Controlling and Covering Financial Risk under Distribution Uncertainty

Shige Peng, Shandong University

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- R. Wiener (1921-1924) Brownian motion (Wiener process)
- Lévy, A. N. Kolmogorov (1933) 《Foundation of Probability Theory》
- K. Itô, (1942–) Itô's stochastic analysis,

Example of distribution uncertainty

Randomly draw a ball from an urn with white and black colors.

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- Allais paradox (1953) to vNM expected utility theory (1944);
- Ellsberg paradox (1961) to Savage's expected utility (1954), Ambiguity aversion (1961);
- Kahneman & Tversky (1979-1992): prospective theory by distorted probability;
- Gilboa & Schmeidler (1989) Maximin expected utility; Hansen & Sargent (2000) Multiplier preference.

Finance: two revolutions

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- Black-Scholes-Merton, 1973 Black-Scholes option pricing formula

"Volatility smile paradox"

- Black-Scholes a revolution， but a biggest paradox

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  - (d) $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \quad \lambda \geq 0$;

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  - $\mathbb{E}[X_i] \downarrow 0$, if $X_i(\omega) \downarrow 0$, $\forall \omega$

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g-Expectation (Peng 1997): The begin of nonlinear expectation theory

Consider the backward stochastic differential equation ([Pardoux-Peng1990])

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Uncertainty of $E_g[\xi] \iff$ the sublinearity of the function g (Chen& Epstein (2002), Econometrica).

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In general $Y_t(\omega) = u(t, \omega(s)_{s\in[0,t]})$, u solves the 'path-dependent' PDE ([P. 2005], [Dupire2009], [Fournie-Cont2013], [P.-Wang2012], [Ekren et. al. 2013], [P.-Song2014])

$$\mathcal{D}_t u(\omega,t) + \frac{1}{2}\Delta_x u(\omega,t) + g(u(\omega,t), \mathcal{D}_x u(\omega,t)) = 0, \quad u(\omega,T) = \varphi(\omega).$$

Path-derivatives for finite dimensional path-PDE [P.2005-2010], [Song-Peng2014]

$$u(\omega, t) = u_k(\omega(t_1), \cdots, \omega(t_k), \omega(t), t), \quad \text{for each } t \in [t_k, t_{k+1}),$$

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\mathcal{D}_t u(\omega, t) :=& \partial_{t+} u_k(x_1, \cdots, x_k, x, t)|_{x_1=\omega(t_1),\cdots,x_k=\omega(t_k),x=\omega(t)} \\
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\mathcal{D}_x^2 u(\omega, t) :=& \partial_x^2 u_k(x_1, \cdots, x_k, x, t)|_{x_1=\omega(t_1),\cdots,x_k=\omega(t_k),x=\omega(t)}, \\
\Delta_x u(\omega, t) :=& \mathrm{tr}[\mathcal{D}_x^2 u(\omega, t)].
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- Parameter estimation with dada: nonlinear statistics

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Nonlinear normal distributions

Definition

A random variable X in $(\Omega, \mathcal{H}, \mathbb{E})$ is normal if

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- $G(a) = \frac{1}{2}[\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-] \quad \overline{\sigma}^2 = \mathbb{E}[X^2], \;\; \underline{\sigma}^2 = -\mathbb{E}[-X^2]$

Nonlinear Law of large number

Theorem (Peng2008-2010)

Let $\{Y_i\}_{i=1}^{\infty}$ *be i.i.d. sequence. Assume*

$$\mathbb{E}[|Y_1|^{1+\delta}] < \infty.$$

Then, for each $\varphi \in C_b(\mathbb{R})$*,*

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\frac{Y_1 + \cdots + Y_n}{n})] = \max_{v\in[\underline{\mu},\overline{\mu}]} \varphi(v).$$

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Theorem (P.2010).

If $(B_t)_{t\geq 0}$ is a G–**Brownian motion** and $\mathbb{E}[B_t] = \mathbb{E}[-B_t] \equiv 0$ then:
$B_{t+s} - B_s \overset{d}{=} N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$, $\forall\ s, t \geq 0$ □

Probability v.s. Nonlinear Expectation

Probability Space	Nonlinear Expectation Space
$(\Omega, \mathcal{F}, P)$	$(\Omega, \mathcal{H}, \mathbb{E})$: (sublinear is basic)
Distributions: $X \overset{d}{=} Y$	
Independence: Y indep. of X	
LLN and CLT	
Normal distributions	
Brownian motion $B_t(\omega) = \omega_t$	
Quadratic variable. $\langle B \rangle_t = t$	
Lévy process	

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Brownian motion $B_t(\omega) = \omega_t$	Nonlinear B.M. $B_t(\omega) = \omega_t$,
Quadratic variable. $\langle B \rangle_t = t$	$\langle B \rangle_t$: still a nonlinear Brownian motion
Lévy process	

Probability v.s. Nonlinear Expectation

Probability Space	Nonlinear Expectation Space
$(\Omega, \mathcal{F}, P)$	$(\Omega, \mathcal{H}, \mathbb{E})$: (sublinear is basic)
Distributions: $X \overset{d}{=} Y$	$X \overset{d}{=} Y$,
Independence: Y indep. of X	Y indep. of X, (non-symm.)
LLN and CLT	nonlinear LLN and CTL
Normal distributions	nonlinear Normal distributions
Brownian motion $B_t(\omega) = \omega_t$	Nonlinear B.M. $B_t(\omega) = \omega_t$,
Quadratic variable. $\langle B \rangle_t = t$	$\langle B \rangle_t$: still a nonlinear Brownian motion
Lévy process	nonlinear Lévy process

Probability v.s. Nonlinear Expectation

Probability Space	**Nonlinear Expectation Space**
Itô's calculus for BM	Itô's calculus for nonlinear BM
SDE $dx_t = b(x_t)dt + \sigma(x_t)dB_t$	
Diffusion: $\partial_t u - \mathcal{L}u = 0$	
Markovian pro. and semi-grou	
Martingales	
$E[X\|\mathcal{F}_t] = E[X] + \int_0^T z_s dB_s$	

Probability v.s. Nonlinear Expectation

Probability Space	Nonlinear Expectation Space
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SDE $dx_t = b(x_t)dt + \sigma(x_t)dB_t$	$dx_t = \cdots + \beta(x_t)d\langle B\rangle_t$
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Diffusion: $\partial_t u - \mathcal{L}u = 0$	$\partial_t u - G(t, x, u, Du, D^2u) = 0$
Markovian pro. and semi-grou	
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Probability v.s. Nonlinear Expectation

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$E[X\|\mathcal{F}_t] = E[X] + \int_0^T z_s dB_s$	$\mathbb{E}[X\|\mathcal{F}_t] = \mathbb{E}[X] + \int_0^t z_s dB_s + K_t$ $K_t = \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds$

Probability Space	Nonlinear Expectation Space
P-almost surely analysis	
$X(\omega)$: P-quasi continuous $\Longleftrightarrow$ X is $\mathcal{B}(\Omega)$-meas.	

Probability Space	Nonlinear Expectation Space
P-almost surely analysis	c-quasi surely analysis
$X(\omega)$: P-quasi continuous $\iff$ X is $\mathcal{B}(\Omega)$-meas.	

Probability Space	Nonlinear Expectation Space
P-almost surely analysis	c-quasi surely analysis
	$c(A) = \sup_\theta E_{P_\theta}[\mathbf{1}_A]$
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From VaR to GVaR

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$$\mathrm{VaR}_\alpha^F(X) = -\inf\{x \mid P(X \le x) > \alpha\}$$
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Can we use G-normal distribution in the place of a linear distribution F?

$$\{F_\theta(x)\}_{\theta\in\Theta} : \text{Uncertain distributions of } N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$$

$$F_G(x) := \max_{\theta\in\Theta} F_\theta(x),$$

$$GVaR_\alpha = \max_{\theta\in\Theta} VaR_\alpha^{F_\theta}(X).$$

Explicit formula of $F_G(x)$

$$\mathrm{GVaR}_\alpha(X) := -\inf\{x \in \mathbb{R} : F_G(x) > \alpha\}.$$

We have

$$F_G(x) = \mathbb{E}_G[\mathbf{1}_{\{X \leq x\}}] = u(x,t)|_{t=1},$$

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u is the solution of the PDE

$$\partial_t u - G(\partial^2_{xx} u) = 0, \tag{1}$$

with the Cauchy

$$u(x,0) = \mathbf{1}_{[0,\infty)}(x). \tag{2}$$

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F_G has the explicit expression:

$$F_G(x) = \int_{-\infty}^{x} \frac{\sqrt{2}}{\sqrt{\pi(\overline{\sigma}+\underline{\sigma})^2}} \left[\exp(\frac{-y^2}{2\overline{\sigma}^2})\mathbf{1}_{y\le 0} + \exp(\frac{-y^2}{2\underline{\sigma}^2})\mathbf{1}_{y>0}\right] dy. \qquad (3)$$

Empirical test of robust VaR

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Comparison with historical simulation VaR (HVaR)

HVaR uses the historical data of $\{X_s\}_{t-l+1\leq s\leq t}$ to simulate the value $\mathrm{VaR}_{\alpha,t}(X_{t+1})$:

$$\mathrm{HVaR}_{\alpha,t}(X_{t+1}) = -\max\{x : \frac{\sharp(X_s \leq x : t-l+1 \leq s \leq t)}{l} \leq \alpha\}.$$

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$$\beta_{\mathrm{HVaR}_\alpha} := \sharp\{t \in [1, T-1] : -\mathrm{HVaR}_{\alpha,t}(X_{t+1}) > X_{t+1}\}/T.$$

- X_{t+1} is assumed to be G-normally distributed:

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- For each $\bar{t}$, use the passed 1 year data $\{X_{\bar{t}-s}\}_{0 \le s \le l-1}$ to estimate two parameters $\underline{\sigma}_{\bar{t}}^2$ and $\overline{\sigma}_{\bar{t}}^2$ at the day $\bar{t}$:

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- Fix a window width $w = 100$ use the moving window

$$\sigma_{\bar{t},w}^2 := \sigma^2(X_{\bar{t}-w+1}, \cdots, X_{\bar{t}}).$$

- Then get the upper and low data variances:

$$\overline{\sigma}_{\bar{t}}^2 = \max\{\sigma_{\bar{t},20}^2, \sigma_{\bar{t}-s,w}^2, ;\ s \in [0, \cdots, l - w]\},$$
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- Then get the upper and low data variances:

$$\overline{\sigma}^2_{\bar{t}} = \max\{\sigma^2_{\bar{t},20}, \sigma^2_{\bar{t}-s,w}, ;\ s \in [0, \cdots, l-w]\},$$
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- $\sigma^2_{\bar{t},w}(X_{\bar{t}-w+1}, \cdots, X_{\bar{t}})$ is the std of $(X_{\bar{t}-w+1}, \cdots, X_{\bar{t}})$.

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$$\mathrm{GVaR}_{\alpha,\bar{t}}(X_{\bar{t}+1}) = -\max\{x :\ \ F_{G_{\bar{t}}}(x) \le \alpha\}.$$

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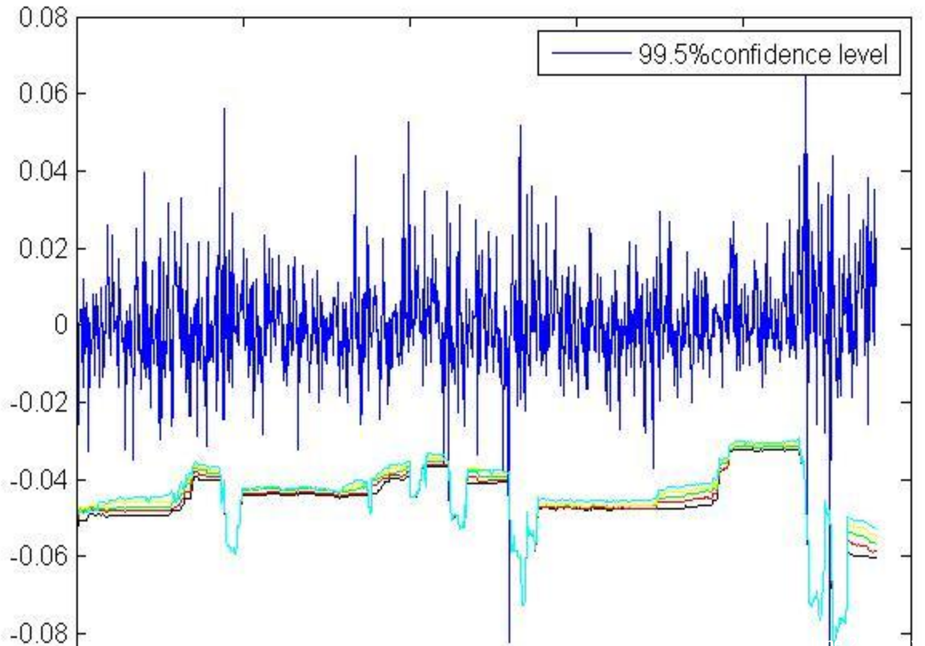
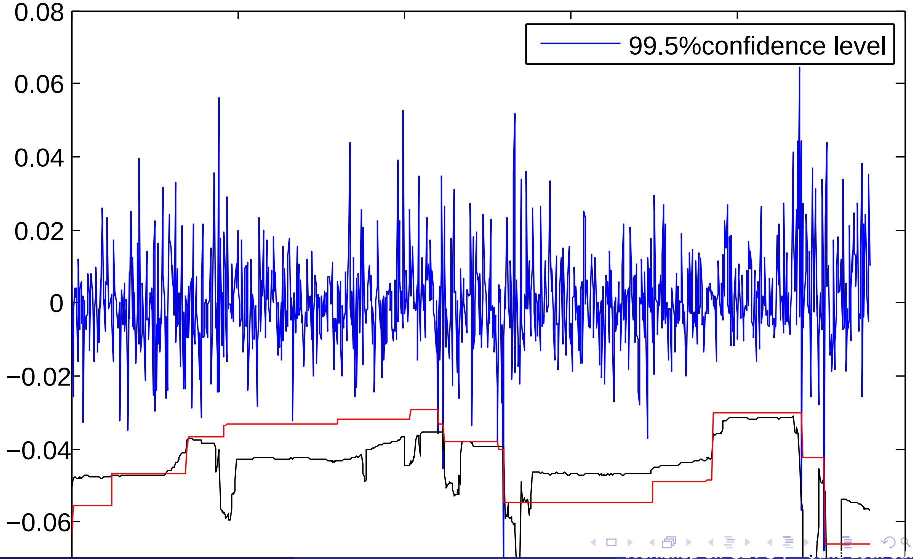
Tab. 1: $\beta_{\mathrm{GVaR}_\alpha}$ with Confidence level $\alpha = 99.5\%$					
Window width	80	90	100	110	120
$\beta_{\mathrm{GVaR}_\alpha}$	0.42	0.42	0.52	0.52	0.63

Tab. 1: $\beta_{\mathrm{GVaR}_\alpha}$ with Confidence level $\alpha = 99.5\%$					
Window width	80	90	100	110	120
$\beta_{\mathrm{GVaR}_\alpha}$	0.42	0.42	0.52	0.52	0.63

$$\beta_{\mathrm{GVaR}_\alpha} := \sharp\{t \in [1, T-1] : -\mathrm{GVaR}_{\alpha,t}(X_{t+1}) > X_{t+1}\}/T.$$


99.5%confidence level
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08

GVaR v.s. HVaR



GVaR v.s. HVaR

Tab.2: Break rate of HVaR v.s. Break rate of GVaR			
Confidence level α	99%	99.5%	99.7%
$\beta_{\mathrm{HVaR}_\alpha}$	0.94%	0.63 %	0.52%
$\beta_{\mathrm{GVaR}_\alpha}$	0.63%	0.52 %	0.31%

GVaR v.s. HVaR

Tab.2: Break rate of HVaR v.s. Break rate of GVaR			
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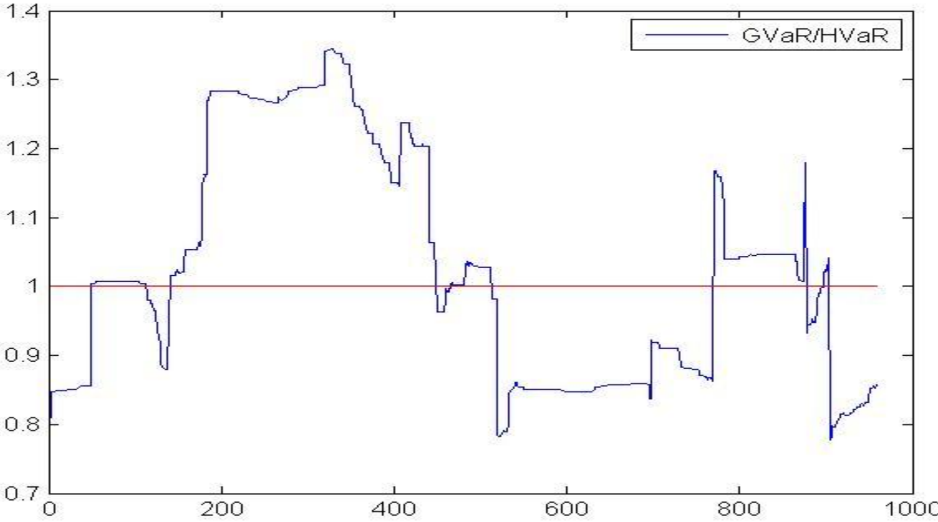
The rates of exceedence of GVaR is significantly lower than those of HVaR for each given α.


GVaR/HVaR
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
200
400
600
800
1000

- The curve $m(t) = \frac{\mathrm{GVaR}_{\alpha,t}(X_{t+1})}{HVaR_{\alpha,t}(X_{t+1})}$ with $\alpha = 99.5$. The red line is the time-average value $M = \int_0^T m(t)dt / T = 1.0826$.

- The curve $m(t) = \frac{\mathrm{GVaR}_{\alpha,t}(X_{t+1})}{HVaR_{\alpha,t}(X_{t+1})}$ with $\alpha = 99.5$. The red line is the time-average value $M = \int_0^T m(t)dt/T = 1.0826$.
- With $\alpha = 99.5\%$, the time-average value of $\frac{\mathrm{GVaR}_{\alpha,t}(X_{t+1})}{\mathrm{HVaR}_{\alpha,t}(X_{t+1})}$ is 1.0826. This means that, with the marginal account average 8.26% higher than HVaR, the rate of exceedence $\beta_{\mathrm{GVaR}_\alpha}$ of GVaR is 16.7% lower than that of HVaR.

Thank you