Control and stabilization of 2×2 hyperbolic systems on graphs

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Outline

Motivation

The 2×2 hyperbolic system

$$(1) \qquad \begin{cases} p_t + Lp_x + Gp = 0 & \text{in } (0,\ell) \times (0,+\infty), \\ q_t + Mp_x + Kq = 0 & \text{in } (0,\ell) \times (0,+\infty), \end{cases}$$

is a model for dissipative wave equation on the real interval $(0, \ell)$.
In electrical engineering:
p represents the voltage V, q the electrical current I at $(\ell - x, t)$.
In biology, models of arterial blood flow:
p is the pressure, q the flow rate at (x, t).

Stability results

The exponential stability of (1) with the boundary conditions

$$q(0,t)=0, \quad p(\ell,t)=\gamma q(\ell,t), \forall t>0,$$

with $\gamma > 0$, can be proved by using Theorem 1 of [Diagne-Bastin-Coron 12]. Using the method of characteristic we can even show that it can be stabilized in a finite time if $G = K = 0$ and $L = M = \gamma = 1$, see [Perrollaz-Rosier 14].

A. Diagne, G. Bastin, and J.-M. Coron.
Lyapunov exponential stability of 1-D linear hyperbolic systems of balance laws.
Automatica J. IFAC, 48(1):109–114, 2012.

V. Perrollaz and L. Rosier.
Finite-time stabilization of 2×2 hyperbolic systems on tree-shaped networks.
SIAM J. Control Optim., 52(1):143–163, 2014.

Motivation again

Recent applications, like electrical circuits, arterial networks, networks of open channels, traffic flows on networks, involve such a system set on networks. In that case some transmission conditions, that translate some physical preservations, have to be imposed at the junctions.

Main goal: Consider 2×2 hyperbolic systems on a network and introduce a general class of transmission conditions, including reasonable ones (like the Kirchhoff law or the one introduced in [Zong-Xu14]) so that with dissipative boundary conditions at the exterior vertices, we obtain an exponentially stable system.

Main questions

- Well-posedness of the problem.
- Strong stability of the solution.
- Uniform stability.
- Exact controllability.

Graphs

All graphs considered here are non empty, finite and simple. Let G be a connected topological graph imbedded in $\mathbb{R}^m$, $m \in \mathbb{N}^*$, with n vertices $V = \{v_i : 1 \leq i \leq n\}$ and N edges $E = \{e_j : 1 \leq j \leq N\}$. Each edge e_j is a Jordan curve in $\mathbb{R}^m$ and is assumed to be parametrized by its arc length parameter x_j, such that the parametrization

$$\pi_j : [0, l_j] \to e_j : x_j \mapsto \pi_j(x_j)$$

is twice differentiable, i. e., $\pi_j \in C^2([0, l_j], \mathbb{R}^m)$ for all $1 \leq j \leq N$. We now define the C^2-network Γ associated with G as the union

$$\Gamma = E \cup V.$$

G. Lumer.
Connecting of local operators and evolution equations on networks.
In *Potential theory, Copenhagen 1979*, volume 787 of *Lecture Notes in Math.*, pages 219–234. Springer, Berlin, 1980.

Notations

The valency of each vertex v is denoted by $\gamma(v)$.
$V_{\rm ext} = \{v \in V : \gamma(v) = 1\}$ the set of boundary (or exterior) vertices,
$V_{\rm int} = V \setminus V_{\rm ext}$, the set of interior vertices.
For each vertex v, $J_v = \{j \in \{1, \dots, N\} : v \in e_j\}$ the set of edges adjacent to v and let N_v be the cardinal of J_v. Note that if $v \in V_{\rm ext}$ then N_v is a singleton that we write $\{j_v\}$. For each vertex v and $j \in N_v$

$$\text{normal vector in } e_j \text{ at } v : \nu_j(v) = \begin{cases} 1 & \text{if} \quad \pi_j(l_j) = v, \\ \text{-1} & \text{if} \quad \pi_j(0) = v, \end{cases}$$

For a function $u : \Gamma \to \mathbb{C}$, we set $u_j = u \circ \pi_j : [0, l_j] \to \mathbb{C}$, its “restriction” to the edge e_j. Finally, differentiations are carried out on each edge e_j with respect to the arc length parameter x_j.
For any $v \in V$, we finally introduce the mappings

$$T_v : \mathbb{C}^{N_v} \to \mathbb{C}^{N_v} : y = (y_j)_{j \in J_v} \to T_v y = (y_j \nu_j(v))_{j \in J_v}.$$

$$\text{“trace” mapping:} \quad \gamma_v : PC(\Gamma) \to \mathbb{C}^{N_v} : u \to \gamma_v u = (u_j(v))_{j \in J_v}.$$

The problem

Fix a C^2-network Γ. For each edge e_j, we also fix $L_j > 0$, $M_j > 0$ and $G_j \geq 0$, $K_j \geq 0$. For each $v \in V_{\rm int}$, we fix a subspace Z_v of $\mathbb{C}^{N_v}$ and denote by $Z_v^\perp$ its orthogonal complement in $\mathbb{C}^{N_v}$ with respect to the euclidean inner product. We finally fix a decomposition of $V_{\rm ext} = V_{\rm ext}^{\rm Dir} \cup V_{\rm ext}^{\rm Diss}$ with two disjoint subsets $V_{\rm ext}^{\rm Dir}$ and $V_{\rm ext}^{\rm Diss}$ and $\alpha \geq 0$.

(2)

$$\begin{cases} \partial_t p_j + L_j \partial_x q_j + G_j p_j = 0 \text{ in } Q_j = (0, l_j) \times (0, \infty), \forall j = 1, \ldots, N, \\ \partial_t q_j + M_j \partial_x p_j + K_j q_j = 0 \text{ in } Q_j, \forall j = 1, \ldots, N, \\ \gamma_v p(\cdot, t) \in Z_v, \forall v \in V_{\rm int}, t > 0, \ \text{(transmission cond.)} \\ T_v \gamma_v q(\cdot, t) \in Z_v^\perp, \forall v \in V_{\rm int}, t > 0, \ \text{(transmission cond.)} \\ q_{j_v}(v, t) = 0, \forall v \in V_{\rm ext}^{\rm Dir}, t > 0, \ \text{(Dirichlet bc)} \\ p_{j_v}(v, t) = \alpha \nu_{j_v}(v) q_{j_v}(v, t), \forall v \in V_{\rm ext}^{\rm Diss}, t > 0, \ \text{(dissipative bc)} \\ p(\cdot, 0) = p_0, q(\cdot, 0) = q_0 \text{ in } \Gamma. \end{cases}$$

Particular cases

1. Kirchoff conditions: [Sherwin 03] for all $v \in V_{\rm int}$

$$\sum_{j \in J_v} \nu_j(v) q_j(v,t) = 0, \text{ (mass conservation)},$$

$$p_j(v,t) = p_k(v,t), \forall j,k \in J_v, \text{ (pressures agree at the junction)}.$$

(3) $$Z_v = \text{ Span } (1,\dots,1)^\top.$$

2. Flow rates partition model: [Zong-Xu 14] for all $v \in V_{\rm int}$, fix one edge $j_v \in J_v$ and take

$$\sum_{j \in J_v} \alpha_j \nu_j(v) p_j(v,t) = 0, \text{ (no loss of energy)},$$

$$q_j(v,t) = \alpha_j q_{j_v}(v,t), \forall j \in J_v, \text{ (flow rates partition)}.$$

with $\alpha_{j_v} = 1$ and for some mass partition coefficients $\alpha_j \in (0,1)$ such that $\sum_{j \in J_v \setminus \{j_v\}} \alpha_j = 1$

(4) $$Z_v^\perp = \text{ Span } (\alpha_j)_{j \in J_v}.$$

The energy space

Introduce the Hilbert space $H = L^2(\Gamma)^2 = \Pi_{j=1}^N L^2(0, l_j)^2$ with the inner product

$$((p, q), (r, s))_H = \sum_{j=1}^{N} \int_0^{l_j} \left(L_j^{-1} p_j(x_j) \bar{r}_j(x_j) + M_j^{-1} q_j(x_j) \bar{s}_j(x_j) \right) dx_j,$$

Define the operator A_α:

$$D(A_\alpha) = \{(p, q) \in PH^1(\Gamma)^2 \text{ satisfying (5) to (8) hereafter } \},$$

$$\gamma_v p \in Z_v, \forall v \in V_{\text{int}}, \tag{5}$$

$$T_v \gamma_v q \in Z_v^\perp, \forall v \in V_{\text{int}}, \tag{6}$$

$$q_{j_v}(v) = 0, \forall v \in V_{\text{ext}}^{\text{Dir}}, \tag{7}$$

$$p_{j_v}(v) = \alpha \nu_{j_v}(v) q_{j_v}(v, t), \forall v \in V_{\text{ext}}^{\text{Diss}}. \tag{8}$$

$$A_\alpha(p, q) = -(Lq' + Gp, Mp' + Kq), \quad \forall (p, q) \in D(A_\alpha).$$

$$Lq' := (L_j q_j')_{j=1}^N, Mp' := (M_j p_j')_{j=1}^N, Gp := (G_j p_j)_{j=1}^N, Kq := (K_j q_j)_{j=1}^N.$$

Cauchy problem/Maximal dissipativity

Problem (2) can be written as a Cauchy problem:

$$U_t = A_\alpha U,\ U(0) = U_0, \tag{9}$$

where $U = (p, q)$. Existence is obtained using semigroup theory:

Theorem

Under the above assumptions, the operator A_α generates a C_0-semigroup of contractions on H.

A_α is m-dissipative, i.e., $\lambda I - A_\alpha$ is surjective for some $\lambda > 0$ and

$$\Re(A_\alpha U, U)_H = -\alpha \sum_{v \in V_{\text{ext}}^{\text{Diss}}} |\gamma_v q|^2 - \sum_{j=1}^{N} \int_0^{l_j} (L_j^{-1} G_j |p_j|^2 + M_j^{-1} K_j |q_j|^2)\, dx_j \leq 0. \tag{10}$$

Using Lumer-Phillips' thm, A_α generates a C_0-sg of contraction.

A tool

Lemma (Pruss, Huang)

Let $(e^{tA})_{t\geq 0}$ *be a* C_0 *semigroup on a Hilbert space H satisfying*

$$\|e^{tA}\| \leq M, \forall t \geq 0,$$

for some $M > 0$*. Then it is exponentially stable, i.e., satisfies*

$$||e^{tA}U_0|| \leq C\, e^{-\omega t}||U_0||_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,$$

for some positive constants C and ω *if and only if*

$$\rho(A) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{11}$$

$$\sup_{\beta\in\mathbb{R}} \|(i\beta - A)^{-1}\|_{\mathcal{L}(H)} < \infty, \tag{12}$$

where $\rho(A)$ *denotes the resolvent set of the operator A.*

A perturbation theorem

Theorem

Let A be the generator of an exponentially stable C_0 semigroup of contraction on a Hilbert space H with a compact resolvent and let B be a bounded operator on H such that $-B$ is a non negative selfadjoint operator. Suppose further that the C_0 semigroup $(e^{t(A+B)})_{t\geq 0}$ generated by $A+B$ on H satisfies

$$\|e^{t(A+B)}\| \leq M, \forall t \geq 0, \tag{13}$$

for some $M > 0$. Then $(e^{t(A+B)})_{t\geq 0}$ is exponentially stable.

Proof

A standard perturbation result yields that $A+B$ generates a C_0 semigroup. Due to the assumption (13), according to the previous Lemma, it suffices to show that $A+B$ satisfies (11) and (12).

(11): As $A+B$ has also a compact resolvent, it suffices to check that $A+B$ has no eigenvalue on $i\mathbb{R}$: Let $\xi \in \mathbb{R}, u \in D(A)$ be s. t.

$$(i\xi - A - B)u = 0.$$

Then taking the inner product in H with u and taking the real part:

$$\Re(Au, u)_H + (Bu, u)_H = 0.$$

As A is dissipative, we deduce that $(Bu, u)_H = 0$, or equivalently

$$\|(-B)^{\frac{1}{2}} u\|_H^2 = 0 \Rightarrow Bu = 0.$$

Hence $(i\xi - A)u = 0$ and by (11) applied to A, we deduce that $u = 0$. Hence $A+B$ satisfies (11).

The second property is proved similarly by using the fact that A satisfies (12).

Consequence

Corollary (Coro 1)

Let $A_\alpha^{(0)}$ be the operator A_α defined before with $G_j = K_j = 0$ for all $j = 1, \ldots, N$. If $A_\alpha^{(0)}$ generates an exponentially stable C_0 semigroup on H, then A_α generates an exponentially stable C_0 semigroup on H.

It suffice to notice that

$$A_\alpha = A_\alpha^{(0)} + B,$$

where B is defined by

$$B(p, q) = -((G_j p_j)_{j=1}^N, (K_j q_j)_{j=1}^N).$$

From now on we then assume that $G_j = K_j = 0$ for all $j = 1, \ldots, N$.

Link wave eq.

Theorem (Thm 2)

Let $(p, q) \in C([0,\infty); D(A_\alpha^2)) \cap C^1([0,\infty); D(A_\alpha)) \cap C^2([0,\infty); H)$ *be a sol. of* (9) *with* $K_j = G_j = 0$. *Then* $\forall t > 0$, $q(\cdot, t)$ *satifies the wave eq.*

$$\partial_t^2 q_j - M_j L_j q_j'' = 0 \text{ in } Q_j, \forall j = 1, \dots, N, \tag{14}$$

the boundary/transmission conditions

$$T_v \gamma_v q(\cdot, t) \in Z_v^\perp, \forall v \in V_{\text{int}}, t > 0, \tag{15}$$

$$\gamma_v (Lq')(v, t) \in Z_v, \forall v \in V_{\text{int}}, t > 0, \tag{16}$$

$$q_{j_v}(v, t) = 0, \forall v \in V_{\text{ext}}^{\text{Dir}}, t > 0, \tag{17}$$

$$L_{j_v} \partial_\nu q_{j_v}(v, t) = -\alpha \partial_t q_{j_v}(v, t), \forall v \in V_{\text{ext}}^{\text{Diss}}, t > 0, \tag{18}$$

and the initial conditions

$$q(\cdot, 0) = q_0, \partial_t q(\cdot, 0) = -L p_0'.$$

Proof

The regularity on (p, q) allows to deduce that (p, q) satisfies (2) strongly, implying that

$$(19) \quad \left\{ \begin{array}{l} \partial_t p_j + L_j \partial_x q_j = 0 \text{ in } Q_j, \forall j = 1, \ldots, N, \\ \partial_t q_j + M_j \partial_x p_j = 0 \text{ in } Q_j, \forall j = 1, \ldots, N. \end{array} \right.$$

Furthermore we can derive the first identity in space and the second one in time and obtain

$$\partial_t p_j' + L_j q_j'' = 0 \text{ in } Q_j,$$
$$\partial_t^2 q_j + M_j \partial_t p_j' = 0 \text{ in } Q_j,$$

eliminating $\partial_t p_j'$ we arrive at (14).

An associated operator

system (14)-(18) $\Leftrightarrow$

$$U_t = \mathcal{A}_\alpha U, U(0) = U_0,$$

with $U = (q, q_t)$ and

$$D(\mathcal{A}_\alpha) = \{(q, r) \in PH^2(\Gamma) \times PH^1(\Gamma) : (q, r) \text{ satisfies (20) to (23)}\},$$

(20) $$T_v \gamma_v q \in Z_v^\perp, \forall v \in V_{\text{int}},$$

(21) $$\gamma_v (Lq')(v) \in Z_v, \forall v \in V_{\text{int}},$$

(22) $$\partial_\nu L_{j_v} q_{j_v}(v) = -\alpha r_{j_v}(v), \forall v \in V_{\text{ext}}^{\text{Diss}},$$

(23) $$q_{j_v}(v) = 0, \forall v \in V_{\text{ext}}^{\text{Dir}}.$$

A remark

Note that the transmission condition (15) can be equivalently written as

(24) $$\gamma_v q \in T_v Z_v^{\perp}, \forall v \in V_{\rm int}.$$

Hence if for all $v \in V_{\rm int}$, we chose $T_v Z_v^{\perp} = \text{Span}\,(1, 1, \ldots, 1)^{\top}$, system (14)-(18) is nothing else than the wave equation on Γ with the so-called Kirchoff condition at the interior node and a standard damping condition at the node of $V_{\rm ext}^{\rm Diss}$. Accordingly owing to [Schmidt 99], if Γ has some cycles or if two exterior nodes are uncontrolled, we cannot expect to have an expential decay from a boundary dissipation.

Therefore in the remainder of the talk, we assume that Γ is a tree, that $V_{\rm ext}^{\rm Dir}$ is reduced to one external vertex, called the root of the tree and that $V_{\rm ext}^{\rm Diss}$ is the set of all other external vertices.

Some notation

To simplify the arguments below, we now parametrize each edge e_j in such a way that the vertex $\pi_j(0)$ is more close to the root than $\pi_j(l_j)$. For all $v \in V_{\rm int}$, we denote by $Y_v := T_v Z_v^\perp \subset \mathbb{C}^{N_v}$ (hence $Y_v^\perp = T_v Z_v$) and assume without loss of generality that the first component of $\mathbb{C}^{N_v}$ corresponds to the edge j such that $\pi_j(l_j) = v$ (or equivalently the edge $j \in J_v$ closer to the root); let us denote this edge j_v. We further denote by Π_v the orthogonal projection on $(1, 0, \ldots, 0)^\top$ in $\mathbb{C}^{N_v}$.

An assumption

Now we make the following assumption:

(25) $$\forall v \in V_{\text{int}} : \Pi_v Y_v \neq \{0\} \text{ and } \Pi_v Y_v^{\perp} \neq \{0\}.$$

The two examples mentioned before satisfy condition (25).

Lemma

If the assumption (25) does not hold, then any $(q, r) \in D(\mathcal{A}_\alpha)$ *satisfies*

(26) $$\exists v \in V_{\text{int}} : q'_{j_v}(v) = 0 \text{ or } q_{j_v}(v).$$

Proof

(25) does not hold if and only if $\exists v \in V_{\rm int} : \Pi_v Y_v = \{0\}$ or $\Pi_v Y_v^\perp = \{0\}$. Fix such a $v \in V_{\rm int}$ and take any $(q, r) \in D(\mathcal{A}_\alpha)$.

1. $\Pi_v Y_v = \{0\} \Leftrightarrow (1, 0, \dots, 0)^\top \in Y_v^\perp$. As $\gamma_v q \in Y_v$, we deduce that

$$q_{j_v}(v) = \gamma_v q \cdot (1, 0, \dots, 0)^\top = 0.$$

On the contrary $\Pi_v Y_v^\perp = \{0\} \Leftrightarrow (1, 0, \dots, 0)^\top \in Y_v$ and as $T_v \gamma_v (Lq')(v) \in Y_v$, we deduce that

$$L_j q'_{j_v}(v) = \pm T_v \gamma_v L q'(v) \cdot (1, 0, \dots, 0)^\top = 0 \Rightarrow q'_{j_v}(v) = 0.$$

Rk If $(q, r) \in D(\mathcal{A}_\alpha)$ satisfies (26) $\Rightarrow \exists$ at least one $v \in V_{\rm int}$ s. t. we have either Dirichlet or Neumann condition on q at the extremity v of j_v. Hence let Γ'_v the subtree made of the edges of Γ that are descendant of v and let $\Gamma_v = \Gamma \setminus \Gamma'_v$. Then the wave system (14)-(18) can be splitted up into similar but decoupled problems in Γ_v and in Γ'_v. But on Γ_v, we have at least two vertices with a non dissipative bc (the root and v), since in such a situation, in general we do not have stability of the system (see [Schmidt 99]), the condition (25) seems to be realistic.

A technical result

Lemma (Le 3)

The condition (25) holds if and only if for all $v \in V_{\mathrm{int}}$, there exist coefficients $a_{v,j}, b_{v,j} \in \mathbb{C}$, for all $j \in J_v \setminus \{j_v\}$ such that

$$y_{j_v} = \sum_{j \in J_v \setminus \{j_v\}} a_{v,j} y_j, \forall y \in Y_v, \tag{27}$$

$$z_{j_v} = \sum_{j \in J_v \setminus \{j_v\}} b_{v,j} z_j, \forall z \in Y_v^{\perp}. \tag{28}$$

Let $v \in V_{\mathrm{int}}$. Then $\Pi_v Y_v^{\perp} \neq \{0\}$ if and only if there exists $z \in Y_v^{\perp}$ such that $z_{j_v} \neq 0$. Since $y \cdot z = 0$, for any $y \in Y_v$, we get

$$y_{j_v} z_{j_v} = -\sum_{j \in J_v \setminus \{j_v\}} y_j z_j,$$

which furnishes (27) with $a_{v,j} = \frac{z_j}{z_{j_v}}$. Conversely if (27) holds, then $(1, (-a_{v,j})_{j \in J_v \setminus \{j_v\}})^{\top}$ belongs to $Y_v^{\perp}$ and therefore $\Pi_v Y_v^{\perp} \neq \{0\}$.

Arendt-Batty's thm

One simple way to prove the strong stability is to use the following theorem.

Theorem (Arendt-Batty)

Let X be a reflexive Banach space and $(T(t))_{t\geq 0}$ be a semigroup generated by A on X. Assume that $(T(t))_{t\geq 0}$ is bounded and that no eigenvalues of A lies on the imaginary axis. If $\sigma(A)\cap i\mathbb{R}$ is countable, then $(T(t))_{t\geq 0}$ is stable.

Since the resolvent of our operator is compact, we only have to analyze the discrete spectrum on the imaginary axis.

W. Arendt and C. J. K. Batty.
Tauberian theorems and stability of one-parameter semigroups.
Trans. Amer. Math. Soc., 305(2):837–852, 1988.

A spectral analysis

Theorem (Thm 4)

Assume that Γ is a tree, that $V_{\text{ext}}^{\text{Dir}}$ is the root of the tree and that $V_{\text{ext}}^{\text{Diss}}$ is the set of all other exterior vertices. If $\alpha > 0$ and condition (25) holds, then

$$i\mathbb{R} \subset \rho(A_\alpha^{(0)}).$$

Pf

Let $\xi \in \mathbb{R}$ and $U = (p, q) \in D(A_\alpha^{(0)})$ be such that $(i\xi - A_\alpha^{(0)})U = 0$, or equivalently satisfying

(29) $$\begin{cases} i\xi p_j + L_j q_j' = 0, \\ i\xi q_j + M_j p_j' = 0, \forall j = 1, \dots, N. \end{cases}$$

Dissipativeness of $A_\alpha^{(0)}$ (inequality (10)) $\Rightarrow$

(30) $$q_{j_v}(v) = p_{j_v}(v) = 0, \forall v \in V_{\rm ext}^{\rm Diss}.$$

For $\xi \neq 0$, by eliminating p_j from the first identity of (29) we find that

$$M_j L_j q_j'' = -\xi^2 q_j, \forall j = 1, \dots, N.$$

Consequently for all $j = 1, \dots, N$, there exist constants $c_{j,1}, c_{j,2} \in \mathbb{C}$ such that

$$\begin{aligned} q_j(x) &= c_{j,1} e^{i\xi x} + c_{j,2} e^{-i\xi x}, \\ p_j(x) &= \frac{i}{L_j}(c_{j,1} e^{i\xi x} - c_{j,2} e^{-i\xi x}), \forall x \in (0, l_j). \end{aligned}$$

Pf ctd

For any $v \in V_{\text{ext}}^{\text{Diss}}$, the boundary conditions (30) yield the system

$$\left\{ \begin{array}{l} c_{j_v,1} e^{i\xi l_{j_v}} + c_{j_v,2} e^{-i\xi l_{j_v}} = 0, \\ c_{j_v,1} e^{i\xi l_{j_v}} - c_{j_v,2} e^{-i\xi l_{j_v}} = 0. \end{array} \right.$$

As the determinant of this system is -2, we deduce that $c_{j_v,1} = c_{j_v,2} = 0$. This means that $p_{j_v} = q_{j_v} = 0$ for all $v \in V_{\text{ext}}$.

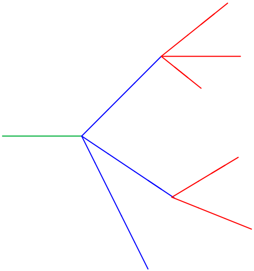

Figure: A tree shaped network: generations of edges.

Now fix an arbitrary node $w \in V_{\text{int}}$ of the last but one generation, in other words, a node having an edge $j \in J_w$ such that the other extremity of e_j is in $V_{\text{ext}}^{\text{Diss}}$. But for such a vertex, we remark that we have just shown that $p_j = q_j = 0$ for all $j \in J_w \setminus \{j_w\}$, and by the fact that $(p, q) \in D(A_\alpha^{(0)})$ and the assumption (25) (with the help of Le 3), we deduce that

$$q_{j_w}(w) = p_{j_w}(w) = 0.$$

Comparing with (30), we can view this node as a (new) dissipative node and reiterating the previous argument we deduce that
$q_{j_w} = p_{j_w} = 0$.
From one generation to the previous one we arrive at the root edge and find $p = q = 0$.

The strong stab

Arendt and Batty $\Rightarrow$

Corollary

Under the assumptions of Thm 4, the semi-group generated by $A_\alpha^{(0)}$ is strongly stable.

An identity with multiplier

Lemma

Let $T > 0$ *and let* $m : \Gamma \to \mathbb{R}$ *be a multiplier with the regularity* $m_j \in C^1([0, l_j])$*, for all* $j = 1, \cdots, N$*. Then for all* (q_0, q_1) *smooth enough the solution* q *of* (14)-(18) *satisfies*

$$\frac{1}{2}\sum_{v\in V}\sum_{j\in J_v}\int_0^T (L_j|q_j'(v,t)|^2 + M_j^{-1}|\partial_t q_j(v,t)|^2)m_j(v)\nu_j(v)\,dt \tag{31}$$

$$= \frac{1}{2}\sum_{j=1}^{N}\int_0^T\int_0^{l_j} m_j'(M_j^{-1}|\partial_t q_j|^2 + L_j|q_j'|^2)\,dx_j dt$$

$$+ \sum_{j=1}^{N} M_j^{-1}\int_0^{l_j}\partial_t q_j m_j q_j'\,dx_j\Big|_0^T.$$

Pf

This identity is obtained as follows: first multiply the equation

$$\partial_t^2 q_j - M_j L_j q_j'' = 0,$$

by $M_j^{-1} m_j q_j'$ and integrate on $(0, l_j) \times (0, T)$, secondly perform some integrations by part in space and time and thirdly take the sum on j.

An observability estimate

Lemma

Let the assumptions of Thm 4 be satisfied. Then there exist a positive constant c' and a positive time T_0 which depend only on l_j, M_j, L_j, the spaces Y_v, for all $v \in V_{\rm int}$ and on the algebraic structure of Γ such that the solution q of (14)-(18) *satisfies*

$$c'(T - T_0)\mathcal{E}(T) \leq \sum_{v \in V_{\rm ext}^{\rm Diss}} \int_0^T |\partial_t q_{j_v}(v,t)|^2 \, dt, \forall T > 0, \tag{32}$$

where the energy is

$$\mathcal{E}(t) = \frac{1}{2} \sum_{j=1}^{N} \int_0^{l_j} (L_j |q_j'(x,t)|^2 + M_j^{-1} |\partial_t q_j(x,t)|^2) \, dx_j.$$

Pf

It suffices to prove (32) for smooth solutions q. Now in the identity (31) we restrict ourselves to a multiplier m such that in the left-hand side of (31) the contribution of the interior nodes is nonpositive and the contribution of the root is zero. So we look for m in the form

$$m_j(x_j) = x_j + \beta_j, \forall j = 1, \ldots, N, \tag{33}$$

with $\beta_j \in \mathbb{R}$ and satisfying $m_{j_r}(r) = 0$ for the root r and

$$\sum_{j \in N_v} M_j^{-1} m_j(v) \nu_j(v) |q_j'(v,t)|^2 \leq 0, \tag{34}$$

$$\sum_{j \in N_i} L_j m_j(v) \nu_j(v) |\partial_t q_j(v,t)|^2 \leq 0, \forall t > 0, v \in V_{\text{int}}. \tag{35}$$

Pf ctd

If these properties hold, then (31) implies that

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{N}\int_0^T\int_0^{l_j}(M_j^{-1}|\partial_t q_j|^2+L_j|q_j'|^2)\,dx_j dt \\
\leq\ & \frac{1}{2}\sum_{v\in V_{\text{ext}}^{\text{Diss}}}\int_0^T(L_j|q_{j_v}'(v,t)|^2+M_j^{-1}|\partial_t q_{j_v}(v,t)|^2)m_{j_v}(v)\,dt \\
-\ & \sum_{j=1}^{N}M_j^{-1}\int_0^{l_j}\partial_t q_j m_j q_j'\,dx_j\Big|_0^T.
\end{aligned}$$

Hence there exists a positive constant C such that

$$\int_0^T\mathcal{E}(t)dt\leq C\sum_{v\in V_{\text{ext}}^{\text{Diss}}}\int_0^T(|q_{j_v}'(v,t)|^2+|\partial_t q_{j_v}(v,t)|^2)\,dt+C(\mathcal{E}(0)+\mathcal{E}(T)).$$

Pf ctd

By using the boundary condition (18), we obtain

$$\int_0^T \mathcal{E}(t)dt \leq C \sum_{v\in V_{\text{ext}}^{\text{Diss}}} \int_0^T |\partial_t q_{j_v}(v,t)|^2\, dt + C(\mathcal{E}(0)+\mathcal{E}(T)).$$

Since the energy is non increasing, we deduce that

$$T\mathcal{E}(T) \leq C \sum_{v\in V_{\text{ext}}^{\text{Diss}}} \int_0^T |\partial_t q_{j_v}(v,t)|^2\, dt + C(\mathcal{E}(0)+\mathcal{E}(T)).$$

As $\mathcal{E}(0)-\mathcal{E}(T) = \alpha \int_0^T \sum_{v\in V_{\text{ext}}^{\text{Diss}}} |\partial_t q_{j_v}(v,t)|^2 dt, \forall T>0$, we arrive at

$$T\mathcal{E}(T) \leq (1+\alpha)C \sum_{v\in V_{\text{ext}}^{\text{Diss}}} \int_0^T |\partial_t q_{j_v}(v,t)|^2\, dt + 2C\mathcal{E}(T).$$

This yields (32) with $T_0 = 2C$.

Exp decay wave eq.

Theorem (Thm 5)

Let the assumptions of Thm 4 be satisfied. Then system (14)-(18) *is exponentially stable, i.e., there exist two positive constants M and* ω *such that*

$$\mathcal{E}(t) \leq Me^{-\omega t}\mathcal{E}(0), \forall t \geq 0.$$

The proof is a direct consequence of (32) and the identity

$$\mathcal{E}(0) - \mathcal{E}(T) = \alpha \int_0^T \sum_{v \in V_{\text{ext}}^{\text{Diss}}} |\partial_t q_{j_v}(v,t)|^2 dt, \forall T > 0$$

that yield for $T > T_0$

$$\mathcal{E}(T) \leq C(T)(\mathcal{E}(0) - \mathcal{E}(T)),$$

with $C(T) = \frac{1}{\alpha c'(T - T_0)}$. Hence the result follows from a standard argument.

Exp decay hyperbolic system

Theorem

Let the assumptions of Thm 4 be satisfied. Then system (2) is exponentially stable, i.e., there exist two positive constants C and ω *such that*

(36)
$$E(t) \leq Me^{-\omega t}E(0), \forall t \geq 0,$$

where the energy is

$$E(t) = \frac{1}{2}\sum_{j=1}^{N}\int_0^{l_j}(L_j^{-1}|p_j(x,t)|^2 + M_j^{-1}|q_j(x,t)|^2)\,dx_j.$$

Pf

For $(P_0, Q_0) \in D(A_\alpha^2)$, let (P, Q) be the solution of (2) with $K_j = G_j = 0$ and initial data (P_0, Q_0). Then by Thm 2, we know that Q is a strong solution of (14)-(18). Hence by Thm 5, we deduce that

$$\mathcal{E}(t) \leq Me^{-\omega t}\mathcal{E}(0), \forall t \geq 0. \tag{37}$$

Now we notice that

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\sum_{j=1}^{N}\int_0^{l_j}(L_j|q_j'(x_j,t)|^2 + M_j^{-1}|\partial_t q_j(x_j,t)|^2)\,dx_j \\ &= \frac{1}{2}\sum_{j=1}^{N}\int_0^{l_j}(L_j|q_j'(x_j,t)|^2 + M_j|p_j'(x_j,t)|^2)\,dx_j = \frac{1}{2}\|A_\alpha^{(0)}(p,q)\|_H^2,\end{aligned}$$

owing to (19) and the definition of $A_\alpha^{(0)}$. Hence (37) is equivalent to

$$\|A_\alpha^{(0)}(P(t),Q(t))\|_H^2 \leq Me^{-\omega t}\|A_\alpha^{(0)}(P_0,Q_0)\|_H^2, \forall t \geq 0. \tag{38}$$

Pf ctd

For any $(p_0, q_0) \in D(A_\alpha)$, let (p, q) be the solution of (2) with $K_j = G_j = 0$. As $A_\alpha^{(0)}$ is an isomorphism (Thm 4), we can consider $(P_0, Q_0) = (A_\alpha^{(0)})^{-1}(p_0, q_0)$ that belongs to $D(A_\alpha^2)$. As the solution (P, Q) of (2) with $K_j = G_j = 0$ and initial data (P_0, Q_0) is given by

$$(P(t), Q(t)) = (A_\alpha^{(0)})^{-1}(p(t), q(t)),$$

the estimate (38) lead to

$$\|(p(t), q(t))\|_H^2 \leq Me^{-\omega t}\|(p_0, q_0)\|_H^2, \forall t \geq 0.$$

This proves (36) for $A_\alpha^{(0)}$ by a density argument.
The general case follows from Coro 1.

Exact controllability

Theorem

Let the assumptions of Thm 4 be satisfied. Then for $T \gg$,
$\forall (p_0, q_0) \in H$, $\exists$ *controls* $u_v \in L^2(0, T)$, $\forall v \in V_{\text{ext}}^{\text{Diss}}$ *s. t. the sol.* (p, q) *of*

$$(39) \quad \begin{cases} \partial_t p_j + L_j \partial_x q_j + G_j p_j = 0 \text{ in } (0, l_j) \times (0, T), \forall j = 1, \ldots, N, \\ \partial_t q_j + M_j \partial_x p_j + K_j q_j = 0 \text{ in } (0, l_j) \times (0, T), \forall j = 1, \ldots, N, \\ \gamma_v p(\cdot, t) \in Z_v, \forall v \in V_{\text{int}}, t \in (0, T), \\ T_v \gamma_v q(\cdot, t) \in Z_v^{\perp}, \forall v \in V_{\text{int}}, t \in (0, T), \\ q_{j_v}(v, t) = 0, \forall v \in V_{\text{ext}}^{\text{Dir}}, t \in (0, T), \\ p_{j_v}(v, t) = u_v(t), \forall v \in V_{\text{ext}}^{\text{Diss}}, t \in (0, T), \\ p(\cdot, 0) = p_0, q(\cdot, 0) = q_0 \text{ in } \Gamma, \end{cases}$$

satisfies $p(T) = q(T) = 0$.

Pf

Use Russell's principle: exponential decay of system (2) $\Rightarrow$ EC.


Thank you