Feedback Control of Navier-Stokes Equations over Subdomains

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Introduction

$$\boldsymbol{w}_t - \nu_0 \Delta \boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\, \boldsymbol{w} = -\nabla p + \mathbf{f}\,(\boldsymbol{x}, t) + \mathbf{U}$$

$$\mathrm{div}\boldsymbol{w} = 0, \quad \boldsymbol{w}|_{\partial\Omega} = 0, \quad \boldsymbol{w}|_{t=0} = \boldsymbol{w}_0\,(\boldsymbol{x})$$

- $\mathbf{x} = (x_1, x_2, x_3) \in \Omega \subset I\!R^3, \quad t \in (0, \infty)$,
- Ω – bounded domain
- $\boldsymbol{w} = \boldsymbol{w}\,(\mathbf{x}, t) = (w_1, w_2, w_3)$ – velocity field
- p – pressure
- $\boldsymbol{w}_t + (\boldsymbol{w} \cdot \nabla)\, \boldsymbol{w}$ – convective derivative,
- $\nu_0 \sim \frac{1}{\mathrm{Re}}$ – kinetic viscosity, Re – Reynolds #,
- $\nu_0 \Delta \boldsymbol{w}$ – diffusion term, “friction,”
- $\boldsymbol{w}|_{t=0} = \boldsymbol{w}_0\,(\mathbf{x})$ – initial data
- $\mathrm{div}\, \boldsymbol{w} = 0$ – incompressibility,
- $\mathbf{f}$ – external body force
- $\boldsymbol{w}|_{\partial\Omega} = 0$ – “no-slip” boundary condition,
- $\mathbf{U}$ – control

unknowns (velocity field, pressure)

$$\boldsymbol{w}_t - \nu_0 \Delta \boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\, \boldsymbol{w} = -\nabla p + \mathbf{f}(\boldsymbol{x}, t) + \mathbf{U}$$

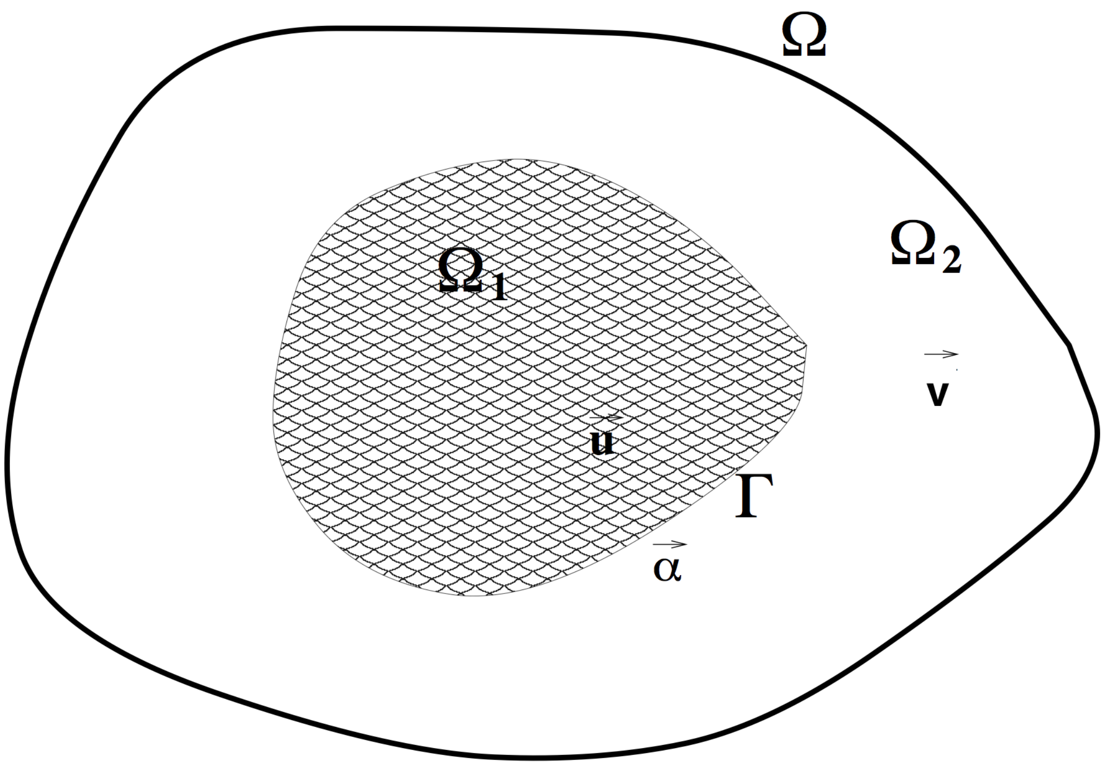
$$\operatorname{div} \boldsymbol{w} = 0, \quad \boldsymbol{w}|_{\partial\Omega} = 0, \quad \boldsymbol{w}|_{t=0} = \boldsymbol{w}_0(\boldsymbol{x})$$

Feedback control over $\Omega_1 \subsetneq \Omega$

$$\begin{aligned} \mathbf{U} = \mathbf{U}(\boldsymbol{x}, t) &= K\boldsymbol{w} \\ &= \nu_1 \left(\int_{\Omega_1} |\boldsymbol{w}_{\boldsymbol{x}}(\boldsymbol{x}, t)|^2 \, d\boldsymbol{x} \right) \Delta \boldsymbol{w}(\boldsymbol{x}, t)\, \chi_{\Omega_1}(\boldsymbol{x}) \end{aligned}$$

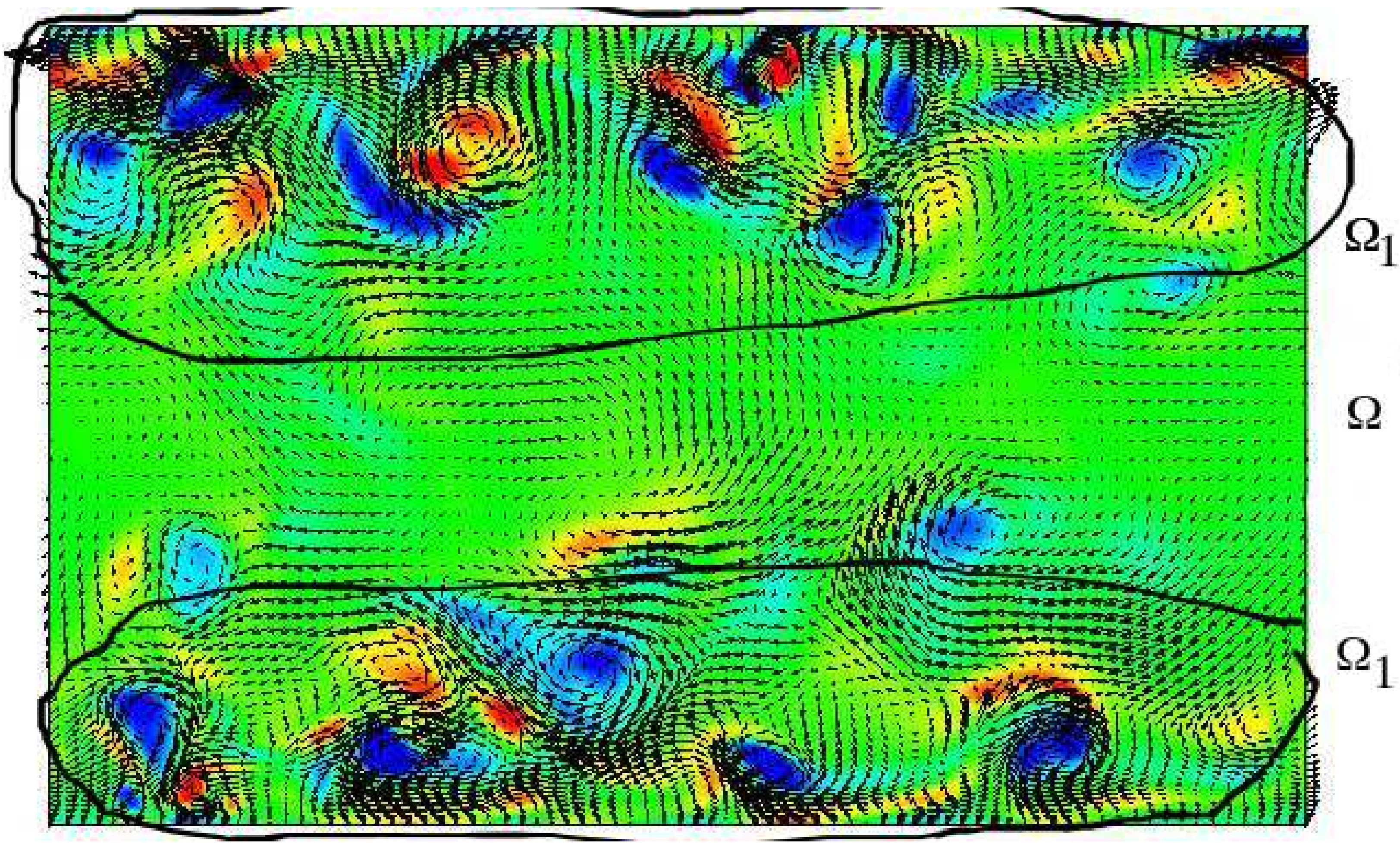
- $\mathcal{E}_{\Omega_1}(t) = \int_{\Omega_1} |\boldsymbol{w}_{\boldsymbol{x}}(\boldsymbol{x}, t)|^2 \, d\boldsymbol{x}$ – energy dissipation rate
- $\boldsymbol{\mathcal{D}}(\boldsymbol{x}, t) = \chi_{\Omega_1}(\boldsymbol{x})\, \Delta \mathbf{w}(\boldsymbol{x}, t)$ – friction
- $\chi_{\Omega_1}(\boldsymbol{x})$ – characteristic function of the set Ω_1
- Output (measurement) $= \mathbf{Y} = \left(\mathcal{E}_{\Omega_1}(t), \boldsymbol{\mathcal{D}}(\boldsymbol{x}, t)\right)$
- Input (control) $= \mathbf{U} = K\mathbf{Y} = \nu_1 \mathcal{E}_{\Omega_1}(t)\, \boldsymbol{\mathcal{D}}(\boldsymbol{x}, t)$
- $\nu_1 > 0$ – control gain

Control and Uncontrolled Domains

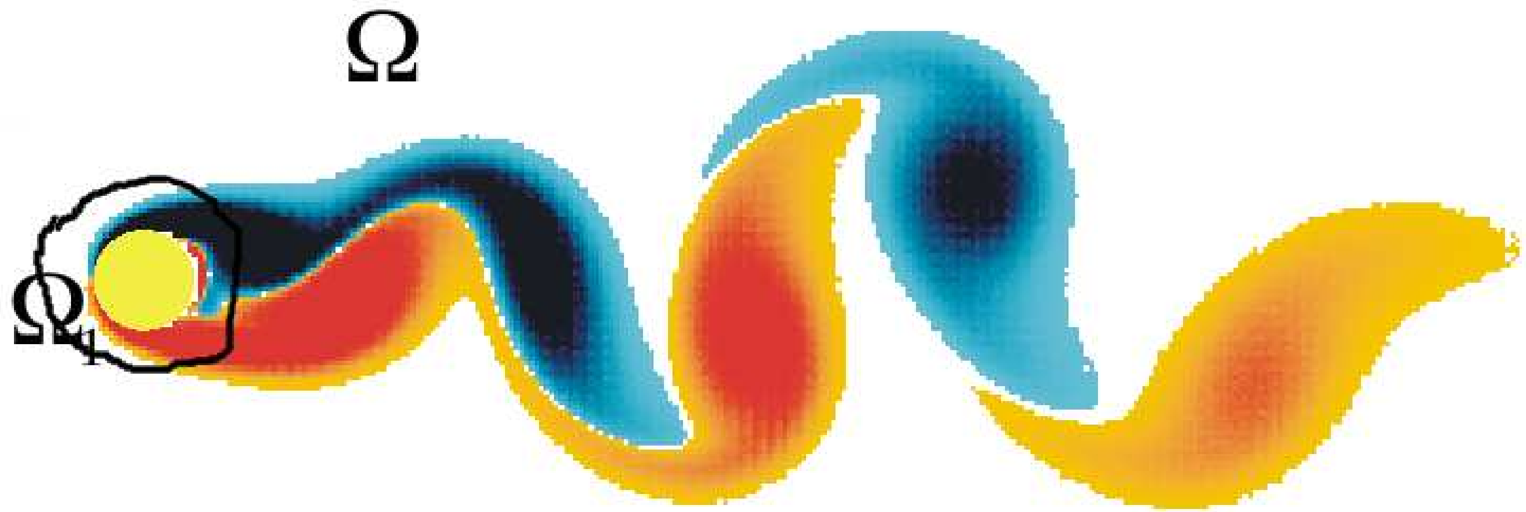


Possible Application I

Stabilizing Channel Flow (boundary layer)



Stabilizing Flow Around Obstacle



Problem Statement

- In $\Omega_1 \times [0,T]$ (controlled domain)

$$\mathbf{v}_t - \left(\nu_0 + \nu_1 \left\|\mathbf{v}_{\boldsymbol{x}}\right\|^2\right) \Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla p_1 + \mathbf{f}\,(\boldsymbol{x},t)$$

$$\mathrm{div}\mathbf{v} = 0, \text{ in } \Omega_1 \times [0,T]\,, \quad \mathbf{v}|_{\partial\Omega} = 0, \quad \mathbf{v}|_{t=0} = \mathbf{v}_0 \text{ in } \Omega_1$$

- In $\Omega_2 \times [0,T]$, (uncontrolled domain) $\Omega_2 = \Omega \setminus \Omega_1$

$$\mathbf{u}_t - \nu_0 \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\nabla p_2 + \mathbf{f}\,(\mathbf{x},t)$$

$$\mathrm{div}\mathbf{u} = 0, \text{ in } \Omega_2 \times [0,T]\,, \quad \mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \text{ in } \Omega_2$$

- Interface Conditions on $\Gamma_T = \Gamma \times [0,T]$, $(\Gamma = \partial\Omega_1 \setminus \partial\Omega)$

$$\left(1 + \frac{\nu_1}{\nu_0}\int\limits_{\Omega_1} |\mathbf{v}_{\mathbf{x}}|^2 dx\right) \mathbf{v}_{x_k}\Big|_{\Gamma_T} = \mathbf{u}_{x_k}\Big|_{\Gamma_T}, \quad k = 1,2,3$$

$$\mathbf{v}\Big|_{\Gamma_T} = \mathbf{u}\Big|_{\Gamma_T}, \qquad p_1\Big|_{\Gamma_T} = p_2\Big|_{\Gamma_T}$$

- spatially nonhomogeneous viscosity coefficient

$$\nu(\boldsymbol{x}) = \nu_0 + \nu_1 \chi_{\Omega_1}(\boldsymbol{x}) \left(\int_{\Omega_1} |\boldsymbol{w}_{\boldsymbol{x}}|^2 \, d\boldsymbol{x} \right)$$

- kinetic energy of the fluid ($\frac{1}{2} \|\boldsymbol{w}\|_{\Omega}^2$) is a decreasing function of time

$$\frac{1}{2} \frac{d}{dt} \|\boldsymbol{w}\|_{\Omega}^2 + \nu_0 \|\boldsymbol{w}_{\boldsymbol{x}}\|_{\Omega}^2 + \nu_1 \|\boldsymbol{w}_{\boldsymbol{x}}\|_{\Omega_1}^4 = 0$$

O.A. Ladyzhenskaya, “New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them,” *Proc. Steklov Inst. Math.*, 102, (1967), 95–118.

$$\Omega_1 = \Omega$$

- Based on statistical fluid mechanics
- Corresponds to the case of control over the whole domain
-
- Interface condition makes it more complicated
- Domains and the solutions are “glued together”.

Main Results

Let $\mathbf{f}$, $\boldsymbol{w}_0$, and ν_0 satisfy the assumptions

$$\mathbf{f}\Big|_{\Omega_1} \in H^1\left([0,T], L^2(\Omega_1)\right) \text{ is arbitrary, and} \tag{1}$$

$$\boldsymbol{w}_0\Big|_{\Omega_1} \in H^2(\Omega_1) \text{ is arbitrary,} \tag{2}$$

$$\mathbf{f}\Big|_{\Omega\setminus\Omega_1} \in L^2\left(Q_T''\right) \quad \text{for } Q_T'' = \Omega\setminus\Omega_1 \times [0,T], \tag{3}$$

$$\boldsymbol{w}_0\Big|_{\Omega\setminus\Omega_1} \in H^1(\Omega\setminus\Omega_1), \tag{4}$$

$$\frac{1}{\nu_0}\left(\|\boldsymbol{w}_0\|_{\Omega\setminus\Omega_1} + \|(\boldsymbol{w}_0)_{\boldsymbol{x}}\|_{\Omega\setminus\Omega_1} + \|\mathbf{f}\|_{Q_T''}\right) \le \rho, \tag{5}$$

where ρ is small enough.

For any $T > 0$, under the hypotheses (1)–(5) there exists a unique strong solution $\boldsymbol{w}\,(\boldsymbol{x}, t)$ to the controlled Navier–Stokes equations, i.e., there exists a unique vector field $\boldsymbol{w} \in C\left([0,T], H^1(\Omega)\right)$, such that
$\mathbf{v} = \boldsymbol{w}|_{\Omega_1} \in C\left([0,T]\,, H^2\,(\Omega_1)\right)$,
$\mathbf{u} = \boldsymbol{w}|_{\Omega\setminus\Omega_1} \in C\left([0,T]\,, H^2\,(\Omega \setminus \Omega_1)\right)$. More precisely

$$\underset{0\le t\le T}{ess\ sup}\{\|\boldsymbol{w}\|_{\Omega}^2 + \|\boldsymbol{w}_{\boldsymbol{x}}\|_{\Omega}^2\} + \int_0^T \Big(\|\Delta\boldsymbol{w}\|_{\Omega}^2 + \|\Delta\boldsymbol{w}\|_{\Omega\setminus\Omega_1}^2\Big)\, dt < \infty.$$

The trivial solution of this system is asymptotically stable in the sense that

$$\|\boldsymbol{w}(\cdot,t)\|_{\Omega} \xrightarrow{t\to\infty} 0, \text{ when } \|\mathbf{f}(\cdot,t)\|_{\Omega} \xrightarrow{t\to\infty} 0.$$

Outline of the Proof

- Consider the equations on the subdomains Ω_1 and $\Omega \setminus \Omega_1$ separately with general boundary conditions
- Solve the Navier–Stokes equations on these subdomains
- Show that there exists boundary condition that satisfies the interface conditions
- Show uniqueness

⋮

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- Show uniqueness

$\vdots$

- Classical energy estimates

Main Steps

- Consider the equations on the subdomains Ω_1 and $\Omega \setminus \Omega_1$ separately with general boundary conditions
- Solve the Navier–Stokes equations on these subdomains
- Show that there exists boundary condition that satisfies the interface conditions
- Show uniqueness

$\vdots$

- Classical energy estimates
- Contraction mapping with bootstrapping

More Details

- $\mathcal{N}(\Gamma) = \left\{ \boldsymbol{\alpha} \,|\, \boldsymbol{\alpha},\, \boldsymbol{\alpha_t} \in C\left([0,T], H^{3/2}(\Gamma) \right) \right\}$ (functions space of interface boundary values)
- Subproblem 1. in $Q'_T = \Omega_1 \times [0,T]$ (controlled)

$$\mathbf{v}_t - \nu_0 \Delta \mathbf{v} - \nu_1 \|\mathbf{v}_{\boldsymbol{x}}\|^2_{\Omega_1} \Delta \mathbf{v} + \mathrm{v}_k \mathbf{v}_{x_k} = -\nabla p_1 + \mathbf{f} \quad \text{in } Q'_T$$
$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q'_T$$
$$\mathbf{v} = \boldsymbol{\alpha} \quad \text{on } \Gamma_T, \quad \mathbf{v}(\cdot, 0) = \mathbf{v}_0 \quad \text{in } \Omega_1$$

- Subproblem 2. in $Q''_T = (\Omega \setminus \Omega_1) \times [0,T]$ (uncontrolled)

$$\mathbf{u}_t - \nu_0 \Delta \mathbf{u} + u_k \mathbf{u}_{x_k} = -\nabla p_2 + \mathbf{f} \text{ in } Q''_T$$
$$\operatorname{div} \mathbf{u} = 0 \text{ in } Q''_T$$
$$\mathbf{u} = \boldsymbol{\alpha} \text{ on } \Gamma_T, \quad \mathbf{u} = 0 \text{ on } \partial\Omega \times [0,T]$$
$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \text{ in } \Omega \setminus \Omega_1$$

Subproblem on the Controlled Domain

$$\mathbf{v}_t - \nu_0 \Delta \mathbf{v} - \nu_1 \|\mathbf{v}_{\boldsymbol{x}}\|^2_{\Omega_1} \Delta \mathbf{v} + \mathrm{v}_k \mathbf{v}_{x_k} = -\nabla p_1 + \mathbf{f} \quad \text{in } Q'_T$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q'_T$$

$$\mathbf{v} = \boldsymbol{\alpha} \quad \text{on } \Gamma_T, \quad \mathbf{v}(\cdot, 0) = \mathbf{v}_0 \quad \text{in } \Omega_1$$

$$\mathcal{E}_0 : \quad H^{k-\frac{1}{2}}(\Gamma) \longrightarrow H^k(\Omega), \qquad \alpha \longrightarrow \widetilde{\boldsymbol{\alpha}} \qquad \text{lifting operator}$$

$$\mathbf{y} = \mathbf{v} - \widetilde{\boldsymbol{\alpha}}$$

$$\mathbf{y}_t - \nu_0 \Delta \mathbf{y} - \nu_1 \left\| (\mathbf{y} + \widetilde{\boldsymbol{\alpha}})_{\mathbf{x}} \right\|^2_{\Omega_1} \Delta (\mathbf{y} + \widetilde{\boldsymbol{\alpha}}) + y_k \mathbf{y}_{x_k}$$
$$+ \widetilde{\alpha}_k \mathbf{y}_{x_k} + \widetilde{\boldsymbol{\alpha}}_{x_k} y_k = -\nabla p_1 + \mathbf{F}_1 \quad \text{in } Q'_T, \qquad (Ctrld)$$
$$\operatorname{div} \mathbf{y} = 0 \text{ in } Q'_T,$$
$$\mathbf{y} = 0 \text{ on } \Gamma_T, \qquad \mathbf{y}(\cdot, 0) = \mathbf{v}_0 - \widetilde{\boldsymbol{\alpha}}(\cdot, 0) \text{ in } \Omega_1,$$

$$\mathbf{F}_1 \equiv \mathbf{f} - \widetilde{\alpha}_k \widetilde{\boldsymbol{\alpha}}_{x_k} + \nu_0 \Delta \widetilde{\boldsymbol{\alpha}} - \widetilde{\boldsymbol{\alpha}}_t$$

Controlled A Priori Estimate I

$$\mathbf{y}_t - \nu_0 \Delta \mathbf{y} - \nu_1 \left\| (\mathbf{y} + \widetilde{\boldsymbol{\alpha}})_{\mathbf{x}} \right\|_{\Omega_1}^2 \Delta \left(\mathbf{y} + \widetilde{\boldsymbol{\alpha}} \right) + y_k \mathbf{y}_{x_k}$$
$$+ \widetilde{\alpha}_k \mathbf{y}_{x_k} + \widetilde{\boldsymbol{\alpha}}_{x_k} y_k = -\nabla p_1 + \mathbf{F}_1 \text{ in } Q_T', \qquad (Ctrld)$$
$$\operatorname{div} \mathbf{y} = 0 \text{ in } Q_T',$$
$$\mathbf{y} = 0 \text{ on } \Gamma_T, \qquad \mathbf{y}(\cdot, 0) = \mathbf{v}_0 - \widetilde{\boldsymbol{\alpha}}(\cdot, 0) \text{ in } \Omega_1,$$

$$\int_{\Omega_1} (Ctrld) \cdot \mathbf{y} d\boldsymbol{x}$$

$$\|\mathbf{y}\|_{\Omega_1}^2 \le \int_0^t \left(\frac{2K_{\widetilde{\alpha}}^2 K_1}{\nu_1} + 202\nu_1 \left\| \widetilde{\boldsymbol{\alpha}}_x \right\|_{\Omega_1}^4 + \frac{\|\mathbf{F}_1\|_{\Omega_1}^2 K_1}{\nu_0} \right) ds$$
$$+ \|\mathbf{y}_0\|_{\Omega_1}^2 \le C(T, \mathbf{y}_0, \mathbf{f}, \Omega_1, \widetilde{\boldsymbol{\alpha}}, \nu_0, \nu_1). \qquad (C - E1)$$

Controlled A Priori Estimate II

$$\begin{aligned}\|\mathbf{y}\|^2_{\Omega_1} &\leq \|\mathbf{y}_0\|^2_{\Omega_1} + \int_0^t \left(\frac{2K^2_{\widetilde{\alpha}} K_1}{\nu_1} + 202\nu_1 \|\widetilde{\boldsymbol{\alpha}}_x\|^4_{\Omega_1} + \frac{\|\mathbf{F}_1\|^2_{\Omega_1} K_1}{\nu_0} \right) ds \\ &\leq C(T, \mathbf{y}_0, \mathbf{f}, \Omega_1, \widetilde{\boldsymbol{\alpha}}, \nu_0, \nu_1). \qquad (C-E1) \Longleftarrow\end{aligned}$$

$$\frac{d}{dt}\|\mathbf{y}\|^2_{\Omega_1} + 2\nu_0 \|\mathbf{y}_\mathbf{x}\|^2_{\Omega_1} + \frac{\nu_1}{2}\|\mathbf{y}_\mathbf{x}\|^4_{\Omega_1} + 4\nu_1 \left(\int_{\Omega_1} \langle \mathbf{y}_{\boldsymbol{x}}, \widetilde{\boldsymbol{\alpha}}_{\boldsymbol{x}} \rangle \, d\boldsymbol{x} \right)^2$$

$$\leq \frac{2K^2_{\widetilde{\alpha}} K_1}{\nu_1} + 202\nu_1 \|\widetilde{\boldsymbol{\alpha}}_{\boldsymbol{x}}\|^4_{\Omega_1} + \frac{\|\mathbf{F}_1\|^2_{\Omega_1} K_1}{\nu_0} + \frac{\nu_0 \|\mathbf{y}\|^2_{\Omega_1}}{K_1} \Longrightarrow \int_0^T \cdot dt$$

$$\|\mathbf{y}\|^2_{\Omega_1} + 2\nu_0 \int_0^t \|\mathbf{y}_\mathbf{x}\|^2_{\Omega_1} \, ds + \frac{\nu_1}{2} \int_0^t \|\mathbf{y}_\mathbf{x}\|^4_{\Omega_1} \, ds + 4\nu_1 \int_0^t \left(\int_{\Omega_1} \langle \mathbf{y}_\mathbf{x}, \widetilde{\boldsymbol{\alpha}}_{\boldsymbol{x}} \rangle \, d\mathbf{x} \right)^2 ds$$

$$\leq C\left(T, \mathbf{y}_0, \mathbf{f}, \Omega_1, \widetilde{\boldsymbol{\alpha}}, \nu_0, \nu_1\right) \qquad (C-E2)$$

$$\mathbf{y}_t - \nu_0 \Delta \mathbf{y} - \nu_1 \left\| (\mathbf{y} + \widetilde{\boldsymbol{\alpha}})_{\mathbf{x}} \right\|^2_{\Omega_1} \Delta (\mathbf{y} + \widetilde{\boldsymbol{\alpha}}) + y_k \mathbf{y}_{x_k}$$

$$+\widetilde{\alpha}_k \mathbf{y}_{x_k} + \widetilde{\boldsymbol{\alpha}}_{x_k} y_k = -\nabla p_1 + \mathbf{F}_1 \quad \text{in } Q'_T, \qquad (Ctrld)$$

$$\operatorname{div} \mathbf{y} = 0 \text{ in } Q'_T,$$

$$\mathbf{y} = 0 \text{ on } \Gamma_T, \qquad \mathbf{y}(\cdot, 0) = \mathbf{v}_0 - \widetilde{\boldsymbol{\alpha}}(\cdot, 0) \text{ in } \Omega_1,$$

$$\int_{\Omega_1} (Ctrld)_t \cdot \mathbf{y}_t d\boldsymbol{x}$$

$$\|\mathbf{y}_t\|^2_{\Omega_1} + \nu_0 \int_0^t \|\mathbf{y}_{\mathbf{x}t}\|^2_{\Omega_1} dt \le C_4 (T, \mathbf{y}_0, \mathbf{f}, \Omega_1, \widetilde{\boldsymbol{\alpha}}, \nu_0, \nu_1). \quad (C - E3)$$

also

$$\frac{d}{dt} \|\mathbf{y}_{\mathbf{x}}\|^2_{\Omega_1} = 2 \int_{\Omega_1} \langle \mathbf{y}_{\mathbf{x}}, \mathbf{y}_{\mathbf{x}t} \rangle \, d\mathbf{x} \le 2 \|\mathbf{y}_{\mathbf{x}}\|_{\Omega_1} \|\mathbf{y}_{\mathbf{x}t}\|_{\Omega_1}$$

$$\frac{d}{dt} \|\mathbf{y}_{\boldsymbol{x}}\|_{\Omega_1} \le \|\mathbf{y}_{\boldsymbol{x}t}\|_{\Omega_1}$$

$$\|\mathbf{y}_{\boldsymbol{x}}\|_{\Omega_1} \le C_5 (T, \mathbf{y}_0, \mathbf{f}, \Omega_1, \widetilde{\boldsymbol{\alpha}}, \nu_0, \nu_1)$$

$$\mathbf{y}_t - \nu_0 \Delta \mathbf{y} - \nu_1 \left\| (\mathbf{y} + \widetilde{\boldsymbol{\alpha}})_{\mathbf{x}} \right\|^2_{\Omega_1} \Delta \left(\mathbf{y} + \widetilde{\boldsymbol{\alpha}} \right) + y_k \mathbf{y}_{x_k}$$
$$+ \widetilde{\alpha}_k \mathbf{y}_{x_k} + \widetilde{\boldsymbol{\alpha}}_{x_k} y_k = -\nabla p_1 + \mathbf{F}_1 \text{ in } Q'_T, \qquad (Ctrld)$$
$$\operatorname{div} \mathbf{y} = 0 \text{ in } Q'_T,$$
$$\mathbf{y} = 0 \text{ on } \Gamma_T, \qquad \mathbf{y}(\cdot, 0) = \mathbf{v}_0 - \widetilde{\boldsymbol{\alpha}}(\cdot, 0) \text{ in } \Omega_1,$$

$$\int_{\Omega_1} (Ctrld) \cdot \widetilde{\Delta} \mathbf{y} d\mathbf{x}$$

$$\widetilde{\Delta} = P\Delta,$$
$$P : L^2(\Omega_1) \longrightarrow \mathcal{H}(\Omega_1) = \left\{ \mathbf{y} \in L^2(\Omega_1), \operatorname{div} \mathbf{y} = 0, \mathbf{y} \cdot \boldsymbol{\eta}|_\Gamma = 0 \right\} \text{ (Stokes op)}$$

$$\|\mathbf{y}_{\boldsymbol{x}}\|^2_{\Omega_1} + \nu_0 \int_0^t \left\| \widetilde{\Delta} \mathbf{y} \right\|^2_{\Omega_1} ds \le \|\mathbf{y_0}_{\boldsymbol{x}}\|^2_{\Omega_1} + \int_0^t F_4(s)\, ds,$$

$$\nu_0 \int_0^t \left\| \widetilde{\Delta} \mathbf{y} \right\|^2_{\Omega_1} ds \quad \le \quad C\left(T, \mathbf{y_0}, \mathbf{f}, \Omega_1, \widetilde{\boldsymbol{\alpha}}, \nu_0, \nu_1\right)$$

Galerkin's Method

$$\mathbf{y}^m = \sum_{i=1}^{m} c_{im}(t)\,\mathbf{a}_i(\boldsymbol{x}), \qquad m = 1, 2, \ldots$$

$$\int_{Q'_T} \mathbf{y}_t^m \cdot \boldsymbol{\phi}\, d\boldsymbol{x}\, dt - \nu_0 \int_{Q'_T} \Delta \mathbf{y}^m \cdot \boldsymbol{\phi}\, d\boldsymbol{x}\, dt$$

$$- \nu_1 \int_{Q'_T} \left\|(\mathbf{y}^m + \widetilde{\boldsymbol{\alpha}})_{\boldsymbol{x}}\right\|^2_{\Omega_1} (\Delta \mathbf{y}^m + \Delta\widetilde{\boldsymbol{\alpha}}) \cdot \boldsymbol{\phi}\, d\boldsymbol{x}\, dt + \int_{Q'_T} y_k^m \mathbf{y}_{x_k}^m \cdot \boldsymbol{\phi}\, d\boldsymbol{x}\, dt$$

$$+ \int_{Q'_T} \widetilde{\alpha}_k \mathbf{y}_{x_k}^m \cdot \boldsymbol{\phi}\, d\boldsymbol{x}\, dt + \int_{Q'_T} \widetilde{\boldsymbol{\alpha}}_{x_k} \cdot y_k^m \boldsymbol{\phi}\, d\boldsymbol{x}\, dt = - \int_{Q'_T} \nabla p_1^m \cdot \boldsymbol{\phi}\, d\boldsymbol{x}\, dt + \int_{Q'_T} \mathbf{F}_1 \cdot \boldsymbol{\phi}\, d\boldsymbol{x}\, dt$$

$$\|\mathbf{y}^m\|^2_{\mathcal{M}_1} \equiv \underset{0 \le t \le T}{ess\ sup} \left\{ \|\mathbf{y}^m\|^2_{\Omega_1} + \|\mathbf{y}^m_{\boldsymbol{x}}\|^2_{\Omega_1} + \|\mathbf{y}^m_t\|^2_{\Omega_1} \right\}$$

$$+ \int_0^T \left\|\widetilde{\Delta}\mathbf{y}^m\right\|^2_{\Omega_1} dt + \int_0^T \|\mathbf{y}^m_{\boldsymbol{x}t}\|^2_{\Omega_1}\, dt \le C\left(T, \mathbf{y_0}, \mathbf{f}, \Omega_1, \widetilde{\boldsymbol{\alpha}}, \nu_0, \nu_1\right) < \infty$$

Subproblem on the Uncontrolled Domain

$$\mathbf{u}_t - \nu_0 \Delta \mathbf{u} + u_k \mathbf{u}_{x_k} = -\nabla p_2 + \mathbf{f} \text{ in } Q_T'' = (\Omega \setminus \Omega_1) \times [0,T],$$
$$\operatorname{div} \mathbf{u} = 0 \text{ in } Q_T'', \quad \mathbf{u} = \boldsymbol{\alpha} \text{ on } \Gamma_T,$$
$$\mathbf{u} = 0 \text{ on } \partial\Omega \times [0,T], \quad \mathbf{u}(\cdot,0) = \mathbf{u}_0 \text{ in } \Omega \setminus \Omega_1.$$

$$\mathbf{y} = \mathbf{v} - \widetilde{\boldsymbol{\alpha}}$$

$$\mathbf{y}_t - \nu_0 \Delta \mathbf{y} + y_k \mathbf{y}_{x_k} + \widetilde{\alpha}_k \mathbf{y}_{x_k} + \widetilde{\boldsymbol{\alpha}}_{x_k} y_k = -\nabla p_2 + \mathbf{F}_1 \text{ in } Q_T''$$

$$\operatorname{div} \mathbf{y} = 0 \text{ in } Q_T'', \quad \mathbf{y} = 0 \text{ on } \partial(\Omega \setminus \Omega_1) \times [0,T],$$
$$\mathbf{y}(\cdot,0) = \mathbf{u}_0 - \widetilde{\boldsymbol{\alpha}}(\cdot,0) \text{ in } \Omega \setminus \Omega_1.$$

We require $\boldsymbol{\alpha}$, $\mathbf{u}_0$, $\mathbf{f}$ to be small enough, and ν_0 to be large enough, so that

$$\nu_0 - 2K_{\widetilde{\alpha}}\sqrt{K_1} \geq 0$$

$$\nu_0^2 K_1 - 4K_{\widetilde{\alpha}}^2 - 4C^2 \left\|\widetilde{\boldsymbol{\alpha}}_{\boldsymbol{x}}\right\|_{\Omega\setminus\Omega_1} \left\|\Delta\widetilde{\boldsymbol{\alpha}}\right\|_{\Omega\setminus\Omega_1} > 0,$$

$$\left\|\mathbf{u}_{\mathbf{0}\boldsymbol{x}} - \widetilde{\boldsymbol{\alpha}}_{\boldsymbol{x}}(\cdot,0)\right\|_{\Omega\setminus\Omega_1}^2 + \frac{4}{\nu_0}\int_0^T \left\|\mathbf{F}_1\right\|_{\Omega\setminus\Omega_1}^2 \, dt$$

$$\leq \frac{\nu_0^2}{6\sqrt{6}C^2}\sqrt{K_1 - \frac{4K_{\widetilde{\alpha}}^2}{\nu_0^2} - \frac{4C^2 \left\|\widetilde{\boldsymbol{\alpha}}_{\boldsymbol{x}}\right\|_{\Omega\setminus\Omega_1} \left\|\Delta\widetilde{\boldsymbol{\alpha}}\right\|_{\Omega\setminus\Omega_1}}{\nu_0^2}}$$

where

$$\mathbf{F}_1 \equiv \mathbf{f} - \widetilde{\alpha}_k \widetilde{\boldsymbol{\alpha}}_{x_k} + \nu_0 \Delta\widetilde{\boldsymbol{\alpha}} - \widetilde{\boldsymbol{\alpha}}_t,$$

Existence of Solution on the Uncontrolled Domain

Using a priori estimates and Galerkin'smethod:
There exists a unique solution $\mathbf{u}(\boldsymbol{x}, t)$ to the Navier–Stokes equations on the uncontrolled domain $\Omega \setminus \Omega_1$, which is in the class $\mathcal{M}_2$ of functions having finite norm

$$\|\mathbf{u}\|^2_{\mathcal{M}_2} \equiv \underset{0\leq t\leq T}{ess\ sup}\{\|\mathbf{u}\|^2_{\Omega\setminus\Omega_1} + \|\mathbf{u}_{\boldsymbol{x}}\|^2_{\Omega\setminus\Omega_1}\} + \int_0^T \|\Delta\mathbf{u}\|^2_{\Omega\setminus\Omega_1}\, dt.$$

Combining the Two Subproblems

$$\boldsymbol{w}(\boldsymbol{x},t) = \begin{cases} \mathbf{v}(\boldsymbol{x},t) & \text{in } Q'_T \\ \mathbf{u}(\boldsymbol{x},t) & \text{in } Q''_T \end{cases}$$

- $\mathbf{u}|_{\Gamma_T} = \mathbf{v}|_{\Gamma_T} = \boldsymbol{\alpha} \in \mathcal{N}(\Gamma)$
- There exists a unique $\boldsymbol{\alpha}$ such that

$$\left(1 + \frac{\nu_1}{\nu_0} \int\limits_{\Omega_1} |\mathbf{v}_{\boldsymbol{x}}|^2 dx \right) \mathbf{v}_{x_k}\Big|_{\Gamma_T} = \mathbf{u}_{x_k}\Big|_{\Gamma_T}, \quad k = 1, 2, 3$$

(Fixed point problem)

Define the mappings

$$\Phi, \Psi : \mathcal{N}(\Gamma) \longrightarrow C\left([0,T], H^{1/2}(\Gamma)\right)$$

$$\begin{array}{ccc} \boldsymbol{\alpha} & \xrightarrow{\quad\Psi\quad} & \Psi(\boldsymbol{\alpha}) \equiv \left(1+\dfrac{\nu_1}{\nu_0}\displaystyle\int_{\Omega_1}|\mathbf{v}_{\boldsymbol{x}}|^2dx\right)\mathbf{v}_{\boldsymbol{x}}|_\Gamma \\ \big\downarrow & \textbf{Subproblem 1.} & \big\uparrow \\ \mathbf{v} & \longrightarrow & \mathbf{v}_{\boldsymbol{x}} \end{array}$$

$$\begin{array}{ccc} \boldsymbol{\alpha} & \xrightarrow{\quad\Phi\quad} & \Phi(\boldsymbol{\alpha}) \equiv \mathbf{u}_{\boldsymbol{x}}|_\Gamma \\ \big\downarrow & \textbf{Subproblem 2.} & \big\uparrow \\ \mathbf{u} & \longrightarrow & \mathbf{u}_{\boldsymbol{x}} \end{array}$$

Interface Condition

$$\Phi\left(\boldsymbol{\alpha}\right) = \Psi\left(\boldsymbol{\alpha}\right)$$

or equivalently (fixed point formulation)

$$\mathcal{K}\left(\boldsymbol{\alpha}\right) \equiv \Phi^{-1} \circ \Psi\left(\boldsymbol{\alpha}\right) = \boldsymbol{\alpha}$$

$\mathcal{K} = \Phi^{-1} \circ \Psi$ is a contraction mapping on sufficiently small time interval $[0, T_0]$.

1. Ψ is Lipschitz continuous:

$$\left\|\Psi\left(\boldsymbol{\alpha}_1\right) - \Psi\left(\boldsymbol{\alpha}_2\right)\right\|_{C\left([0,T],H^{1/2}(\Gamma)\right)} \leq C \left\|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2\right\|_{\mathcal{N}(\Gamma)}$$

- embedding theorems
- trace theorem
- extension theorem

2. Φ^{-1} exists and is Lipschitz continuous.

$$\|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2\|_{\mathcal{N}(\Gamma)} \leq C(T) \left\|\Phi(\boldsymbol{\alpha}_1) - \Phi(\boldsymbol{\alpha}_2)\right\|_{C\left([0,T],H^{1/2}(\Gamma)\right)}$$

$$C(T) \xrightarrow{T \to 0} 0$$

(a) Φ is one–to–one, hence invertible
(b) Φ^{-1} is a contraction mapping for small T_0

- embedding theorems
- trace theorem
- extension theorem
- generalized Poincaré inequality (Lawson)
- pressure estimates

Finding Fixed Point (Continued)

- $\mathcal{K} = \Phi^{-1} \circ \Psi$ is a contraction mapping for sufficiently small time interval $[0, T_0]$
- $C(T)$ does not depend on the data
- Time steps of length T_0 cover an arbitrary interval $[0, T]$

One of many estimates

$$\begin{aligned} \|\mathbf{w}(\cdot,t)\|_{\Omega} &\leq \|\mathbf{w}(\cdot,0)\|_{\Omega} e^{-\nu_0 K_1 t} \\ &\quad + \int_0^t \|\mathbf{f}(\boldsymbol{x},s)\|_{\Omega} e^{-\nu_0 K_1 (t-s)} ds \end{aligned}$$

Numerical Simulations

- Promising preliminary results in a 2D channel with control in the boundary layer.
- Plans for 3D problems using GPUs.