Null controllability of one-dimensional parabolic equations by the flatness approach

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This talk is based on joint work with:

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Outline

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth open set, $\Gamma_0 \subset \partial\Omega$ be a (nonempty) open set, and $T > 0$.

We are concerned with the **null controllability problem**:
given θ_0, find a function $u = u(t,x)$ s.t. the solution of

$$
\begin{array}{rcll}
\theta_t - \Delta\theta & = & 0 & (t,x) \in (0,T) \times \Omega, \\
\dfrac{\partial\theta}{\partial\nu} & = & 1_{\Gamma_0} u(t,x) & (t,x) \in (0,T) \times \Omega, \\
\theta(0,x) & = & \theta_0(x), & x \in \Omega
\end{array}
$$

satisfies

$$
\theta(T,x) = 0 \qquad x \in \Omega
$$

Huge literature...

- Duality methods (observability estimate for the adjoint eq.)
  - **Fattorini-Russell '71**, **Luxembourg-Korevarr '71**, **Dolecki '73** (1D, using biorthogonal families and complex analysis)
  - **Lebeau-Robbiano '95**, **Imanuvilov-Fursikov 96'** (ND, $\forall(\Omega, \Gamma_0, T)$, using Carleman estimates)
- Direct methods
  - **Jones '77**, **Littman '78** (construction of a fundamental solution with compact support in time, $\Gamma_0 = \partial\Omega$)
  - **Littman-Taylor 2007** (solution of ill-posed problems)
  - **Laroche-Martin-Rouchon 2000** (approximate controllability using a flatness approach)

Here, we shall revisit the flatness approach, obtain the **null controllability**, and show its relevance to numerics.

- Roughly: parameterisation of the trajectory by some (flat) output; introduced in 1995 by **M. Fliess, J. Lévine, Ph. Martin, P. Rouchon** for (linear or nonlinear) ODE; very useful for motion planning of mechanical systems
- Method applied then by **Laroche-Martin-Rouchon** in 2000 to derive the approximate controllability of (i) the 1D heat eq; (ii) the beam equation; (iii) the linearized KdV equation.
- The first control problem considered reads:

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad x \in (0,1) \\ \theta_x(t,0) &= 0, \quad \theta_x(t,1) = u(t), \\ \theta(0,x) &= \theta_0. \end{aligned}$$

- They proved in 2000 that for **"nice"** initial data decomposed as

$$\theta_0(x) = \sum_{i \geq 0} y_i \frac{x^{2i}}{(2i)!}$$

  with

$$|y_i| \leq C \frac{i!^s}{R^i}, \qquad i \geq 0$$

  with $s \in (1,2)$, C, $R > 0$, the system can be driven to 0 with a control $u(t)$ that is Gevrey of order s.

Take $y = \theta(t, 0)$ as output. It is **flat**, in the sense that the map $\theta \to y$ is a **bijection** between appropriate spaces of functions.

Seek a formal solution (analytic in x) in the form

$$\theta(t, x) = \sum_{i \geq 0} a_i(t) \frac{x^i}{i!}$$

Plugging this sum in the heat eq. gives $\sum_{i \geq 0} [a_{i+2} - a_i{}'] \frac{x^i}{i!} = 0$ ($' = d/dt$), and hence

$$a_{i+2} = a_i', \quad i \geq 0.$$

Since $a_0(t) = \theta(t, 0) = y(t)$ and $a_1(t) = 0$, we arrive to

$$a_{2i+1} = 0, \quad a_{2i} = y^{(i)}, i \geq 0,$$

$$\theta(t, x) = \sum_{i \geq 0} y^{(i)}(t) \frac{x^{2i}}{(2i)!}, \quad u(t) = \theta_x(t, 1) = \sum_{i \geq 1} \frac{y^{(i)}(t)}{(2i-1)!}$$

- Since $\theta(t,x) = \sum_{i\geq 0} y^{(i)}(t)\frac{x^{2i}}{(2i)!}$, it remains to find $y \in C^\infty([0,T])$ s.t. the series converges and

$$y^{(i)}(0) = y_i, \quad y^{(i)}(T) = 0, \quad i \geq 0.$$

  Impossible to do with an analytic function, but possible with a function Gevrey of order $s > 1$!

- Definition: a function $y \in C^\infty([0,T])$ is **Gevrey of order** $s \geq 0$ if there exist $R, C > 0$ such that

$$|y^{(p)}(t)| \leq C\frac{p!^s}{R^p}, \quad \forall p \in \mathbb{N}, \ \forall t \in [0,T]$$

  The larger s, the less regular y is ($s = 1 \iff y \in C^\omega$)

First main result (Automatica, 2014)

Theorem

Let $\theta_0 \in L^2(0,1)$ and $T > 0$. Pick $\tau \in (0,T)$ and $s \in (1,2)$. There exists $y \in C^\infty([\tau,T])$ Gevrey of order s on $[\tau,T]$ such that, setting

$$u(t) = \begin{cases} 0 & \textit{if } 0 \le t \le \tau \\ \sum_{i\ge 0} \frac{y^{(i)}(t)}{(2i-1)!} & \textit{if } \tau < t \le T, \end{cases}$$

the solution θ of

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad x \in (0,1) \\ \theta_x(t,0) = 0,\ \theta_x(t,1) &= u(t), \\ \theta(0,x) &= \theta_0(x) \end{aligned}$$

satisfies $\theta(T,.) = 0$.

We apply a null control to smooth out the state and reach the class of states for which Laroche-Martin-Rouchon' result is valid.

Decomposing the initial state θ_0 as a Fourier series of cosines $\theta_0(x) = \sum_{n\geq 0} c_n \sqrt{2}\cos(n\pi x)$, we obtain

$$\theta(\tau, x) = \sum_{n\geq 0} c_n e^{-n^2\pi^2\tau} \sqrt{2}\cos(n\pi x) = \sum_{i\geq 0} y_i \frac{x^{2i}}{(2i)!}$$

Lemma

$$|y_i| \leq C\frac{i!}{\tau^i} \qquad \forall i \geq 0$$

for some constant $C > 0$, so that $x \to \theta(\tau, x)$ is Gevrey of order $1/2$.

Proposition

(Flatness property) *Let* $s \in (1,2)$ *and* $y \in C^\infty([t_1,t_2])$ $(-\infty < t_1 < t_2 < \infty)$ *be Gevrey of order s on* $[t_1,t_2]$. *Let*

$$\theta(t,x) := \sum_{i\geq 0} \frac{x^{(2i)}}{(2i)!} y^{(i)}(t).$$

Then θ *is Gevrey of order* s *in* t *and* $s/2$ *in* x *on* $[t_1,t_2]\times[0,1]$, *and it solves the ill-posed problem*

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad (t,x)\in[t_1,t_2]\times[0,1], \\ \theta(t,0) &= y(t), \\ \theta_x(t,0) &= 0. \end{aligned}$$

It remains to design a function $y \in C^\infty([\tau,T])$ Gevrey of order $s \in (1,2)$ such that

$$y^{(i)}(\tau) = y_i, \qquad y^{(i)}(T) = 0, \qquad \forall i \geq 0.$$

This is Laroche-Martin-Rouchon' result.

- Let

$$\bar{y}(t) = \sum_{i\geq 0} y_i \frac{(t-\tau)^i}{i!}$$

Since $|y_i| \leq Ci!/\tau^i$, $\bar{y}$ is analytic on $[\tau, \tau + R]$ if $R < \tau$.
(Actually, $\bar{y}$ can be extended to $(\tau, +\infty)$ as an analytic function)

- For any $s \in (1,2)$, we introduce the "Gevrey step function"

$$\phi_s(t) = \begin{cases} 1 & \text{if } t \leq 0 \\ \frac{e^{-(1-t)^{-\kappa}}}{e^{-(1-t)^{-\kappa}} + e^{-t^{-\kappa}}} & \text{if } 0 < t < 1 \\ 0 & \text{if } t \geq 1. \end{cases}$$

where $\kappa = (s-1)^{-1}$. Then ϕ_s is Gevrey of order s.

- For y, we pick $s \in (1,2)$, $0 < R \leq T - \tau$ (where $0 < \tau < T$), and set

$$y(t) := \phi_s(\frac{t-\tau}{R})\bar{y}(t), \qquad t \in [\tau, T].$$

- We are interested in describing the states θ_1 that can be reached at time T from 0 (as in **Fattorini-Russell '71**):

$$\begin{array}{rcll} \theta_t - \theta_{xx} & = & 0, & x \in (0,1),\ t \in (0,T) \\ \theta(t,0) = h_0(t),\ \theta(t,1) & = & h_1(t), & t \in (0,T) \qquad [h_0, h_1 \in L^2(0,T)] \\ \theta(0,x) & = & 0 & \\ \theta(T,x) & = & \theta_1(x) & \end{array}$$

- $\theta_1(x) = \sum_{n\geq 1} c_n \sin(n\pi x)$ is a **reachable state** if

$$\exists \varepsilon > 0 \text{ s.t.} \qquad \sum_{n=1}^{\infty} |c_n| n^{-1} e^{(1+\varepsilon)n\pi} < \infty \qquad \text{(Fattorini-Russell, 1971)}$$

$$\sum_{n=1}^{\infty} |c_n|^2 n e^{2n\pi} < \infty \qquad \text{(Ervedoza-Zuazua, 2011)}$$

- Both (FR) and (EZ) imply that θ_1 has to satisfy the condition

$$\theta_1^{(2p)}(0) = \theta_1^{(2p)}(1) = 0 \quad \forall p \in \mathbb{N}.$$

Very conservative!! (no nontrivial polynomial function concerned!!)

Notation: $H(\Omega)$ denotes the set of (complex) analytic functions in the domain $\Omega \subset \mathbb{C}$.

Theorem

1. *If $\theta_1 \in H(\{z;\ |z-\frac{1}{2}| < \frac{R}{2}\}$ with $R > R_0 := e^{(2e)^{-1}} \sim 1.2$, then θ_1 is reachable from 0 in any time $T > 0$.*
2. *Conversely, any reachable state belongs to*

$$H(\{z = x + iy;\ |x - \frac{1}{2}| + |y| < \frac{1}{2}\})$$

Examples (1) Any polynomial function is reachable!!
(2) Let

$$\theta_1(x) := \frac{1}{(x-\frac{1}{2})^2 + a^2}.$$

Then θ_1 is

- reachable if $|a| > R_0/2 \sim 0.6$
- not reachable if $|a| < 0.5$

Method of proof: flatness approach + (new) **Borel-Ritt** theorem
We have similar results with **only one** control, of Dirichlet or Neumann type.

- Consider now the equation

$$(a(x)\theta_x)_x + b(x)\theta_x + c(x)\theta - \rho(x)\theta_t = 0$$

  where $a, b, c, \rho \in L^1(0,1)$.

- **Alessandrini-Escauriaza (2006)** proved the null controllability of this equation (with internal or Dirichlet boundary control) when $a, b, c, \rho \in L^\infty(0,1)$ with

$$a(x) > K > 0 \quad \text{and } \rho(x) > K > 0 \quad \text{a.e. in } (0,1)$$

  Method of proof: **Lebeau-Robbiano '95** + complex variable analysis.

- We shall see that this result can be **extended** to parabolic equations with singular or degenerate coefficients by using the flatness approach. (for degenerate eq., we refer to **Cannarsa-Martinez-Vancostenoble** 2004,...)

Let a, b, c, ρ with

$$a(x) > 0 \text{ and } \rho(x) > 0 \quad \text{for a.e. } x \in (0,1)$$
$$(\frac{1}{a}, \frac{b}{a}, c, \rho) \in [L^1(0,1)]^4$$
$$\exists K \geq 0, \ \frac{c(x)}{\rho(x)} \leq K \quad \text{for a.e. } x \in (0,1)$$
$$\exists p \in (1, \infty], \ \ a^{1-\frac{1}{p}} \rho \in L^p(0,1).$$

Theorem

Let (a, b, c, ρ) be as above, and $(\alpha_0, \beta_0) \neq (0,0)$, $(\alpha_1, \beta_1) \neq (0,0)$. Let $\theta_0 \in L^1_{\rho(x)dx}(0,1)$ and $T > 0$. Pick $\tau \in (0,T)$ and $s \in (1, 2 - p^{-1})$. Then there exists a control $h = h(t)$ Gevrey of order s on $[0,T]$ such that the solution θ of

$$\begin{aligned}
(a(x)\theta_x)_x + b(x)\theta_x + c(x)\theta - \rho(x)\theta_t &= 0, \qquad x \in (0,1) \\
\alpha_0\theta(t,0) + \beta_0(a\theta_x)(t,0) &= 0, \\
\alpha_1\theta(t,1) + \beta_1(a\theta_x)(t,1) &= h(t), \\
\theta(0,x) &= \theta_0(x)
\end{aligned}$$

satisfies $\theta(T,.) = 0$.

- $(a(x)\theta_x)_x - \theta_t = 0$, with $a(x) > 0$ a.e. and

$$a, 1/a \in L^1(0,1)$$

Possible: $a(x) \sim (x - x_0)^r$ with
  - $-1 < r < 0$ (**singular**) or
  - $0 < r < 1$ (**weakly degenerate**)

at a **single point** $x_0 \in [0,1]$, or at a **sequence** of points. Ex:

$$a(x) = |\sin(x^{-1})|^r, \quad -1 < r < 1$$

- Transmission pb. for the heat eq. (piecewise constant coef.)

$$\begin{aligned}
\rho_0\theta_t &= a_0\theta_{xx}, \; 0 < x < X \\
\rho_1\theta_t &= a_1\theta_{xx}, \; X < x < 1 \\
\theta(t, X^-) &= \theta(t, X^+) \\
a_0\theta_x(t, X^-) &= a_1\theta_x(t, X^+)
\end{aligned}$$

- **Step 1.** Using changes of variables, we can put the system in the **canonical form**

$$\begin{aligned} \theta_{xx} - \rho(x)\theta_t &= 0, \qquad x \in (0,1) \\ \alpha_0\theta(t,0) + \beta_0\theta_x(t,0) &= 0, \\ \alpha_1\theta(t,1) + \beta_1\theta_x(t,1) &= h(t), \\ \theta(0,x) &= \theta_0(x) \end{aligned}$$

  where $\rho \in L^p(0,1)$, $1 < p \leq \infty$.

- **Step 2.** In the time interval $(0,\tau)$, we apply a **null control to smooth out the state**, while in the interval (τ, T) we apply a **non-trivial control to reach 0** at time $t = T$.
- The trajectory reads then

$$\begin{aligned} \theta(x,t) &= \sum_{n\geq 0} e^{-\lambda_n t} e_n(x), \quad x \in (0,1), \ t \in [0,\tau], \\ \theta(x,t) &= \sum_{i\geq 0} y^{(i)}(t) g_i(x), \quad x \in (0,1), \ t \in [\tau, T]. \end{aligned}$$

- For $t \in (0, \tau)$, $\theta(x, t) = \sum_{n \geq 0} e^{-\lambda_n t} e_n(x)$.
  (e_n, λ_n) is the nth pair of eigenfunction/eigenvalue for

$$\begin{aligned} -e_n'' &= \lambda_n \rho e_n, \quad x \in (0,1) \\ \alpha_0 e_n(0) + \beta_0 e_n'(0) &= 0, \\ \alpha_1 e_n(1) + \beta_1 e_n'(1) &= 0, \end{aligned}$$

- For $t \in (\tau, T)$, $\theta(x, t) = \sum_{i \geq 0} y^{(i)}(t) g_i(x)$, where the **generating function** g_i is defined inductively as follows:

$$\begin{aligned} g_0'' &= 0, \quad x \in (0,1) \\ \alpha_0 g_0(0) + \beta_0 g_0'(0) &= 0, \\ \beta_0 g_0(0) - \alpha_0 g_0'(0) &= 1 \end{aligned}$$

  for $i = 0$, and g_i, for $i \geq 1$, is the solution to the **Cauchy problem**

$$\begin{aligned} g_i'' &= \rho g_{i-1}, \quad x \in (0,1) \\ g_i(0) &= 0, \\ g_i'(0) &= 0 \end{aligned}$$

- We can prove $||g_i||_{W^{2,p}(0,1)} \leq C / \left(R^i (i!)^{2-1/p}\right)$

- To ensure that the two expressions of θ agree at $t = \tau$, we have to relate the eigenfunctions e_n to the generating functions g_i.
- We have
$$e_n(x) = \zeta_n \sum_{i \geq 0} (-\lambda_n)^i g_i(x) \qquad (*)$$
  with $\zeta_n \in \mathbb{R}$.
- For $\rho \equiv 1$ and $(\alpha_0, \beta_0, \alpha_1, \beta_1) = (0, 1, 0, 1)$ [Neumann control at $x = 1$], $(*)$ is nothing but the classical Taylor expansion of $\cos(n\pi x)$ around $x = 0$:
$$\cos(n\pi x) = \sum_{i \geq 0} (-1)^i \frac{(n\pi x)^{2i}}{(2i)!}$$

Consider the control system

$$(S) \qquad \begin{cases} \theta_t - \Delta\theta = 0, & (t,x) \in (0,T)\times\Omega \\ \frac{\partial\theta}{\partial\nu}(t,x',1) = u(t,x'), & (t,x') \in (0,T)\times\omega \\ \frac{\partial\theta}{\partial\nu}(t,x) = 0 & (t,x) \in (0,T)\times\partial\Omega\setminus\omega\times\{0\} \\ \theta(0,x) = \theta(x), & x\in\Omega := \omega\times(0,1) \end{cases}$$

Model for the temperature along a rod which is controlled by taking as input the heat flux at one extremity.

Theorem

Let $\Omega = \omega\times(0,1) \subset \mathbb{R}^{N-1}\times\mathbb{R}$, $\theta_0 \in L^2(\Omega)$ and $T>0$ be given. Pick any $\tau\in(0,T)$ and any $s\in(1,2)$. Then there exists a sequence $(y_j)_{j\geq 0}$ of functions in $C^\infty([\tau,T])$ which are Gevrey of order s on $[\tau,T]$ and such that, setting

$$u(t,x') = \begin{cases} 0 & \text{if } 0\leq t\leq\tau, \\ \sum_{i,j\geq 0} e^{-\lambda_j t}\frac{y_j^{(i)}(t)}{(2i-1)!}e_j(x') & \text{if } \tau\leq t\leq T, \end{cases}$$

we have

$$\theta(T,.) = 0.$$

Here, (e_j,λ_j) denotes the j^{th} pair of eigenfunction/eigenvalue for the Neumann Laplacian on $\omega\subset\mathbb{R}^{N-1}$.

Assume given $T > 0$, $\tau \in (0, T)$, $s \in (1, 2)$, and $\theta_0 \in L^2(\Omega)$ decomposed as

$$\theta_0(x', x_N) = \sum_{j,n\geq 0} c_{j,n} e_j(x') \sqrt{2} \cos(n\pi x_N).$$

The exact solution θ of the previous control problem such that $\theta(T, .) = 0$ reads

$$\begin{aligned} \theta(t, x', x_N) &= \sum_{j,n\geq 0} c_{j,n} e^{-(\lambda_j + n^2\pi^2)t} e_j(x') \sqrt{2} \cos(n\pi x_N), \qquad 0 \leq t \leq \tau, \\ \theta(t, x', x_N) &= \sum_{j\geq 0} e^{-\lambda_j t} e_j(x') \sum_{i\geq 0} y_j^{(i)}(t) \frac{x_N^{2i}}{(2i)!}, \qquad \tau \leq t \leq T, \end{aligned}$$

where

$$\begin{aligned} y_j(t) &= \phi(t) \sum_{n\geq 0} c_{j,n} e^{-n^2\pi^2 t}, \qquad \tau \leq t \leq T, \\ \phi(t) &= \phi_s\left(\frac{t-\tau}{T-\tau}\right), \qquad \tau \leq t \leq T. \end{aligned}$$

In practice, only **partial sums** can be computed. They prove to give **accurate** approximations of both the trajectory and the control. **Exponentially small** errors!

Initial state: $\theta_0 := 1_{(1/2,1)}(x) - 1_{(0,1/2)}(x)$

Parameters: $\tau = 0.3$, $R = 0.2$, $T = \tau + R = 0.5$, $s = 1.6$

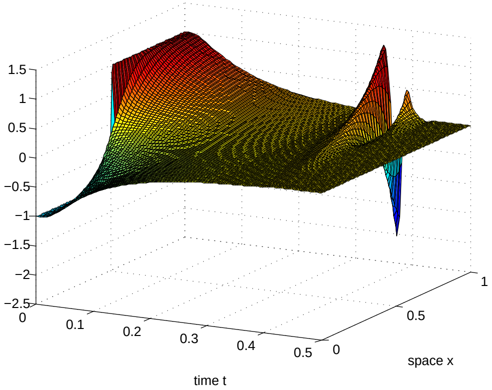


Fig.1. $\bar{\theta}(t, x)$

Computations by Philippe Martin

Initial state: $\theta_0 := 1_{(1/2,1)}(x) - 1_{(0,1/2)}(x)$

Parameters: $\tau = 0.3$, $R = 0.2$, $T = \tau + R = 0.5$, $s = 1.6$

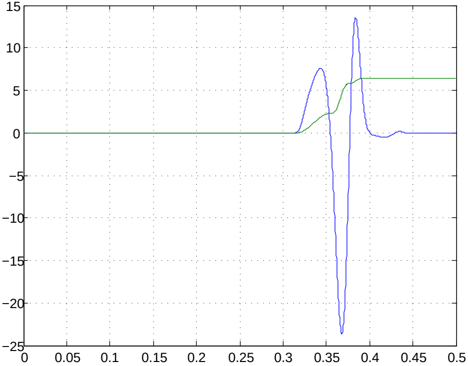
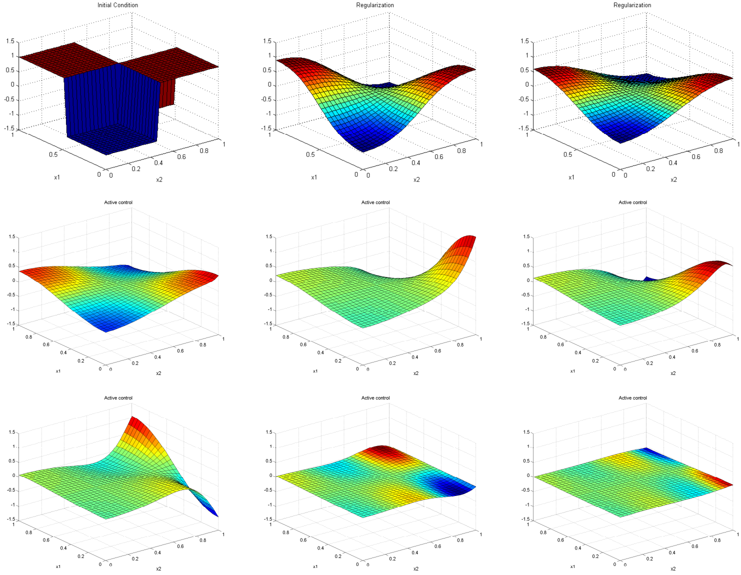


Fig. 2. $\bar{u}(t)$ (blue) and $||\bar{u}(t)||_{L^2(0,t)}$ (green)

Computations by Philippe Martin

Initial state: $\theta_0 := \left(1_{(1/2,1)}(x_1) - 1_{(0,1/2)}(x_1)\right)\left(1_{(0,1/2)}(x_2) - 1_{(1/2,1)}(x_2)\right)$

Parameters: $\tau = 0.05$, $R = 0.25$, $T = \tau + R = 0.3$, $s = 1.65$



Computations by Philippe Martin

Initial state: $\theta_0 := \left(1_{(1/2,1)}(x_1) - 1_{(0,1/2)}(x_1)\right)\left(1_{(0,1/2)}(x_2) - 1_{(1/2,1)}(x_2)\right)$
Parameters: $\tau = 0.05$, $R = 0.25$, $T = 0.35$, $s = 1.65$

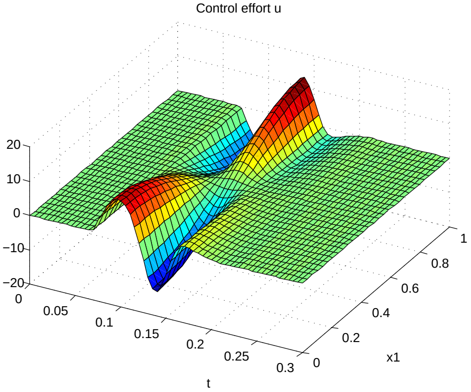


Fig. 4. $\bar{u}(t, x_1)$

Computations by Philippe Martin

Transmission pb. for the heat eq. (piecewise constant coef.)

$$\begin{aligned}
\rho_0\theta_t &= a_0\theta_{xx}, \quad 0 < x < X \\
\rho_1\theta_t &= a_1\theta_{xx}, \quad X < x < 1 \\
\theta(t, X^-) &= \theta(t, X^+) \\
a_0\theta_x(t, X^-) &= a_1\theta_x(t, X^+)
\end{aligned}$$

Parameters: $X = 1/2$, $(a_0, \rho_0, a_1, \rho_1) = (10/19, 15/8, 10, 1/8)$

Initial state: $\theta_0 := \frac{1}{2}1_{(1/2,1)}(x) - \frac{1}{2}1_{(0,1/2)}(x)$

Parameters: $\tau = 0.3$, $T = 0.35$, $s = 1.6$,

$(a_0, \rho_0, a_1, \rho_1) = (10/19, 15/8, 10, 1/8)$

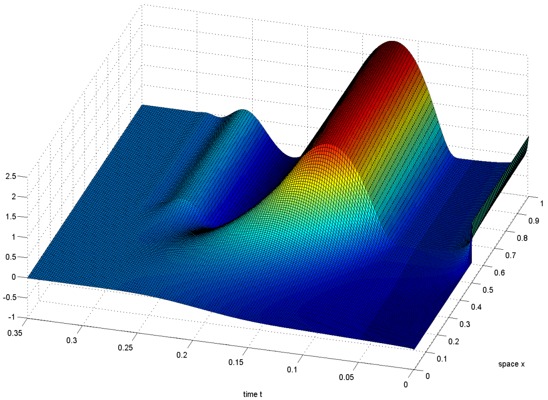


Fig.1. $\bar{\theta}(t, x)$

Computations by Philippe Martin

- The flatness approach allows to **prove in a very simple way** the null controllability of the heat equation (in cylinders), with explicit controls and trajectories easy to approximate.
- Extension to **general parabolic equations with discontinuous coefficients** done.
- Future direction of research
  - Extension to any pair (Ω, Γ_0) in 2D
  - Exact controllability results for linear/nonlinear equations
  - Numerical investigation of the cost of the control in terms of: T (time control), $\tau = T - R$ (free evolution), s (Gevrey regularity)

Thank you!