Variational Inequality, Financial Engineering and Optimal Control of Obstacle

Qihong Chen

Shanghai University of Finance and Economics

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Outline

Introduction

In recent years, variational inequalities and free boundary problems have been extensively used in many parts of Engineering, Mechanics, Chemistry, and Bioscience etc., even Economics and Finance.

This talk is concerned with various optimal control problems (especially, the obstacle control problem) for systems governed by a variational inequality.

Obstacle Problem

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This leads to

$$\begin{cases} y \geq \varphi \\ -y'' \geq 0 \\ -y''(y-\varphi) = 0 \end{cases}$$

which is a (stationary) differential complementarity system.

Complementarity Problem (CP)

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$$\text{Find } y \in H^2 \cap H_0^1$$

$$s.t. \begin{cases} y \geq \varphi \\ -\Delta y \geq f \\ (y - \varphi)(-\Delta y - f) = 0 \end{cases}$$

Variational Inequality (VI)

It is easily seen that any solution of (CP) satisfies the following

Variational Inequality (VI)

$$\text{Find } y \in H_0^1$$

$$s.t. \begin{cases} y \in K_\varphi \equiv \{z \in H_0^1 | z \geq \varphi \ a.e. \ in \ \Omega\} \\ \langle \nabla y, \nabla(z-y)\rangle \geq \langle f, z-y\rangle \qquad \forall z \in K_\varphi \end{cases}$$

On the other hand, any solution of (VI) with second order differentiability must be a solution of (CP).

So, in this sense, (VI) is a weak formulation of (CP).

Fully Nonlinear Partial Differential Equation (FNPDE)

(CP) can be equivalently written as the following

Fully Nonlinear Partial Differential Equation (FNPDE)

$$\text{Find } y \in H^2 \cap H_0^1$$

$$s.t.\ \min\{-\Delta y - f, y - \varphi\} = 0$$

$$(i.e. \quad -\Delta y - f + y - \varphi - |\Delta y + f + y - \varphi| = 0)$$

Differential Inclusion (DI)

Also, (CP) can be converted to the following

Differential Inclusion (DI)

$$\text{Find } y \in H^2 \cap H_0^1$$

$$s.t. \; -\Delta y + \beta(y - \varphi) \ni f$$

$$where \; \beta(r) = \begin{cases} \{0\} & r > 0 \\ (-\infty, 0] & r = 0 \\ \emptyset & r < 0 \end{cases}$$

Free Boundary Problem (FBP)

Inherently, (CP) can be formulated as the following

Free Boundary Problem (FBP)

$$\text{Find } y \in H^2 \cap H_0^1$$

$$s.t. \left\{ \begin{array}{c} -\Delta y = f \ \ in \ \Omega^+ \equiv \{\{x \in \Omega | y(x) > \varphi(x)\} \\ \nabla y \ \ is \ continuous \ on \ \partial\Omega^+ \cap \Omega \end{array} \right.$$

Stefan Problem

A typical FBP is the Stefan problem, which is a model for the melting or solidification of a pure material by heat transfer. Because the solid/liquid interface is a priori unknown, we must solve a free boundary problem, which is inherently nonlinear.

A

simple dimensionless model for the Stefan problem is

$$\begin{cases} u_t = u_{xx} \quad s(t) < x < +\infty & (modeling\ the\ flow\ of\ heat) \\ u(s(t),t) = 0 & (the\ phase-change\ temperature\ is\ 0) \\ -u_x(s(t),t) = \dot{s}(t) & (the\ energy\ is\ conserved) \end{cases}$$

Define a new variable $v(x,t)$ by the so-called Baiocchi transformation

$$v(x,t) = \int_{s^{-1}(x)}^{t} u(x,\tau) d\tau$$

where $s^{-1}(x)$ is the inverse of $s(t)$. Then the Stefan problem can be transformed into the following

Linear Complementarity Problem (LCP)

$$\begin{cases} v(x,t) \geq 0 \\ -v_t + v_{xx} + 1 \geq 0 \\ v(-v_t + v_{xx} + 1) = 0 \end{cases}$$

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Pricing of American Option

Variational inequalities often arise in Financial Engineering. A well-known example is the pricing model for American option.With European options, the payoff can only occur at the expiration date T. But for American options, the payoff can occur at any time τ on or before the expiration date T.

Suppose that $V(S,t)$ is the value of an American option depending on an underlying asset price S and with payoff $\varphi(S,t)$. Then the usual Black-Scholes model leads to the following

Linear Complementarity Problem (LCP)

$$\begin{cases} V(S,t) \geq \varphi(S,t) \\ -L_{BS}V \geq 0 \\ (V-\varphi)L_{BS}V = 0 \end{cases}$$

with $V(S,T) = \varphi(S,T)$, V and V_S being continuous.

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- 3.

$$(V - \varphi)L_{BS}V = 0$$

Either early exercise is optimal, in which case the option value is equal to that of the payoff, or the option satisfies Black-Scholes equation.

It is interesting that the determination of value for an American option is a version of the Stefan problem!

Local Volatility

Based on the assumption of constant volatility, the famous Black-Scholes formula can be used to evaluate the European option simply and quickly, with estimating or forecasting volatility constant as an input. The value of the option is monotonic in the volatility parameter. Then we can invert the B-S formula to determine the volatility (called implied volatility) from the market option price. If the model were perfectly realistic, the implied value would be the same for all options on the same underlying with different strikes and maturities. Unfortunately, this is not the case. Implied Black-Scholes volatilities vary with strikes and time to the maturity, which are known as thesmile effect and the term structure respectively.

There have been various attempts to extend the Black-Scholes theory to account for the volatility smile and the term structure. One class of models introduces a non-traded source of risk such as jumps or stochastic volatility. Another class of models introduces deterministic volatility function (calledlocal volatility) that varies with spot price and time.

Recovering Local Volatility from Market Option Prices in Optimal Control Framework

As is well known, when the stock price S follows a general diffusion of the form

$$\frac{dS}{S} = \mu dt + \sigma dW_t \tag{1}$$

where μ is the drift, σ is the stock volatility, and W_t is a standard Wiener process, the European call option premium $u = u(s, t; K, T)$ satisfies the following Black-Scholes equation

$$\begin{cases} u_t + \frac{1}{2}s^2\sigma^2 u_{ss} + (r-q)su_s - ru = 0 \quad (s,t) \in R^+ \times [0,T), \\ u(0,t) = 0, \\ u(s,T) = (s-K)^+ \end{cases} \tag{2}$$

where T is the maturity, K is the strike, r is the risk-free interest rate, q is the dividend yield (on the stock) and σ is the stock volatility as in (1), which is the only parameter (in the pricing model) that is not directly observable .

In the recent years, many options become liquid. With such plentiful data, option market has become a rich source of information. So it is natural to recover the unknown volatility from the observed market price of options for different K and/or T at current time t^* with current stock price s^*. In the continuous-time setting, this amounts to the following:

Problem I

Determine the coefficient σ, so that the solution of (2) fits the current market prices of options at (s^*, t^*) for different strikes K and/or maturity T.

From the mathematical point of view, this is an inverse problem of PDE. But it is not a standard one, since it requires us to determine the coefficient σ of the pricing equation by means of a series of observed values of the solution corresponding to certain parameters (K and/or T, usually discrete) at a fixed point (s^*, t^*). Such a problem is ill-posed in general.

Performing some calculations and using the well-known properties of Green function, we find that as a function of (K, T), $u(s, t; K, T)$ satisfies

$$\begin{cases} u_\tau = \frac{1}{2}K^2\sigma^2 u_{KK} - (r-q)Ku_K - qu & (K,\tau) \in R^+ \times R^+, \\ u|_{K=0} = e^{-q\tau}s, \\ u|_{\tau=0} = (s-K)^+ \end{cases} \tag{3}$$

where $\tau = T - t$ is the time remaining to maturity.

In principle, we can infer the volatility function from the complete knowledge of the option price. That is to say, if the current market prices of the options are known for all conceivable strikes K and maturity T, then the volatility function can be found directly from the equation in (3).

For simplicity, we assume that the volatility is time-independent and use only option prices with different strikes and a fixed maturity date.

The problem that we are involved in can be formulated as a typical inverse problem of parabolic equation with terminal observation.

Problem II

Find $\sigma(K)$ such that the solution of (3) satisfies

$$u(s^*, t^*; K, \tau)|_{\tau=\tau^*} = u^*(K)$$

where $\tau^* = T - t^*$ and $u^*(K)$ is the current market price of options for different $K > 0$ at current time t^* with stock price s^*.

For the sake of compatibility, $u^*(K)$ is assumed to be a continuous function satisfying

$$\lim_{K\to 0} u^*(K) = e^{-q\tau^*} s^*, \qquad \lim_{K\to\infty} u^*(K) = 0. \tag{4}$$

The above problem is not well-posed either. However, with the standardized form, we are able to solve it in an optimal control framework.

To remove the singularity at $K = 0$, we make the change of variables

$$y = \ln \frac{K}{s}, \qquad v = \frac{1}{s} e^{q\tau} u$$

in (3), which leads to a Cauchy problem

$$\begin{cases} v_\tau = a(y)(v_{yy} - v_y) - (r - q)v_y & (y, \tau) \in Q = R \times (0, \tau^*], \\ v|_{\tau=0} = (1 - e^y)^+ \end{cases} \tag{5}$$

where

$$v(y, \tau) = \frac{1}{s} e^{q\tau} u(K, \tau), \quad a(y) = \frac{1}{2}\sigma^2(K).$$

Under the above transformation, Problem II becomes

Problem II'

Find $a(y)$ such that the solution of (5) satisfies

$$v(y, \tau)|_{\tau=\tau^*} = v^*(y)$$

where $v^*(y) = \frac{1}{s^*} e^{q\tau^*} u^*(K)$ satisfying (by (4))

$$\lim_{y \to -\infty} v^*(y) = 1, \qquad \lim_{y \to +\infty} v^*(y) = 0. \tag{6}$$

We further impose the following condition on the given data:

$$\int_{\mathbb{R}} |v^*(y) - H(-y)|^2 dy < +\infty \tag{7}$$

where $H(\cdot)$ is the well-known Heviside function.
Let

$$A = \{a \in C(\mathbb{R}) |\ 0 < a_0 \leq a(y) \leq a_1, \quad \nabla a \in L^2(\mathbb{R})\}$$

be the control set, where a_0 and a_1 are the lower and upper bounds of half volatility square respectively.
The known theory for parabolic equations guarantees that for any given $a \in A$, there is a unique solution $v(y,\tau)$ to the Cauchy problem (5) with the property:

$$|v(y,\tau) - H(-y)| = O(e^{-|y|}) \qquad (y \to \infty). \tag{8}$$

(8), together with (7), ensures that

$$\int_{\mathbb{R}} |v(y,\tau) - v^*(y)|^2 dy < \infty, \quad \forall \tau \in [0, \tau^*].$$

This makes it possible for us to define a meaningful cost functional

$$J(a) = \frac{1}{2}\|v(\cdot,\tau^*) - v^*(\cdot)\|^2_{L^2(\mathbb{R})} + \frac{N}{2}\|\nabla a\|^2_{L^2(\mathbb{R})}, \qquad a \in A \tag{9}$$

where v is the solution of (5) corresponding to a.

Now we can introduce the following optimal control problem:

Problem III

Find an $\bar{a} \in A$, such that

$$J(\bar{a}) = \inf_{a\in A} J(a).$$

The unusual feature of our problem is that the control variable lies in the coefficient of the second-order partial derivative in the pricing PDE and moreover, the equation is in non-divergence form.

It is easily seen that, $J(a)$ is a non-negative lower semicontinuous functional and A is bounded in Hölder space $C^{\frac{1}{2}}(R)$. Based on the state analysis, we can establish the existence for the above optimal control problem.

Theorem

Existence

Problem III admits at least one optimal control $\bar{a} \in A$.

The optimality condition for our optimal control problem is

Theorem

Optimality Condition

Let $\bar{a} \in A$ be an optimal control of Problem III and $\bar{v}$ is the corresponding solution of (3). Then there exists a function $\bar{\varphi}$ solving the adjoint equation

$$\begin{cases} \varphi_\tau + (\bar{a}\varphi)_{yy} + (\bar{a}\varphi)_y + (r-q)\varphi_y = 0, \\ \varphi|_{\tau=\tau^*} = \bar{v}(y,\tau^*) - v^*(y). \end{cases} \tag{10}$$

such that $\bar{a}$ is a weak solution of the elliptic bilateral variational inequality

$$\begin{cases} a_0 \le a \le a_1, \\ (-a_{yy} + f(y;\bar{v},\bar{\varphi}))(a-a_0) \le 0, \\ (-a_{yy} + f(y;\bar{v},\bar{\varphi}))(a-a_1) \le 0 \end{cases} \tag{11}$$

where

$$f(y;\bar{v},\bar{\varphi}) = \frac{1}{N}\int_0^{\tau^*} \bar{\varphi}(\bar{v}_{yy} - \bar{v}_y)d\tau. \tag{12}$$

Then, in the optimal control framework, the local volatility can be successfully recovered from market option prices.

It turns out that, as an optimal control, the unknown local volatility solves an elliptic bilateral variational inequality coupled with a forward & backward parabolic system.([9])

More Examples

There are more examples of variational inequalities, which have played important roles in some recent studies on mathematical finance.

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More Examples

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- In [10], the pricing model of a callable convertible security is formulated as a parabolic bilateral variational inequality in order to study the optimal policies of the holder's conversion and issuer's calling for the callable American warrants and callable convertible bonds.
- In [11], the optimal investment problem with finite-horizon and transaction costs for a constant relative risk aversion investor is linked with a parabolic bilateral problem involving two free boundaries which correspond to the optimal buying and selling boundaries, respectively.

- In [12], to investigate the impact of macroeconomic conditions on irreversible investments under a regime switching model, the authors' main effort is to rigorously justify the existence and uniqueness of optimal threshold-type policies. Variational inequalities are used to characterize the optimal strategy by an abstract, nonconstructive reasoning.

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- In [13], the authors examine irreversible investment decisions in duopoly games with a variable economic climate. The problem is formulated as a stopping-time game under Stackelberg leader-follower competition, in which both players determine their respective optimal market entry date. By extending the variational inequality approach, they solve for the free boundaries, and obtain optimal investment strategies for each player.

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Optimal control for Variational Inequalities

The investigation of optimal control for nonlinear partial differential equations involving nonsmooth terms (including variational inequalities and free boundary problems) has attracted much attention in the literature.

The first work on the optimal control of variational inequalities was probably that of Mignot [14] in 1976. Later, many authors made contributions to this topic in different aspects, especially in the study of various version of the maximum principle ([15]-[19]).

We note that, for a long time, the optimality conditions for control problems governed by partial differential equations have been proved under the convexity condition of the control domain. (Thus, the Pontryagin original result is not included as a special case.) However, in many practical problems, the control domain usually is not necessarily convex (e.g. in the switching control problem).

It was Li and Yao who proved a maximum principle for evolutionary distributed parameter systems with a nonconvex control domain ([20]), which covered Pontryagin's result.

Later, Yong developed a method for deriving a Pontryagin type maximum principle for problems governed by semilinear elliptic partial differential equations and variational inequalities ([21]).

These pioneering works have established a foundation of infinite-dimensional optimal control theory with nonconvex control domains.

For evolutionary systems, there are many contributions devoted to the derivation of Pontryagin principle , in which an abstract evolution equation setting was commonly used.

We note that by using the abstract framework for evolutionary systems, some pointwise information on the state may be lost.

In [22], Pontryagin maximum principle for semilinear and quasilinear parabolic equations with pointwise state constraints was studied by using the framework of partial differential equation instead of the abstract framework.

This idea has also been followed by several authors ([24]-[27]). However, the case in which the nonlinear term is multivalued was not considered in these studies.

State-constrained Optimal Control for Evolutionary Variational Inequalities ([23])

Problem (C)

$$\min J(y,u) \equiv \int_Q L(x,t,y(x,t),u(x,t))dxdt$$

where (y,u) is a pair satisfying

$$\begin{cases} y \in W_2^{2,1}(Q) \cap L^2(0,T;H_0^1(\Omega)), & y|_{t=0} = y_0 \quad \text{in } \Omega \\ y \geq 0 & \text{in } Q \\ y_t - \Delta y - f(x,t,y,u) \geq 0 & \text{in } Q \\ (y_t - \Delta y - f)y = 0 & \text{in } Q \end{cases}$$

and the state-constraint of form

$$G(y) \in S.$$

Let us make the following assumptions.

(C$_1$) $\Omega \subset \mathbb{R}^n$ is a bounded region with $C^{1,1}$ boundary $\partial\Omega$; U is a Polish space (a separable complete metric space) and

$$\mathcal{U} = \{u \colon Q \to U \,|\, u(\cdot,\cdot) \text{ is measurable}\}.$$

(C$_2$) For some $\alpha \in (0,1)$ and any $p > 1$,

$$y_0 \in C_0^{\alpha}(\overline{\Omega}) \cap W^{2-2/p,p}(\Omega)$$

and

$$y_0 \geq 0 \qquad \text{a.e. in}\, \Omega.$$

(C$_3$) The function $f : \Omega \times [0,T] \times \mathbb{R} \times U \to \mathbb{R}$ have the following properties: $f(\cdot,\cdot,y,u)$ is measurable on $\Omega \times [0,T]$, and $f(x,t,\cdot,u)$ is in $C^1(\mathbb{R})$ with $f(x,t,\cdot,\cdot)$ and $f_y(x,t,\cdot,\cdot)$ continuous on $\mathbb{R} \times U$. Moreover, there exists a constant $M > 0$ such that

$$-M \leq f_y \leq 0 \qquad \text{on} \quad \Omega \times [0,T] \times \mathbb{R} \times U$$

$$|f(x,t,0,u)| \leq M \qquad \text{on} \quad \Omega \times [0,T] \times U.$$

(C_4) The function $L\colon \Omega \times [0, T] \times \mathbb{R} \times U \to \mathbb{R}$ satisfies: $L(\cdot, \cdot, y, u)$ is measurable on $\Omega \times [0, T]$, $L(x, t, \cdot, u)$ is in $C^1(\mathbb{R})$ with $L(x, t, \cdot, \cdot)$ and $L_y(x, t, \cdot, \cdot)$ continuous on $\mathbb{R} \times U$, and for any $R > 0$, there exists a constant $M_R > 0$ such that

$$|L| + |L_y| \leq M_R \qquad \text{on} \quad \Omega \times [0, T] \times [-R, R] \times U.$$

(C_5) Z is a Banach space with the dual Z^* being strictly convex. $S \subset Z$ is convex and closed, and is of finite codimension in Z. The map $G\colon\ C_0(\bar{Q}) \to Z$ is continuously Fréchet differentiable, where

$$C_0(\bar{Q}) = \{\eta \in C(\bar{Q}) |\ \eta|_\Sigma = 0\}.$$

(C_6) *Stability* of the optimal cost with respect to small perturbation of the state constraint.

Theorem

Pontryagin Principle *Let (C_1)–(C_6) hold and the following compatibility condition hold for the set S, the map G and the initial state y_0:*

$$supp\, G'(\eta)^*\partial d_S(G(\eta)) \subset Q \cup (\Omega \times \{T\})$$

$$\forall \eta \in C_0(\bar{Q}) \,\text{with}\, G(\eta) \in S, \quad \eta|_{t=0} = y_0.$$

Let $(\bar{y}, \bar{u})$ be an optimal pair for Problem (C). Then there exist $\bar{z} \in L^q(0, T; W_0^{1,q}(\Omega))$ with $1 < q < \frac{n+2}{n+1}$, $\bar{\lambda} \in [0, 1]$, $\bar{\mu} \in \mathcal{M}_0(\bar{Q})$ and $\bar{\nu} \in \partial d_S(G(\bar{y})) \subset Z^*$ such that

$$\bar{\lambda} + \|\bar{\nu}\|_{Z^*} > 0,$$

$$\begin{cases} -\bar{z}_t - \Delta \bar{z} - f_y(x, t, \bar{y}, \bar{u})\bar{z} = \bar{\lambda} L_y(x, t, \bar{y}, \bar{u}) - \bar{\mu} + G'(\bar{y})^*\bar{\nu}|_Q & \text{in } Q \\ \bar{z}|_\Sigma = 0 \\ \bar{z}|_{t=T} = G'(\bar{y})^*\bar{\nu}|_{\Omega\times\{T\}} \end{cases}$$

$$< \bar{\nu}, \ \eta - G(\bar{y}) > \le 0 \quad \forall \eta \in S \quad \text{(the transversality condition)}$$

$$supp\,\bar{\mu} \subset \{(x,t) \in Q |\, \bar{y}(x,t) = 0\}$$

and

$$\begin{aligned} & H(x,t,\bar{y}(x,t),\bar{u}(x,t),\bar{\lambda},\bar{z}(x,t)) \\ = \; & \min_{u\in U} H(x,t,\bar{y}(x,t),u,\bar{\lambda},\bar{z}(x,t)) \qquad \text{a.e.}(x,t) \in Q \end{aligned}$$

where

$$H(x,t,y,u,\lambda,z) = zf(x,t,y,u) + \lambda L(x,t,y,u)$$

for any $(x,t,y,u,\lambda,z) \in \Omega \times [0,T] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R}$.

Moreover, if

$$supp\,\bar{\mu} \cap supp\, G'(\bar{y})^* \bar{\nu} = \emptyset; \qquad \mathcal{N}(G'(\bar{y})^*) = \{0\},$$

then

$$(\bar{\lambda}, \bar{z}) \neq 0.$$

This work ([23]) may be viewed as a continuation of [22] (which gives Pontryagin principles for semilinear and quasilinear parabolic equations) and [21] (which concerns the elliptic version of the one considered here). In the derivation of Pontryagin principle, the co-state variable must satisfy the adjoint equation (usually in some weak sense). When the state system is a variational inequality as the case considered here, it is important to make a good approximation of the inequality by a convenient parabolic equation. Furthermore, to prove the convergence of the approximate co-state, we have to establish some uniform estimation for a family of parabolic PDE's involving measure data. Hence, there are more technical difficulties encountered in this study than that in [21] and [22].

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Minimax Control problem

If the cost functional, which is to be minimized, is the *sup* norm of some function of the state and the control, such a problem is usually referred to as a *minimax control problem* (since the problem is to minimize a "maximum").

Minimax control problems sometimes seem to arise more naturally in applications than the standard problem involving integral cost, especially when one is attempting to minimize the maximum deviation from the desired goal. Unfortunately, such problems were not well-studied (especially for infinite-dimensional systems).

As we know, the minimax control problem for ordinary differential equations was studied by E.N. Barron and the Pontryagin maximum principle for finite-dimensional minimax problem was proved in [28].

The first infinite-dimensional version of Pontryagin principle for minimax problem was presented in [29] (by J.Yong) with the state equation being a second order semilinear elliptic partial differential equation.

However, at our best knowledge, the minimax control problems for variational inequalities have never been discussed before. The major novelty of such a problem lies in the simultaneous presence of the nonsmooth state equation (variational inequality) and the nonsmooth cost functional (the sup norm). The *nonsmoothness* of the cost leads to more complicated necessary conditions for minimax control problems. That is one of the reasons for the lack of investigation.

Minimax Control for Elliptic Variational Bilateral Problem ([30])

Problem (ME)

$$\min J(y,u) \equiv \sup_{x\in\Omega} L(x, y(x), u(x))$$

where (y,u) is a pair satisfying

$$\begin{cases} y \in H^2(\Omega) \cap H_0^1(\Omega) & \\ \varphi \le y \le \psi & \text{in} \quad \Omega \\ (Ay - f(x,y,u))(y-\varphi) \le 0 & \text{in} \quad \Omega \\ (Ay - f(x,y,u))(y-\psi) \le 0 & \text{in} \quad \Omega \end{cases}$$

With respect to the control domain and the data involved, we make the following assumptions.

(ME$_1$) $\Omega \subset \mathbb{R}^n$ is a bounded region with $C^{1,1}$ boundary $\partial\Omega$; U is a Polish space (a separable complete metric space) and

$$\mathcal{U} = \{u\colon \Omega \to U |\, u(\cdot) \text{ is measurable}\}.$$

(ME$_2$) Operator A is defined by

$$Ay(x) = -\sum_{i,j=1}^{n} D_j(a_{ij}(x) D_i y(x))$$

with $a_{ij} \in C^1(\bar{\Omega})$, $\quad a_{ij} = a_{ji}, \quad 1 \leq i, j \leq n$, and for some $\lambda > 0$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda \sum_{i=1}^{n} |\xi_i|^2, \quad \forall x \in \Omega,\ (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n.$$

(ME$_3$) The function $f\colon \Omega\times\mathbb{R}\times U\to\mathbb{R}$ have the following properties: $f(\cdot,y,u)$ is measurable on Ω, and $f(x,\cdot,u)$ is in $C^1(\mathbb{R})$ with $f(x,\cdot,\cdot)$ and $f_y(x,\cdot,\cdot)$ continuous on $\mathbb{R}\times U$. Moreover, there exists a constant $K>0$, such that

$$-K\le f_y\le 0 \qquad \text{on} \quad \Omega\times\mathbb{R}\times U$$

and

$$|f(x,0,u)|\le K \qquad \text{on} \quad \Omega\times U.$$

(ME$_4$) The function $L\colon \Omega\times\mathbb{R}\times U\to\mathbb{R}$ is continuous, $L(x,\cdot,u)$ is in $C^1(\mathbb{R})$ with $L_y(x,\cdot,\cdot)$ continuous on $\mathbb{R}\times U$, and for any $R>0$, there exists a constant $K_R>0$, such that

$$|L|+|L_y|\le K_R \qquad \text{on} \quad \Omega\times[-R,R]\times U.$$

Moreover, there exists a nondecreasing continuous function $\omega\colon[0,+\infty)\to[0,+\infty)$ with $\omega(0)=0$, such that

$$\begin{aligned}|L(\tilde{x},\tilde{y},u)\,-\,L(x,y,u)| \quad &\le \omega(|\tilde{x}-x|+|\tilde{y}-y|)\\ &\forall (x,y,u),\ (\tilde{x},\tilde{y},u)\in\Omega\times\mathbb{R}\times U.\end{aligned}$$

Theorem

Pontryagin Principle for Problem (ME) *Let (ME_1)–(ME_4) hold and $(\bar{y}, \bar{u})$ be an optimal pair for Problem (ME). Then there exist $\bar{z} \in W^{1,p'}(\Omega)$ with $p' = \frac{p}{p-1} \in (1, \frac{n}{n-1})$ and $\bar{\lambda}, \bar{\mu} \in L^\infty(\Omega)^*$, satisfying*

$$\begin{cases} A\bar{z} - f_y(x, \bar{y}, \bar{u})\bar{z} = \bar{\lambda} L_y(x, \bar{y}, \bar{u}) + \bar{\mu} & \text{in}\, \Omega \\ \bar{z}|_{\partial\Omega} = 0 \end{cases}$$

and

$$\bar{\lambda}(\Omega) \equiv < \bar{\lambda}, \chi_\Omega > \geq a > 0$$

such that

$$\bar{z}(x) f(x, \bar{y}(x), \bar{u}(x)) = \min_{u \in U_0(x)} \bar{z}(x) f(x, \bar{y}(x), u) \qquad \text{a.e.} x \in \Omega_0$$

where

$$\begin{cases} \Omega_0 = \{x \in \Omega |\; L(x, \bar{y}(x), \bar{u}(x)) < \bar{J}\} \\ U_0(x) = \{u \in U |\; L(x, \bar{y}(x), u) < \bar{J}\} \qquad x \in \Omega \end{cases}$$

Moreover, in the case $m(\Omega_0) > 0$, for any $0 < \sigma < m(\Omega_0)$, there exists a measurable set $S_\sigma \subset \Omega_0$ with $m(S_\sigma) \geq \sigma$, such that

$$\bar{\lambda}(S_\sigma) = 0.$$

We note that in general, the above $\bar{\lambda}$ is only a finitely additive measure and is not necessarily in $\mathcal{M}(\bar{\Omega})$. If $\bar{\lambda}$ happens to be in $\mathcal{M}(\bar{\Omega})$, then there exists a measurable set $S \subset \Omega_0$ with $m(\Omega_0 \setminus S) = 0$, such that

$$\bar{\lambda}(S) = 0.$$

This means that the support of $\bar{\lambda}$ is disjoint with Ω_0.

Minimax Control for Parabolic Variational Bilateral Problem ([31])

Problem (MP)

$$\min J(y,u) \equiv \sup_{(x,t)\in Q} L(x,t,y(x,t),u(x,t))$$

where (y,u) is a pair satisfying

$$\begin{cases} y \in W_2^{2,1}(Q) \cap L^2(0,T;H_0^1(\Omega)), & y|_{t=0} = y_0 \quad \text{in}\,\Omega \\ \varphi \le y \le \psi & \text{in}\ Q \\ (y_t - \Delta y - f(x,t,y,u))(y-\varphi) \le 0 & \text{in}\ Q \\ (y_t - \Delta y - f(x,t,y,u))(y-\psi) \le 0 & \text{in}\ Q \end{cases}$$

Let us make the following assumptions.

(MP$_1$) $\Omega \subset \mathbb{R}^n$ is a bounded region with $C^{1,1}$ boundary $\partial\Omega$; U is a Polish space (a separable complete metric space) and

$$\mathcal{U} = \{u\colon Q \to U |\, u(\cdot,\cdot) \text{ is measurable}\}.$$

(MP$_2$) For some $\alpha \in (0,1)$ and any $p > 1$,

$$y_0 \in C_0^{\alpha}(\bar{\Omega}) \cap W^{2-1/p,p}(\Omega).$$

(MP$_3$) The function $f\colon \Omega \times [0, T] \times \mathbb{R} \times U \to \mathbb{R}$ has the following properties: $f(\cdot,\cdot,y,u)$ is measurable on $\Omega \times [0, T]$, $f(x,t,\cdot,u)$ is in $C^1(\mathbb{R})$ with $f(x,t,\cdot,\cdot)$ and $f_y(x,t,\cdot,\cdot)$ continuous on $\mathbb{R} \times U$. Moreover, there exists a constant $K > 0$, such that

$$|f_y| \leq K \qquad \text{on} \quad \Omega \times [0, T] \times \mathbb{R} \times U$$

and

$$|f(x,t,0,u)| \leq K \qquad \text{on} \quad \Omega \times [0, T] \times U.$$

(MP$_4$) The function $L\colon \Omega \times [0, T] \times \mathbb{R} \times U \to \mathbb{R}$ is continuous, $L(x,t,\cdot,u)$ is in $C^1(\mathbb{R})$ with $L_y(x,t,\cdot,\cdot)$ continuous on $\mathbb{R} \times U$, and for any $R > 0$, there exists a constant $K_R > 0$, such that

$$|L| + |L_y| \leq K_R \qquad \text{on} \quad \Omega \times [0, T] \times [-R, R] \times U.$$

Moreover, there exists a nondecreasing continuous function $\omega\colon [0,+\infty) \to [0,+\infty)$ with $\omega(0) = 0$, such that

$$\begin{aligned} |L(\tilde{x},\tilde{t},\tilde{y},u) - L(x,t,y,u)| &\leq \omega(|\tilde{x}-x| + |\tilde{t}-t| + |\tilde{y}-y|) \\ &\forall (x,t,y,u),\ (\tilde{x},\tilde{t},\tilde{y},u) \in \Omega \times [0, T] \times \mathbb{R} \times U \end{aligned}$$

Theorem

Pontryagin Principle for Problem (MP) *Let (MP_1)–(MP_4) hold and $(\bar{y}, \bar{u})$ be an optimal pair for Problem (MP). Then there exist $\bar{z} \in L^q(0, T; W_0^{1,q}(\Omega))$ with $1 < q < \frac{n+2}{n+1}$ and $\bar{\lambda}, \bar{\mu} \in L^\infty(Q)^*$ satisfying*

$$\begin{cases} -\bar{z}_t - \triangle \bar{z} - f_y(x,t,\bar{y},\bar{u})\bar{z} = \bar{\lambda} L_y(x,t,\bar{y},\bar{u}) + \bar{\mu} \quad \text{in } Q \\ \bar{z}|_\Sigma = 0, \qquad \bar{z}|_{t=T} = 0 \end{cases}$$

and

$$\bar{\lambda}(Q) \equiv < \bar{\lambda}, \chi_Q > \geq a > 0$$

such that, for a.e.$(x,t) \in Q_0$,

$$\bar{z}(x,t) f(x,t,\bar{y}(x,t),\bar{u}(x,t)) = \min_{u \in U_0(x,t)} \bar{z}(x,t) f(x,t,\bar{y}(x,t),u)$$

where

$$\begin{cases} Q_0 = \{(x,t) \in Q |\, L(x,t,\bar{y}(x,t),\bar{u}(x,t)) < \bar{J}\} \\ U_0(x,t) = \{u \in U |\, L(x,t,\bar{y}(x,t),u) < \bar{J}\} \qquad (x,t) \in Q \end{cases}$$

One of the motivation for the minimax control of variational inequalities is the following:

Consider the deformation of a membrane constrained by one or two obstacles. We would like to design the shape of the membrane so that the largest deviation of the perpendicular displacement y from the desired one, say y_d, is minimized. In the elliptic case, we could take $L(x, y, u) = |y - y_d(x)|^2$.

The above result of optimality condition gives a new version of Pontryagin principle for nonsmooth infinite-dimensional minimax control problem.

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Optimal Control of Obstacle

When the governing system is an obstacle variational inequality, the obstacle itself can also be regarded as the control (as first suggested in [32]). Such a case is referred to as an *obstacle control problem*.

To the best of our knowledge, such problem was first studied in [33], for an elliptic (homogeneous) variational inequality. By virtue of the properties of the super-harmonic functions, the existence and uniqueness as well as characterizations of the optimal pair are established in [33].

To study the obstacle control problem for more general system, an indirect obstacle control model is suggested in [34]. Motivated by [33] and [34], the regularity of the obstacle control problem has been investigated in [37].

The work in [33] has been extended by adding a non-zero source term to the right hand side of the state equation and it has been found in [39] that such an extension is not trivial.

Obstacle Control of Quasilinear Elliptic Variational Inequalities ([35])

Problem (OE)

$$\min J(y,\varphi,\psi) \equiv \int_\Omega \left\{ \frac{1}{2}(y-y_d)^2 + \frac{1}{p}[|\Delta\varphi|^p + |\Delta\psi|^p] \right\} dx$$

where (y,φ,ψ) is a triple satisfying

$$\begin{cases} y \in H^2(\Omega) \cap H_0^1(\Omega) & \\ \varphi \le y \le \psi & \text{in} \quad \Omega \\ -divA(x,\nabla y)(y-\varphi) \le 0 & \text{in} \quad \Omega \\ -divA(x,\nabla y)(y-\psi) \le 0 & \text{in} \quad \Omega \end{cases}$$

Here, $A(x,\eta) = (a_1(x,\eta),\cdots,a_n(x,\eta))$ is nonlinear in η, and the input control is the pair of upper and lower obstacles.

We make the following two assumptions on $A(x,\eta)$:

(OE$_1$) For any $\eta = (\eta_1, \cdots, \eta_n) \in \mathbb{R}^n$, $a_j(\cdot,\eta)$ is a measurable function on Ω with $a_j(\cdot,0)=0$ and for any $x\in\Omega$, $a_j(x,\cdot)$ belongs to $C^1(\mathbb{R}^n)$, $j=1,\cdots,n$.

(OE$_2$) For any $p \geq 2$, $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} \frac{\partial a_j}{\partial \eta_i}(x,\eta)\xi_i\xi_j \geq \Lambda_1(k+|\eta|)^{p-2}|\xi|^2$$

$$\sum_{i,j=1}^{n} \left|\frac{\partial a_j}{\partial \eta_i}(x,\eta)\right| \leq \Lambda_2|\eta|^{p-2}$$

where $k \in (0,1]$, Λ_1 and Λ_2 are some positive constants.

The following lemma is an immediate consequence of assumptions (OE_1) and (OE_2).

Lemma

Under Assumptions (OE_1)-(OE_2), there are positive constants k_1 and k_2 depending only on n, Λ_1 and Λ_2 such that for any $x \in \Omega$, $\eta = (\eta_1, \cdots, \eta_n) \in \mathbb{R}^n$ and $\eta' = (\eta'_1, \cdots, \eta'_n) \in \mathbb{R}^n$

(a)

$$\sum_{j=1}^{n} (a_j(x,\eta) - a_j(x,\eta'))(\eta_j - \eta'_j) \geq k_1 |\eta - \eta'|^p.$$

(b)

$$\sum_{j=1}^{n} |a_j(x,\eta)| \leq k_2 |\eta|^{p-1}.$$

Theorem

Optimality Condition *Assume* (OE_1) *and* (OE_2)*. Let* $(\bar{y}, \bar{\varphi}, \bar{\psi})$ *be an optimal triple to Problem (OE). Then there exists* $\bar{z} \in H_0^1(\Omega)$ *and* $\bar{\mu} \in H^{-1}(\Omega) \cap \mathcal{M}(\bar{\Omega})$ *satisfying*

$$\begin{cases} -div\left(\frac{\partial A}{\partial \eta}(x, \nabla \bar{y})^T \nabla \bar{z}\right) + \bar{\mu} = \bar{y} - y_d & in\ \Omega \\ \bar{z}|_{\partial\Omega} = 0 \end{cases}$$

and

$$supp\, \bar{\mu} \subset \{x \in \Omega| \quad \bar{y}(x) = \bar{\varphi}(x) \quad or \quad \bar{y}(x) = \bar{\psi}(x)\} \qquad (as\ p > \frac{n}{2})$$

such that

$$\Delta(|\Delta\bar{\varphi}|^{p-2}\Delta\bar{\varphi} + |\Delta\bar{\psi}|^{p-2}\Delta\bar{\psi}) + \bar{\mu} = 0 \qquad in\ \Omega$$

where $\mathcal{M}(\bar{\Omega})$ *is the set of all regular signed measures on* $\bar{\Omega}$*.*

Obstacle Control of Semilinear Parabolic Variational Inequalities ([36])

Theorem

Problem (OP)

$$\min J(y,\varphi,\psi) \equiv \int_Q \left\{ \frac{1}{2}(y-y_d)^2 + \frac{1}{p}[|\varphi_t|^p + |\Delta\varphi|^p + |\psi_t|^p + |\Delta\psi|^p] \right\} dxdt$$

where (y,φ,ψ) *is a triple satisfying*

$$\begin{cases} y \in W_2^{2,1}(Q) \cap L^2(0,T;H_0^1(\Omega)), \quad y|_{t=0} = y_0 & \text{in } \Omega \\ \varphi \le y \le \psi & \text{in } Q \\ (y_t - \Delta y - f(x,t,y))(y-\varphi) \le 0 & \text{in } Q \\ (y_t - \Delta y - f(x,t,y))(y-\psi) \le 0 & \text{in } Q \end{cases}$$

We assume that

(OP$_1$)

$$y_0 \in C_0^{\alpha}(\bar{\Omega}) \cap W^{2-\frac{1}{p},p}(\Omega)$$

for some $\alpha \in (0,1)$ and any $p > 1$.

(OP$_2$) the function $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ has the following properties:

- $f(\cdot,\cdot,y)$ is measurable on $\Omega \times [0,T]$;
- $f(x,t,\cdot)$ is a decreasing function in $C^1(\mathbb{R})$, so it is monotone;
- $f(x,t,\cdot)$ is a Lipschitz function, and thus, there exists a constant $K > 0$, such that

$$|f(x,t,y_1) - f(x,t,y_2)| \le K\,|y_1 - y_2| \quad \forall\, y_1,\ y_2 \in \mathbb{R};\ (x,t) \in \Omega \times [0,T]$$

$$|f(x,t,0)| \le K \qquad \forall\,(x,t) \in \Omega \times [0,T].$$

Theorem

Optimality Condition *Assume (OP_1) and (OP_2). Let $(\bar{y}, \bar{\varphi}, \bar{\psi})$ be an optimal triple to Problem (OP). Then for $p > (n+2)/2$, there exist $\bar{z} \in L^2(0, T; H_0^1(\Omega))$ and $\bar{\mu} \in \mathcal{M}_0(\bar{Q})$ satisfying*

$$\begin{cases} -\bar{z}_t - \Delta\bar{z} - f_y(x,t,\bar{y})\bar{z} = \bar{y} - y_d - \bar{\mu} & \text{in } Q \\ \bar{z}|_\Sigma = 0 \\ \bar{z}|_{t=T} = 0 \end{cases}$$

such that

$$\begin{aligned} -\int_Q [(|\bar{\varphi}_t|^{p-2}\bar{\varphi}_t + |\bar{\psi}_t|^{p-2}\bar{\psi}_t)w_t + & \quad (|\Delta\bar{\varphi}|^{p-2}\Delta\bar{\varphi} + |\Delta\bar{\psi}|^{p-2}\Delta\bar{\psi})\Delta w]dxdt \\ = \langle\bar{\mu}, w\rangle & \quad \forall w \in \dot{W}_p^{2,1}(Q) \end{aligned}$$

and

$$\int_Q [\bar{y}_t - \Delta\bar{y} - f(x,t,\bar{y})]\bar{z}\,dxdt = 0.$$

Moreover,

$$-\bar{y}_t + \Delta\bar{y} + f(x,t,\bar{y}) = \bar{\lambda}_\varphi + \bar{\lambda}_\psi$$

with

$$\langle\bar{\lambda}_\varphi, \bar{y} - \bar{\varphi}\rangle = 0; \qquad \langle\bar{\lambda}_\psi, \bar{y} - \bar{\psi}\rangle = 0$$

$$\operatorname{supp}\bar{\lambda}_\varphi \subset \{(x,t) \in Q |\, \bar{y}(x,t) = \bar{\varphi}(x,t)\}$$

$$\operatorname{supp}\bar{\lambda}_\psi \subset \{(x,t) \in Q |\, \bar{y}(x,t) = \bar{\psi}(x,t)\}$$

and

$$\bar{\mu} = \bar{\mu}_\varphi + \bar{\mu}_\psi$$

with

$$\langle\bar{\mu}_\varphi, \bar{y} - \bar{\varphi}\rangle = 0; \qquad \langle\bar{\mu}_\psi, \bar{y} - \bar{\psi}\rangle = 0$$

$$\operatorname{supp}\bar{\mu}_\varphi \subset \{(x,t) \in Q |\, \bar{y}(x,t) = \bar{\varphi}(x,t)\}$$

$$\operatorname{supp}\bar{\mu}_\psi \subset \{(x,t) \in Q |\, \bar{y}(x,t) = \bar{\psi}(x,t)\}.$$

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Thanks!