Exponential stability of a linearised pde implies local exponential stability?

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Introduction

There are two important theorems for checking stability of a non-linear ode or pde. The first one is known as La Salle's theorem.

La Salle's Theorem

Theorem

Consider the abstract differential equation on the Banach space X

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0.$$

If the following hold:

- *there exists a Lyapunov function,* $V : X \mapsto [0, \infty)$*, such that along all classical solutions,*

$$\dot{V}(x(t)) \leq 0;$$

- *For very initial condition* $x_0 \in X$*, the solution lies in a compact subset of* X *(solutions are pre-compact),*

then the solutions converge to the largest invariant subset contained in $\{x \in X \mid \dot{V}(x) = 0\}$

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We show by means of simple examples that both theorems cannot be extended.

La Salle's theorem

We show by a simple linear example that without the assumption of pre-compactness of the trajectories, La Salle's theorem does no longer hold.

Example

Let X be the Hilbert space $L^2(0,\infty)$ equipped with the inner product

$$\langle f, g\rangle := \int_0^\infty f(\zeta)\overline{g(\zeta)}(e^{-\zeta}+1)d\zeta,$$

and let the operators $T(t) : X \to X$, $t \geq 0$, be defined by

$$(T(t)f)(\zeta) := f(\zeta - t) \text{ for } \zeta > t \text{ and zero otherwise }.$$

Hence $T(t)$ is shifting the function f to the right. It is not hard to show that $(T(t))_{t\geq 0}$ is a C_0-semigroup on X.

Example, lack of stability

For

$$(T(t)f)(\zeta) := f(\zeta - t) \text{ for } \zeta > t \text{ and zero otherwise .}$$

there holds

$$\begin{aligned}\|T(t)f\|^2 &= \int_t^\infty |f(\zeta - t)|^2 (e^{-\zeta} + 1) d\zeta \\ &= \int_0^\infty |f(\zeta)|^2 (e^{-\zeta + t} + 1) d\zeta \\ &\geq \int_0^\infty |f(\zeta)|^2 (e^{-\zeta} + 1) d\zeta = \|f\|^2\end{aligned}$$

we see that $T(t)$ is not asymptotically stable. In fact, for every non-zero f, $T(t)f$ does not converge to zero. So this semigroup is not strongly/asymptotically stable.

Example, Lyapunov function

The generator A and its domain $D(A)$ of the semigroup are

$$Af = -\frac{df}{d\zeta}$$

$$D(A) = \{f \in X \mid f \text{ is abs. continuous, } \frac{df}{d\zeta} \in X, \text{ and } f(0) = 0\}.$$

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For the standard Lyapunov function $V(x) = \|x\|^2$ and $x \in D(A)$,

$$\begin{aligned}
\dot{V}(x) &= \int_0^\infty \left[-x(\zeta)'\overline{x(\zeta)} - x(\zeta)\overline{x(\zeta)'}\right](e^{-\zeta}+1)d\zeta \\
&= \left[-|x(\zeta)|^2(e^{-\zeta}+1)\right]_0^\infty - \int_0^\infty |x(\zeta)|^2 e^{-\zeta} d\zeta \\
&= -\int_0^\infty |x(\zeta)|^2 e^{-\zeta} d\zeta,
\end{aligned}$$

where we have used the boundary condition.

Example, conclusion

So we have that

$$\dot{V}(x) \leq \int_0^\infty |x(\zeta)|^2 e^{-\zeta} d\zeta.$$

Since this expression is non-zero for every $x \neq 0$, we have that

$$\dot{V}(x) < 0, \qquad x \neq 0.$$

Hence we have a strict Lyapunov function, whereas the system is not (asymptotically) stable.

Linearisation

Next we construct an example showing that when linearising, the Fréchet derivative is needed and not the (weaker) Gateaux derivative.

Consider first the following scalar differential equations

$$\dot{x}_n(t) = (-1 + 3\sqrt[n]{|x_n(t)|})x_n(t), \quad n \in \mathbb{N}. \tag{1}$$

We summarise some results for these scalar ode's in a lemma.

Lemma

The differential equation (1) has the following properties.

- *The equilibrium's are $\pm 3^{-n}$ and zero.*
- *The right hand-side of (3) is locally Lipschitz continuous, and for $|x_n| \leq r$ the Lipschitz constant can be majorized by $3(1 + \frac{1}{n})\sqrt[n]{r}$.*
- *For $x_n(0) \in (-3^{-n}, 3^{-n})$ the state converges to zero, and for $|x_n(0)| > 3^{-n}$ the state diverges.*
- *For $|x_n(0)| > 3^{-n}$ there is a finite escape time.*
- *The linearization of (1) around zero is $\dot{x}_n(t) = -x_n(t)$ and thus exponentially stable.*

Example

On the state space $X = \ell^2(\mathbb{N})$ we consider the differential equation

$$\dot{x}(t) = -x(t) + f(x(t)), \qquad x(0) = x_0 \tag{2}$$

with f given by

$$(f(x))_n = 3\sqrt[n]{|x_n|} \cdot x_n. \tag{3}$$

Hence our system is a diagonal (non-linear) system with on the diagonal the equation (1), i.e.,

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The results for these scalar equations are used to characterise the behaviour of the non-linear system (2).

Main properties

Theorem

The non-linear system on $X = \ell^2(\mathbb{N})$

$$\dot{x}(t) = -x(t) + f(x(t)), \qquad x(0) = x_0$$

with f *given by*

$$(f(x))_n = 3\sqrt[n]{|x_n|} \cdot x_n.$$

has the following properties.

1. f *is (locally) Lipschitz continuous from* X *to* X.
2. f *is Gateaux differentiable, but not Fréchet at the origin. The Gateaux derivative at the origin is zero.*
3. *The origin is an unstable equilibrium point.*

Proof of instability

We choose $N \in \mathbb{N}$ and $x(0) = (x_{0n})_{n\in\mathbb{N}}$ with $x_{0n} = 0$ for $n \neq N$ and $x_{0N} = 2^{-N}$.
By our lemma we have that for $|x_{0N}| > 3^{-N}$ the N-th equation blows up. So, in particular, we see that for this initial condition the N-th equation of (2) is unstable. Thus the state $x(t)$ diverges.

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Since for $N \to \infty$, there holds $\|x(0)\| \to 0$, we see that there exists an initial state arbitrarily close to zero which is unstable.
Thus the non-linear system is not stable in the origin.

Proof of Gateaux derivative being zero

For $x \in X$ and $\varepsilon \in \mathbb{R} \setminus \{0\}$ we have

$$\begin{aligned} \left\| \frac{f(0+\varepsilon x) - f(0)}{\varepsilon} - 0 \right\|^2 &= \sum_{n=1}^{\infty} 9 \sqrt[n]{\varepsilon^2 x_n^2} x_n^2 \qquad (4) \\ &= 9 \sum_{n=1}^{\infty} \sqrt[n]{\varepsilon^2} \sqrt[n]{x_n^2} x_n^2 \end{aligned}$$

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Next take a $\delta \in (0,1)$ and choose N such that $\sum_{n=N}^{\infty} x_n^2 \leq \delta$. In particular, this implies that $\sqrt[n]{x_n^2} \leq 1$ for $n \geq N$. Now choose ε such that $|\varepsilon| < 1$ and $\sum_{n=1}^{N-1} \sqrt[n]{\varepsilon^2}\sqrt[n]{x_n^2}\, x_n^2 \leq \delta$. Combining these two gives that for this ε there holds that

$$\|\frac{f(0+\varepsilon x)-f(0)}{\varepsilon}-0\|^2 \leq 9(\delta+\delta).$$

Proof of Gateaux derivative being zero

So given x and given δ there exists an $\varepsilon > 0$ such that

$$\|\frac{f(0+\varepsilon x) - f(0)}{\varepsilon} - 0\|^2 \leq 18\delta.$$

Since δ is arbitrarily, this show that

$$\lim_{\varepsilon \to 0} \|\frac{f(0+\varepsilon x) - f(0)}{\varepsilon} - 0\|^2 = 0$$

and so 0 is the Gateaux derivative of f.

f is not Fréchet differentiable

If f would be Fréchet differentiable, then its derivative would equal the Gateaux derivative, and thus zero.
However, we have

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Since $\|x\| \to 0$ for $N \to \infty$, we see that

$$\limsup_{\|x\|\to 0} \frac{\|f(x)\|}{\|x\|} > \frac{3}{2} > 0.$$

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- We show that the Gateaux derivative of f is zero. This implies that the (Gateaux) linearization of (2) is $\dot{x}(t) = -x(t)$.
- The example is not uniformly Lipschitz continuous, and almost every solution will have finite escape time. The following simple adaptation gives a uniformly Lipschitz continuous differential equation on X,

$$\dot{x}_n(t) = \frac{(-1 + 3\sqrt[n]{|x_n(t)|})x_n(t)}{1 + x_n(t)^2},$$

whose (Gateaux) linearization around zero is again $\dot{x}(t) = -x(t)$ but which is not (locally) stable.