talk, we assume that \( \Gamma \) is a tree, that \( V_{\text{Dir}}^{\text{ext}} \) is reduced to one external vertex, called the root of the tree and that \( V_{\text{Diss}}^{\text{ext}} \) is the set of all other external vertices.
To simplify the arguments below, we now parametrize each edge $e_j$ in such a way that the vertex $\pi_j(0)$ is more close to the root than $\pi_j(l_j)$. For all $v \in V_{\text{int}}$, we denote by $Y_v := T_v Z_v^\perp \subset \mathbb{C}^{N_v}$ (hence $Y_v^\perp = T_v Z_v$) and assume without loss of generality that the first component of $\mathbb{C}^{N_v}$ corresponds to the edge $j$ such that $\pi_j(l_j) = v$ (or equivalently the edge $j \in J_v$ closer to the root); let us denote this edge $j_v$. We further denote by $\Pi_v$ the orthogonal projection on $(1, 0, \ldots, 0)^\top$ in $\mathbb{C}^{N_v}$. 
Now we make the following assumption:

\[(25) \quad \forall v \in V_{\text{int}} : \Pi_v Y_v \neq \{0\} \text{ and } \Pi_v Y_v^\perp \neq \{0\}.\]

The two examples mentioned before satisfy condition (25).

Lemma

*If the assumption (25) does not hold, then any \((q, r) \in D(A_\alpha)\) satisfies*

\[(26) \quad \exists v \in V_{\text{int}} : q'_j(v) = 0 \text{ or } q_j(v).\]
Proof

(25) does not hold if and only if \( \exists v \in V_{\text{int}} : \Pi_v Y_v = \{0\} \) or \( \Pi_v Y_v^\perp = \{0\} \). Fix such a \( v \in V_{\text{int}} \) and take any \((q, r) \in D(A_\alpha)\).

1. \( \Pi_v Y_v = \{0\} \iff (1, 0, \ldots, 0)^\top \in Y_v^\perp \). As \( \gamma_v q \in Y_v \), we deduce that

\[
q_{j_v}(v) = \gamma_v q \cdot (1, 0, \ldots, 0)^\top = 0.
\]

On the contrary \( \Pi_v Y_v^\perp = \{0\} \iff (1, 0, \ldots, 0)^\top \in Y_v \) and as \( T_v \gamma_v (Lq')(v) \in Y_v \), we deduce that

\[
L_j q'_{j_v}(v) = \pm T_v \gamma_v Lq'(v) \cdot (1, 0, \ldots, 0)^\top = 0 \implies q'_{j_v}(v) = 0.
\]

Rk If \((q, r) \in D(A_\alpha)\) satisfies (26) \( \iff \exists \) at least one \( v \in V_{\text{int}} \) s. t. we have either Dirichlet or Neumann condition on \( q \) at the extremity \( v \) of \( j_v \). Hence let \( \Gamma'_v \) the subtree made of the edges of \( \Gamma \) that are descendant of \( v \) and let \( \Gamma_v = \Gamma \setminus \Gamma'_v \). Then the wave system (14)-(18) can be split up into similar but decoupled problems in \( \Gamma_v \) and in \( \Gamma'_v \). But on \( \Gamma_v \), we have at least two vertices with a non dissipative bc (the root and \( v \)), since in such a situation, in general we do not have stability of the system (see [Schmidt 99]), the condition (25) seems to be realistic.
A technical result

Lemma (Le 3)

The condition (25) holds if and only if for all \( v \in V_{\text{int}} \), there exist coefficients \( a_{v,j}, b_{v,j} \in \mathbb{C} \), for all \( j \in J_v \setminus \{ j_v \} \) such that

\[
y_{j_v} = \sum_{j \in J_v \setminus \{ j_v \}} a_{v,j} y_j, \forall y \in Y_v,
\]

(27)

\[
z_{j_v} = \sum_{j \in J_v \setminus \{ j_v \}} b_{v,j} z_j, \forall z \in Y_v^\perp.
\]

(28)

Let \( v \in V_{\text{int}} \). Then \( \Pi_v Y_v^\perp \neq \{0\} \) if and only if there exists \( z \in Y_v^\perp \) such that \( z_{j_v} \neq 0 \). Since \( y \cdot z = 0 \), for any \( y \in Y_v \), we get

\[
y_{j_v} z_{j_v} = -\sum_{j \in J_v \setminus \{ j_v \}} y_j z_j,
\]

which furnishes (27) with \( a_{v,j} = \frac{z_j}{z_{j_v}} \). Conversely if (27) holds, then

\[
(1, (-a_{v,j})_{j \in J_v \setminus \{ j_v \}})^\top \quad \text{belong to } Y_v^\perp \quad \text{and therefore } \Pi_v Y_v^\perp \neq \{0\}.\]
One simple way to prove the strong stability is to use the following theorem.

**Theorem (Arendt-Batty)**

Let $X$ be a reflexive Banach space and $(T(t))_{t \geq 0}$ be a semigroup generated by $A$ on $X$. Assume that $(T(t))_{t \geq 0}$ is bounded and that no eigenvalues of $A$ lies on the imaginary axis. If $\sigma(A) \cap i\mathbb{R}$ is countable, then $(T(t))_{t \geq 0}$ is stable.

Since the resolvent of our operator is compact, we only have to analyze the discrete spectrum on the imaginary axis.

Theorem (Thm 4)

Assume that $\Gamma$ is a tree, that $V_{\text{Dir}}^{\text{ext}}$ is the root of the tree and that $V_{\text{Diss}}^{\text{ext}}$ is the set of all other exterior vertices. If $\alpha > 0$ and condition (25) holds, then

$$i\mathbb{R} \subset \rho(A_{\alpha}^{(0)}).$$
Let $\xi \in \mathbb{R}$ and $U = (p, q) \in D(A^{(0)}_\alpha)$ be such that $(i\xi - A^{(0)}_\alpha)U = 0$, or equivalently satisfying

\begin{align}
\begin{cases}
  i\xi p_j + L_j q'_j = 0, \\
  i\xi q_j + M_j p'_j = 0, \quad \forall j = 1, \ldots, N.
\end{cases}
\end{align}

Dissipativeness of $A^{(0)}_\alpha$ (inequality (10)) $\Rightarrow$

\begin{align}
q_{jv}(v) = p_{jv}(v) = 0, \quad \forall v \in V^{\text{Diss}}_{\text{ext}}.
\end{align}

For $\xi \neq 0$, by eliminating $p_j$ from the first identity of (29) we find that

\begin{align}
M_j L_j q''_j = -\xi^2 q_j, \quad \forall j = 1, \ldots, N.
\end{align}

Consequently for all $j = 1, \ldots, N$, there exist constants $c_{j,1}, c_{j,2} \in \mathbb{C}$ such that

\begin{align}
q_j(x) &= c_{j,1} e^{i\xi x} + c_{j,2} e^{-i\xi x}, \\
p_j(x) &= \frac{i}{L_j} (c_{j,1} e^{i\xi x} - c_{j,2} e^{-i\xi x}), \quad \forall x \in (0, l_j).
\end{align}
For any $v \in V_{\text{ext}}^{\text{Diss}}$, the boundary conditions (30) yield the system

$$\left\{ \begin{array}{l}
c_{jv,1} e^{i\xi l_{jv}} + c_{jv,2} e^{-i\xi l_{jv}} = 0, \\
c_{jv,1} e^{i\xi l_{jv}} - c_{jv,2} e^{-i\xi l_{jv}} = 0.
\end{array} \right.$$ 

As the determinant of this system is $-2$, we deduce that $c_{jv,1} = c_{jv,2} = 0$. This means that $p_{jv} = q_{jv} = 0$ for all $v \in V_{\text{ext}}$.

**Figure:** A tree shaped network: generations of edges.
Now fix an arbitrary node $w \in V_{\text{int}}$ of the last but one generation, in other words, a node having an edge $j \in J_w$ such that the other extremity of $e_j$ is in $V_{\text{ext}}^\text{Diss}$. But for such a vertex, we remark that we have just shown that $p_j = q_j = 0$ for all $j \in J_w \setminus \{j_w\}$, and by the fact that $(p, q) \in D(A^{(0)}_\alpha)$ and the assumption (25) (with the help of Le 3), we deduce that

$$q_{j_w}(w) = p_{j_w}(w) = 0.$$ 

Comparing with (30), we can view this node as a (new) dissipative node and reiterating the previous argument we deduce that $q_{j_w} = p_{j_w} = 0$. From one generation to the previous one we arrive at the root edge and find $p = q = 0$. 
Arendt and Batty \( \Rightarrow \)

**Corollary**

*Under the assumptions of Thm 4, the semi-group generated by \( A_{\alpha}^{(0)} \) is strongly stable.*
Uniform stability results

An identity with multiplier

Lemma

Let $T > 0$ and let $m : \Gamma \to \mathbb{R}$ be a multiplier with the regularity $m_j \in C^1([0, l_j])$, for all $j = 1, \cdots, N$. Then for all $(q_0, q_1)$ smooth enough the solution $q$ of (14)-(18) satisfies

\begin{align}
\frac{1}{2} \sum_{v \in V} \sum_{j \in J_v} \int_0^T (L_j |q'_j(v, t)|^2 + M_j^{-1} |\partial_t q_j(v, t)|^2) m_j(v) \nu_j(v) \, dt \\
= \frac{1}{2} \sum_{j=1}^N \int_0^T \int_0^{l_j} m_j'(M_j^{-1} |\partial_t q_j|^2 + L_j |q_j'|^2) \, dx_j \, dt \\
+ \sum_{j=1}^N M_j^{-1} \left. \int_0^{l_j} \partial_t q_j m_j q'_j \, dx_j \right|_0^T.
\end{align}
This identity is obtained as follows: first multiply the equation

$$\partial_t^2 q_j - M_j L_j q_j'' = 0,$$

by $M_j^{-1} m_j q_j'$ and integrate on $(0, l_j) \times (0, T)$, secondly perform some integrations by part in space and time and thirdly take the sum on $j$. 
Lemma

Let the assumptions of Thm 4 be satisfied. Then there exist a positive constant \( c' \) and a positive time \( T_0 \) which depend only on \( l_j, M_j, L_j \), the spaces \( Y_v \), for all \( v \in V_{\text{int}} \) and on the algebraic structure of \( \Gamma \) such that the solution \( q \) of (14)-(18) satisfies

\[
    c'(T - T_0)\mathcal{E}(T) \leq \sum_{v \in V_{\text{ext}}} \int_0^T |\partial_t q_{jv}(v, t)|^2 \, dt, \forall T > 0,
\]

where the energy is

\[
    \mathcal{E}(t) = \frac{1}{2} \sum_{j=1}^N \int_0^{l_j} (L_j |q_j'(x, t)|^2 + M_j^{-1} |\partial_t q_j(x, t)|^2) \, dx_j.
\]
It suffices to prove (32) for smooth solutions $q$. Now in the identity (31) we restrict ourselves to a multiplier $m$ such that in the left-hand side of (31) the contribution of the interior nodes is nonpositive and the contribution of the root is zero. So we look for $m$ in the form

\begin{equation}
    m_j(x_j) = x_j + \beta_j, \forall j = 1, \ldots, N,
\end{equation}

with $\beta_j \in \mathbb{R}$ and satisfying $m_j(r)(r) = 0$ for the root $r$ and

\begin{equation}
    \sum_{j \in N_v} M_j^{-1} m_j(v)v_j(v)|q_j'(v, t)|^2 \leq 0,
\end{equation}

\begin{equation}
    \sum_{j \in N_i} L_j m_j(v)v_j(v)|\partial_t q_j(v, t)|^2 \leq 0, \forall t > 0, v \in V_{\text{int}}.
\end{equation}
If these properties hold, then (31) implies that

\[
\frac{1}{2} \sum_{j=1}^{N} \left[ \int_{0}^{T} \left( M_j^{-1} |\partial_t q_j|^2 + L_j |q'_j|^2 \right) dx_j dt \right]
\leq \frac{1}{2} \sum_{v \in V_{\text{Diss}}_{\text{ext}}} \left[ \int_{0}^{T} \left( L_j |q'_j(v, t)|^2 + M_j^{-1} |\partial_t q_j(v, t)|^2 \right) m_j(v) dt \right]
\]

\[
- \sum_{j=1}^{N} M_j^{-1} \left. \int_{0}^{t_j} \partial_t q_j m_j q'_j \right|_{0}^{T} dx_j.
\]

Hence there exists a positive constant \( C \) such that

\[
\int_{0}^{T} \mathcal{E}(t) dt \leq C \sum_{v \in V_{\text{Diss}}_{\text{ext}}} \int_{0}^{T} \left( |q'_j(v, t)|^2 + |\partial_t q_j(v, t)|^2 \right) dt + C(\mathcal{E}(0) + \mathcal{E}(T)).
\]
By using the boundary condition (18), we obtain

$$\int_0^T \mathcal{E}(t) \, dt \leq C \sum_{v \in V_{\text{ext}}^{\text{Diss}}} \int_0^T |\partial_t q_{jv}(v, t)|^2 \, dt + C(\mathcal{E}(0) + \mathcal{E}(T)).$$

Since the energy is non-increasing, we deduce that

$$TE(T) \leq C \sum_{v \in V_{\text{ext}}^{\text{Diss}}} \int_0^T |\partial_t q_{jv}(v, t)|^2 \, dt + C(\mathcal{E}(0) + \mathcal{E}(T)).$$

As $\mathcal{E}(0) - \mathcal{E}(T) = \alpha \int_0^T \sum_{v \in V_{\text{ext}}^{\text{Diss}}} |\partial_t q_{jv}(v, t)|^2 \, dt, \forall T > 0$, we arrive at

$$TE(T) \leq (1 + \alpha) C \sum_{v \in V_{\text{ext}}^{\text{Diss}}} \int_0^T |\partial_t q_{jv}(v, t)|^2 \, dt + 2C\mathcal{E}(T).$$

This yields (32) with $T_0 = 2C$. 
Theorem (Thm 5)

Let the assumptions of Thm 4 be satisfied. Then system (14)-(18) is exponentially stable, i.e., there exist two positive constants $M$ and $\omega$ such that

$$\mathcal{E}(t) \leq M e^{-\omega t} \mathcal{E}(0), \forall t \geq 0.$$ 

The proof is a direct consequence of (32) and the identity

$$\mathcal{E}(0) - \mathcal{E}(T) = \alpha \int_0^T \sum_{v \in V_{\text{ext}}} \sum_{j \in V_{\text{Diss}}} |\partial_t q_{jv}(v, t)|^2 dt, \forall T > 0$$

that yield for $T > T_0$

$$\mathcal{E}(T) \leq C(T)(\mathcal{E}(0) - \mathcal{E}(T)),$$

with $C(T) = \frac{1}{\alpha c'(T - T_0)}$. Hence the result follows from a standard argument.
Let the assumptions of Thm 4 be satisfied. Then system (2) is exponentially stable, i.e., there exist two positive constants $C$ and $\omega$ such that

$$E(t) \leq Me^{-\omega t} E(0), \quad \forall t \geq 0,$$

where the energy is

$$E(t) = \frac{1}{2} \sum_{j=1}^{N} \int_{0}^{l_j} \left( L_j^{-1} |p_j(x, t)|^2 + M_j^{-1} |q_j(x, t)|^2 \right) dx_j.$$
For \((P_0, Q_0) \in D(A^2_\alpha)\), let \((P, Q)\) be the solution of (2) with \(K_j = G_j = 0\) and initial data \((P_0, Q_0)\). Then by Thm 2, we know that \(Q\) is a strong solution of (14)-(18). Hence by Thm 5, we deduce that

\[
E(t) \leq Me^{-\omega t}E(0), \forall t \geq 0.
\]

Now we notice that

\[
E(t) = \frac{1}{2} \sum_{j=1}^{N} \int_{0}^{l_j} (L_j |q_j'(x_j, t)|^2 + M_j^{-1} |\partial_t q_j(x_j, t)|^2) \, dx_j
\]

\[
= \frac{1}{2} \sum_{j=1}^{N} \int_{0}^{l_j} (L_j |q_j'(x_j, t)|^2 + M_j |p_j'(x_j, t)|^2) \, dx_j = \frac{1}{2} \|A^{(0)}_\alpha(p, q)\|_H^2,
\]

owing to (19) and the definition of \(A^{(0)}_\alpha\). Hence (37) is equivalent to

\[
\|A^{(0)}_\alpha(P(t), Q(t))\|_H^2 \leq Me^{-\omega t} \|A^{(0)}_\alpha(P_0, Q_0)\|_H^2, \forall t \geq 0.
\]
For any \((p_0, q_0) \in D(A_\alpha)\), let \((p, q)\) be the solution of (2) with \(K_j = G_j = 0\). As \(A^{(0)}_\alpha\) is an isomorphism (Thm 4), we can consider \((P_0, Q_0) = (A^{(0)}_\alpha)^{-1}(p_0, q_0)\) that belongs to \(D(A^2_\alpha)\). As the solution \((P, Q)\) of (2) with \(K_j = G_j = 0\) and initial data \((P_0, Q_0)\) is given by

\[(P(t), Q(t)) = (A^{(0)}_\alpha)^{-1}(p(t), q(t)),\]

the estimate (38) lead to

\[\| (p(t), q(t)) \|^2_H \leq M e^{-\omega t} \| (p_0, q_0) \|^2_H, \forall t \geq 0.\]

This proves (36) for \(A^{(0)}_\alpha\) by a density argument. The general case follows from Coro 1.
Let the assumptions of Thm 4 be satisfied. Then for $T \gg$, $\forall (p_0, q_0) \in H$, $\exists$ controls $u_v \in L^2(0, T)$, $\forall v \in V_{ext}^{Diss}$ s. t. the sol. $(p, q)$ of

\[
\begin{align*}
\partial_t p_j + L_j \partial_x q_j + G_j p_j &= 0 \text{ in } (0, l_j) \times (0, T), \forall j = 1, \ldots, N, \\
\partial_t q_j + M_j \partial_x p_j + K_j q_j &= 0 \text{ in } (0, l_j) \times (0, T), \forall j = 1, \ldots, N, \\
\gamma_v p(\cdot, t) &\in Z_v, \forall v \in V_{int}, t \in (0, T), \\
T_v \gamma_v q(\cdot, t) &\in Z_v^\perp, \forall v \in V_{int}, t \in (0, T), \\
q_{jv}(v, t) &= 0, \forall v \in V_{ext}^{Dir}, t \in (0, T), \\
p_{jv}(v, t) &= u_v(t), \forall v \in V_{ext}^{Diss}, t \in (0, T), \\
p(\cdot, 0) &= p_0, q(\cdot, 0) = q_0 \text{ in } \Gamma,
\end{align*}
\]

satisfies $p(T) = q(T) = 0$. 

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Use Russell’s principle: exponential decay of system (2) $\Rightarrow$ EC.