Pseudorational Transfer Functions - a class of infinite-dimensional systems

Yutaka Yamamoto

Professor Emeritus

Kyoto University, Universite de Paris 11

http://www-ics.acs.i.kyoto-u.ac.jp/~yy/

Agenda

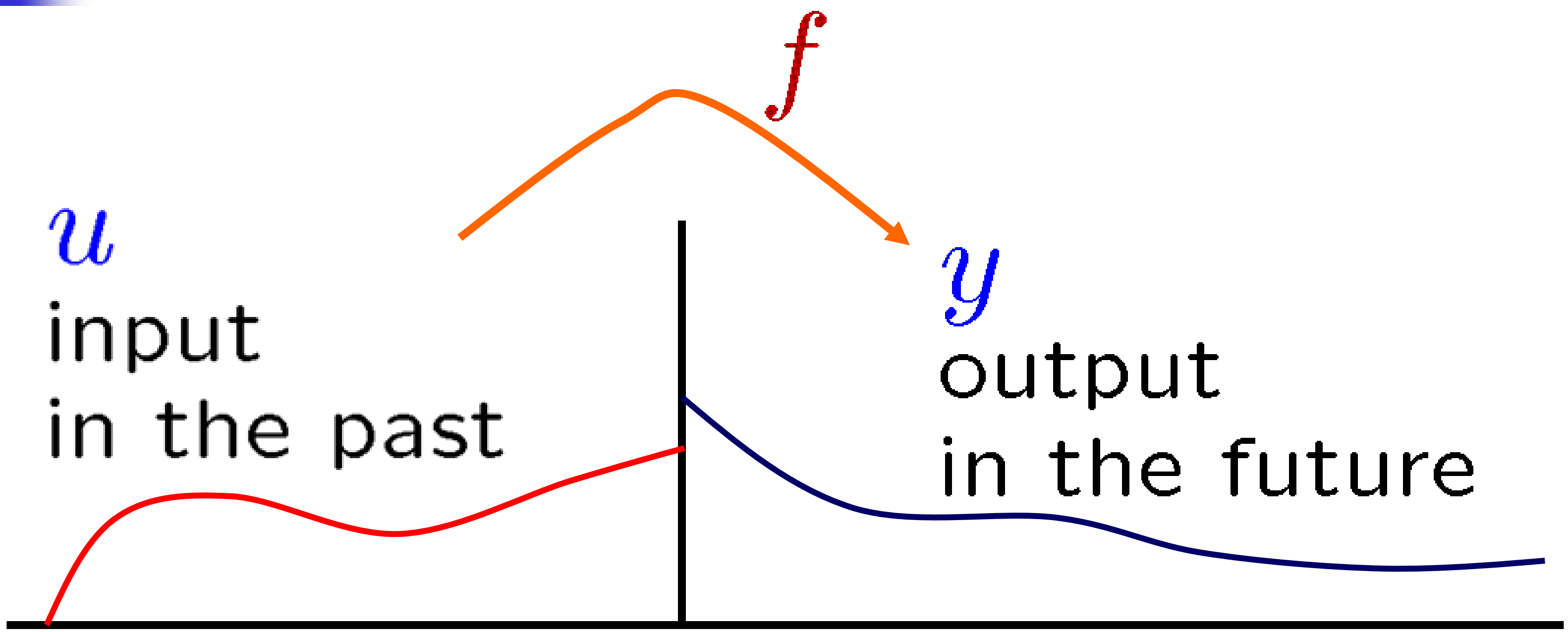
- An algebraic approach to inf-dim. systems
- Class "pseudorational"
- Standard state space realizations
- Spectra, stability, reachability conditions
- New characterization of invariant subspaces of H^2
- $\to$ Optimal sensitivity
- New result on Bezout condition

A general introduction - How I came up with this class?

1978-1980

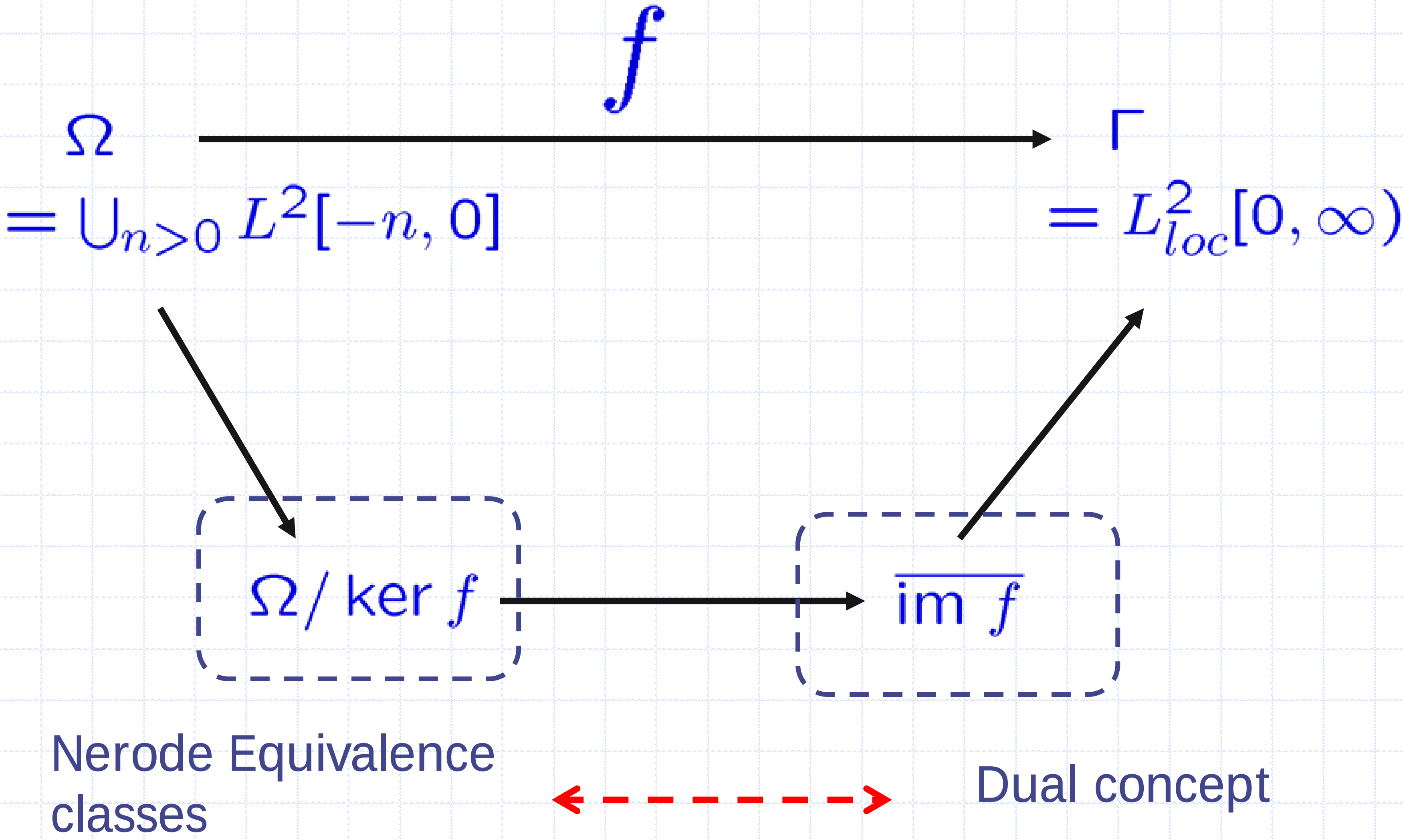
- Struggling with realization theory
- How to “construct” a “canonical” realization from i/o data.

Hankel (I/o) operator



f represents only the causal part

Canonical Contruction



A standard realization construction (Nerode equivalence)

- $\Omega / \ker f$ or $\overline{\mathrm{im}\ f}$
- $\overline{\mathrm{im}\ f} \subset L^2_{loc}[0, \infty)$
- How to compute this?
- crucial property:

$$\overline{\mathrm{im}\ f} \cong \overline{\mathrm{im}\ f}|_{[0,T]}$$

- $\Rightarrow$ Banach space

Some ideas

- Module structure of realizations (Kalman)
- $\alpha(z) \cdot X = 0$, α: polynomial
- $\Rightarrow$ finite length
- In the time domain,
- $q * X = 0 \Leftrightarrow G = q^{-1} * p$

- q has finite length?
- Distribution with compact support
- $s^3 + 2s^2 + 1, s = \delta'$, finite-dim.
- Distribution with support at {0}
- More of such kind, having support bounded in $(-\infty, 0]$

Examples

- Delay-differential systems
- Wave equations
- Systems with finite impulse response
- $1-\frac{s}{2}+\frac{s^2}{3!}-\frac{s^3}{4!}\cdots=\frac{1-e^{-s}}{s}$

Pseudorationality

Motivation

Want to build an algebraic theory for

$$\begin{aligned} \dot{x}(t) &= x(t-1) + u(t) \\ y(t) &= x(t) \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \delta' * x &= \delta_1 * x + u \\ y &= x \end{aligned}$$

$$\Rightarrow \quad y = (\delta'_{-1} - \delta)^{-1} * \delta_{-1} * u$$

Laplace transform

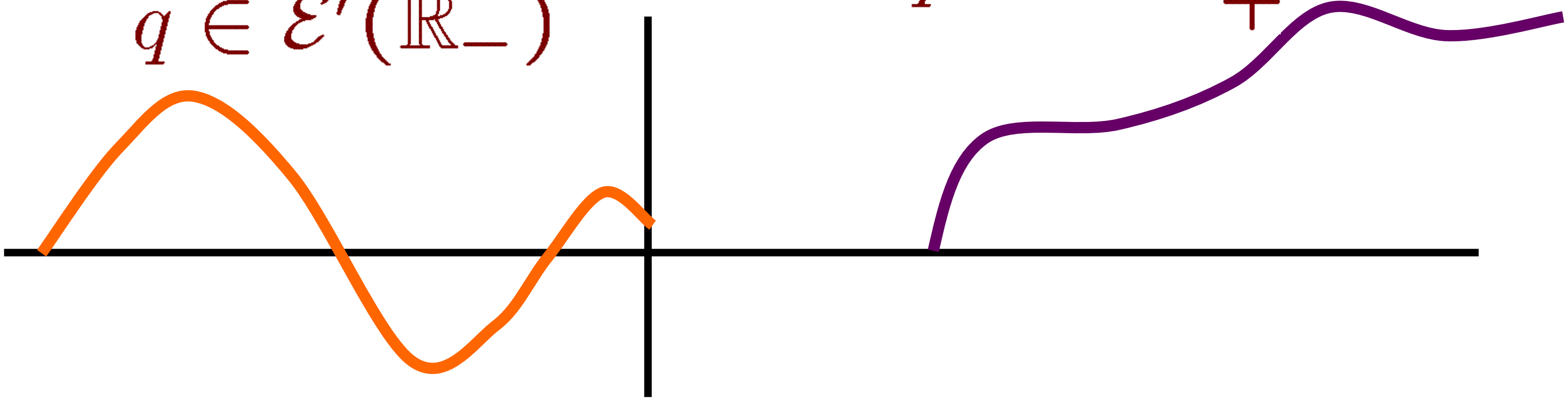
$$\widehat{y}(s) = \frac{e^s}{se^s - 1}\widehat{u}(s)$$

- Ratio of entire functions of exponential type
- Generalization $\rightarrow$ **Ratio of distributions with compact support**

Pseudorational impulse responses (YY '82)

P: *pseudorational* if $\exists q, p \in \mathcal{E}'(\mathbb{R}_-)$

- q^{-1} exists over $\mathcal{D}'_+(\mathbb{R})$,
- $\operatorname{ord} q^{-1} = -\operatorname{ord} q$,
- $P = q^{-1} * p$



Why is this class nice?

- p, q: compact support $\Rightarrow$ bounded-time construction for the state space
- $p(s), q(s)$: entire function of exp. type – analyticity crucial in various computations

Typical Examples

- Delay systems, retarded, neutral; commensurable or non-commensurate, point- or distributed delays
- Continuous-time FIR systems
- Some wave equations
- Closed under pole-zero cancellation
- Compatible with behavioral representation $q * y = p * u$

Paley-Wiener Theorem

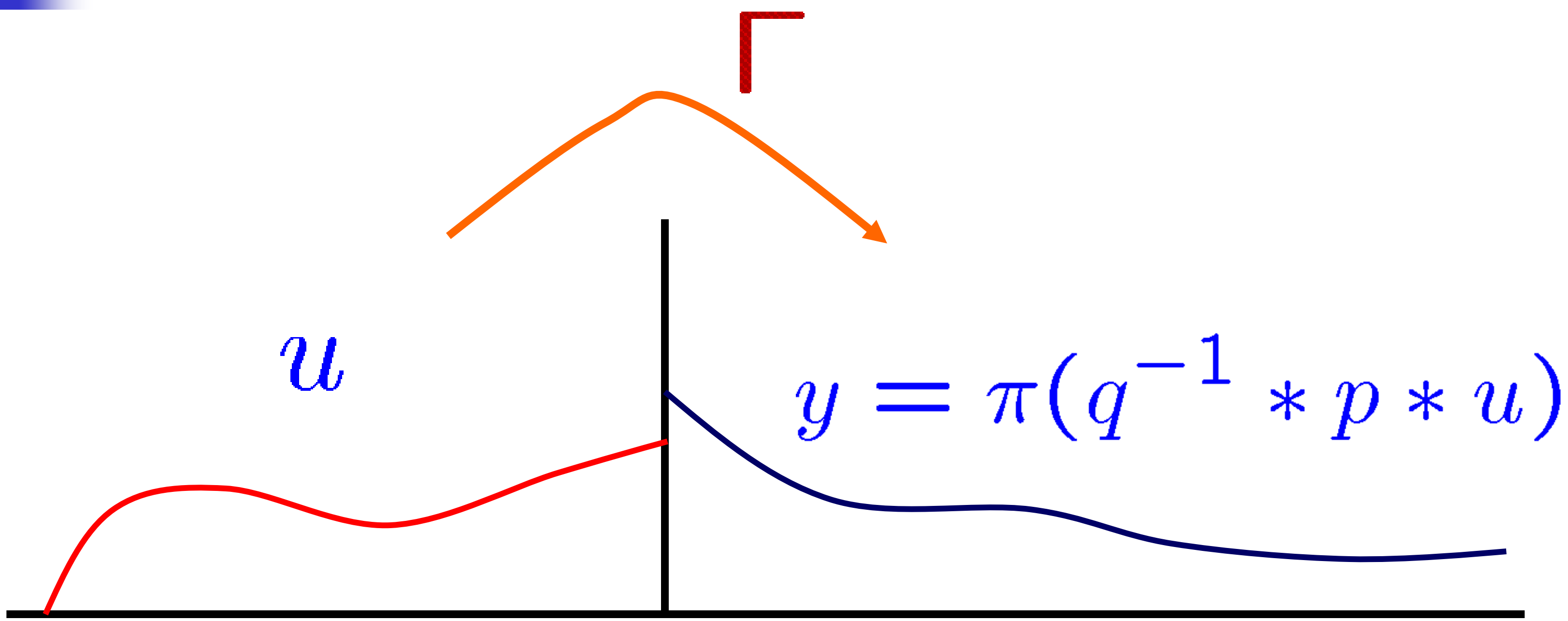
$p \in \mathcal{E}'(\mathbb{R}_-) \iff \hat{p}$: entire function and

$$|\hat{p}(s)| \leq \begin{cases} C(1+|s|)^m e^{\mathrm{Re}\, s}, & \Re s \geq 0 \\ C(1+|s|)^m, & \Re s < 0 \end{cases}$$

For example,

$$p(s) = se^s - 1$$

Input/output (Hankel) operator



State space construction

$$X^q := \{x \in L^2_{loc}[0,\infty) | q * x \in \mathcal{E}'(\mathbb{R}_-)\}$$

- Left shift invariant subspace of $L^2{}_{loc}$
- Space annihilated by $q\,*$
  - Left shift $\sigma_t \to A$
  - $B := \; q^{-1} * p$
  - C: evaluation at $t = 0$
- Shift realization $\Sigma^{q,p}$

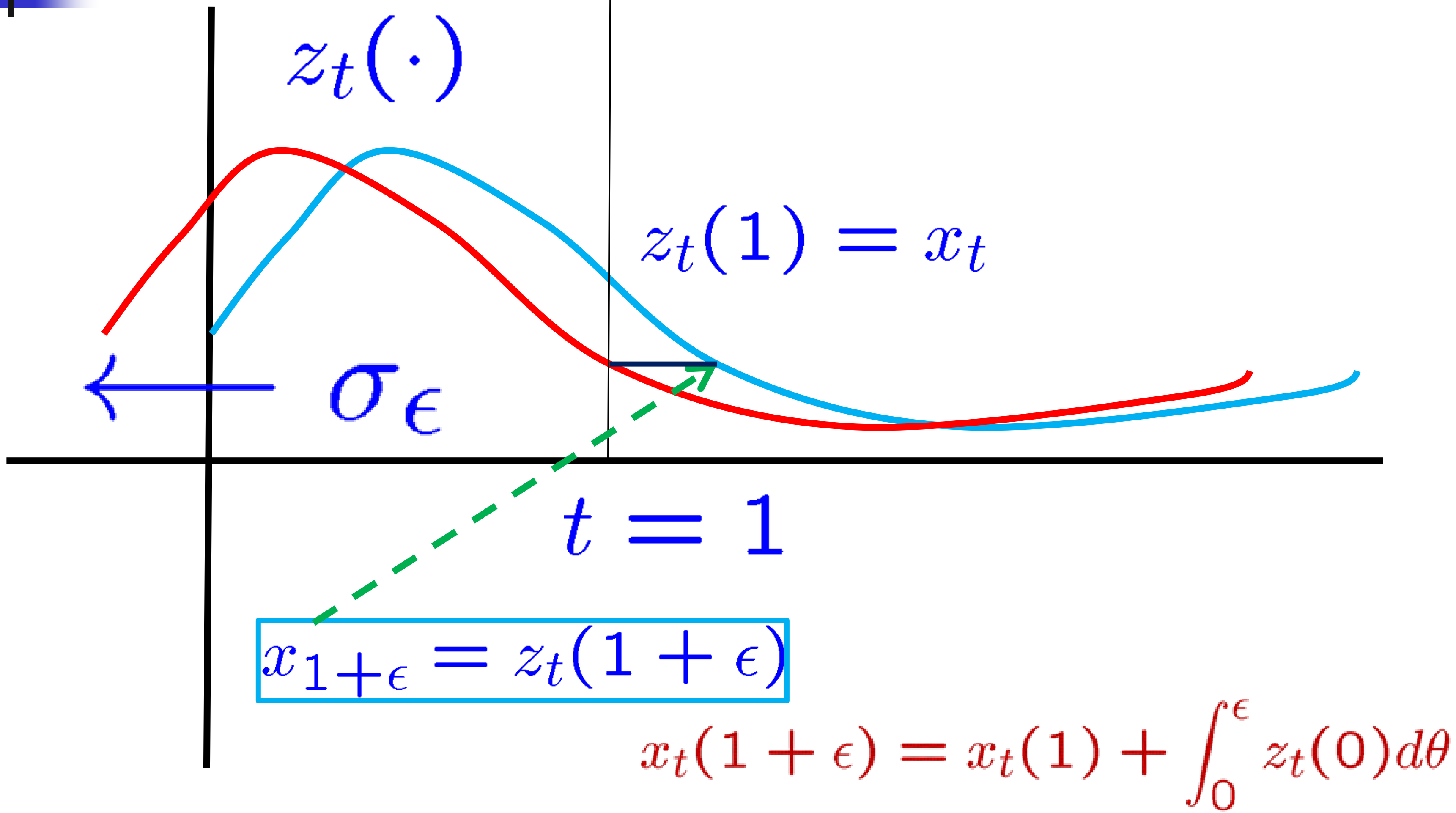
Example

- Take $q := \delta'_{-1} - \delta$

$$X^q = \{x | \frac{dx}{dt}(t+1) - x(t) = 0,\ t > 0\}$$

- This space is completely determined by the initial data $x(\cdot) \in L^2[0,1]$ and $x(1)$
- That is, $X^q \cong \mathbb{R} \times L^2[0,1]$.

$\Rightarrow$ can compute the usual realization in M_2


$z_t(\cdot)$
$z_t(1) = x_t$
σ_ϵ
$t = 1$
$x_{1+\epsilon} = z_t(1+\epsilon)$
$x_t(1+\epsilon) = x_t(1) + \int_0^\epsilon z_t(0)d\theta$

- State spacee: $X^q \cong \mathbb{R} \times L^2[0,1]$.

State: $(x_t, z_t(\theta)) \in \mathbb{R} \times L^2[0,1]$

$$\frac{d}{dt}\begin{bmatrix} x_t \\ z_t(\cdot) \end{bmatrix} = \begin{bmatrix} z_t(0) \\ \frac{\partial}{\partial\theta} z_t(\theta) \end{bmatrix}$$

$$y(t) = x_t$$

Spectrum, stability

- $\sigma(A) = \{\lambda \in \mathbb{C} | \hat{q}(\lambda) = 0\}$

  point spectrum only

  Eigenfunctions: $e^{\lambda t}$

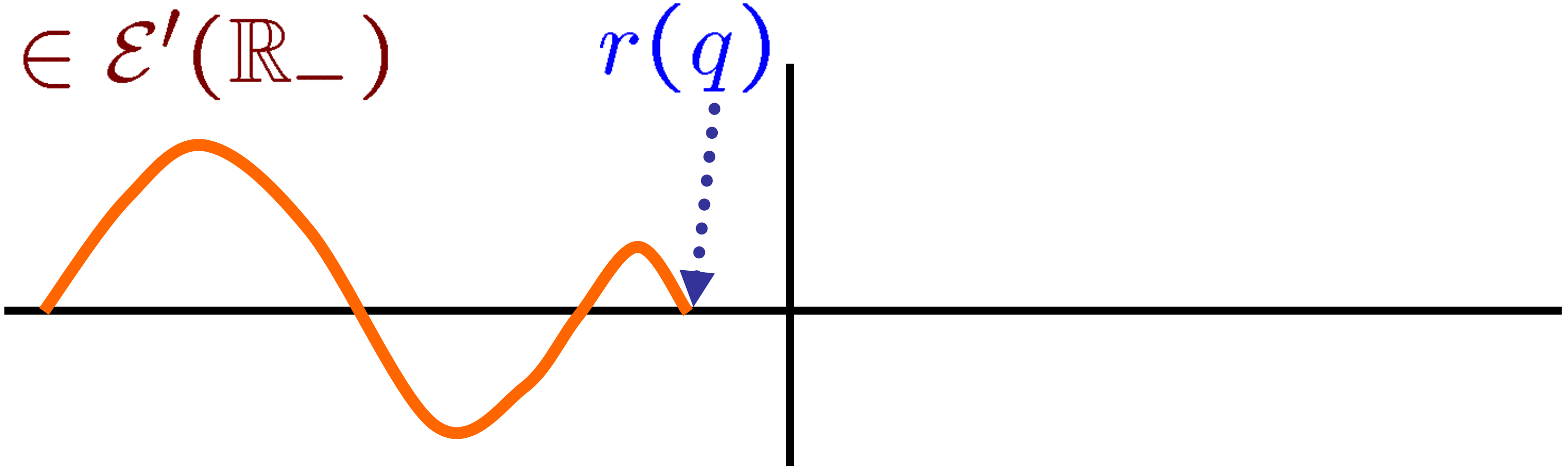
- $||\sigma_t|| \leq Ce^{-\beta t} \iff \sigma(A) \subset \{s | \Re s < -c\}$

Eigenfunction completeness

$$X^q = \overline{\mathrm{span}\{e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots\}} \iff r(q) = 0$$

$$r(q) := \sup\{t | t \in \mathrm{supp}\, q\}$$

$q \in \mathcal{E}'(\mathbb{R}_-)$



Some coprimeness conditions

(p, q): approximately coprime if

$$q * x_n + p * y_n \to \delta$$

$$\Longleftrightarrow$$

- no common zero and
- $\max\{r(q), r(p)\} = 0$

i.e., one of p, q “touches” the origin 0

Invariant subspaces and H^∞ theory

Observe

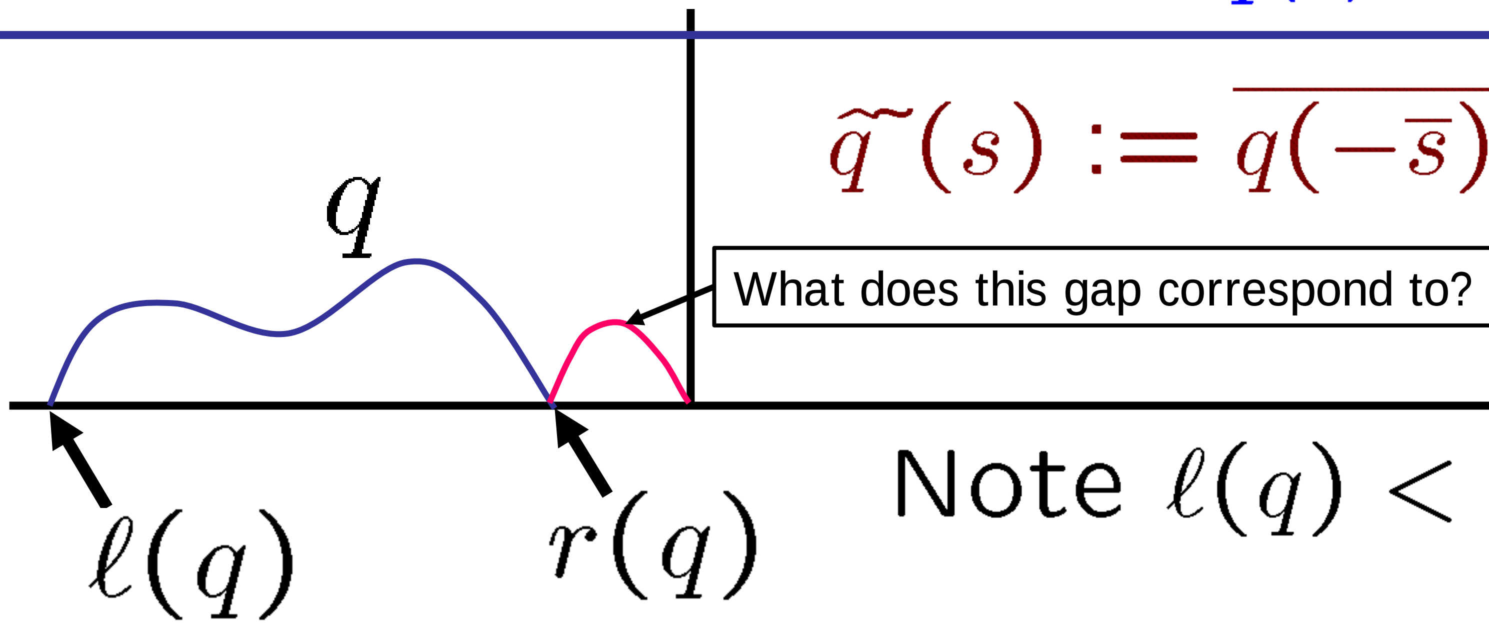
- X^q: left shift invariant
- $\Rightarrow X^q = H(m) = (mH^2)^\perp$
- by the Beurling-Lax theorem for some inner m.
- Question: given q, what is m?

Theorem:

Suppose $1/\hat{q}(s) \in H^\infty$.

$$\hat{X}^q = H(m) \text{ if } m := e^{-\ell(q)s}\,\frac{\tilde{\hat{q}}(s)}{\hat{q}(s)}$$

$$\tilde{\hat{q}}(s) := \overline{q(-\bar{s})}$$



Note $\ell(q) < 0$

Titchmarsh's theorem (local version)

$$\ell(p * q) = \ell(p) + \ell(q)$$

Examples

- $\hat{q} = e^s - \alpha,\ 0 < \alpha < 1$

$$m = e^s \frac{e^{-s} - \alpha}{e^s - \alpha} = \frac{1 - \alpha e^s}{e^s - \alpha}$$

- $\hat{q} = se^s - \beta,\ 0 < \beta < 1$

$$m = e^s \frac{-se^{-s} - \beta}{se^s - \beta} = -\frac{\beta e^s + s}{se^s - \beta}$$

Optimal sensitivity:

- $\gamma_{\text{opt}}(P) := \inf_{\psi \in H^\infty} \|W - P\psi\|_\infty$

Suppose $\hat{p}_1\hat{p}_2/\hat{q}$, $1/\hat{p}_1$ and $1/\hat{p}_2^{\sim} \in H^\infty$

p_1: stable, p_2: antistable

$$\gamma_{\text{opt}}\left(\frac{\hat{p}_1\hat{p}_2}{\hat{q}}\right) =$$

$$\max\{\gamma : \det\left(e^{LH_\gamma}\hat{p}_2^{\sim}(H_\gamma)\hat{p}_2(H_\gamma)^{-1}|_{22}\right) = 0\}$$

where $L := -\ell(q) + \ell(p_1) - r(p_2)$

You don't have to factor P to get m!

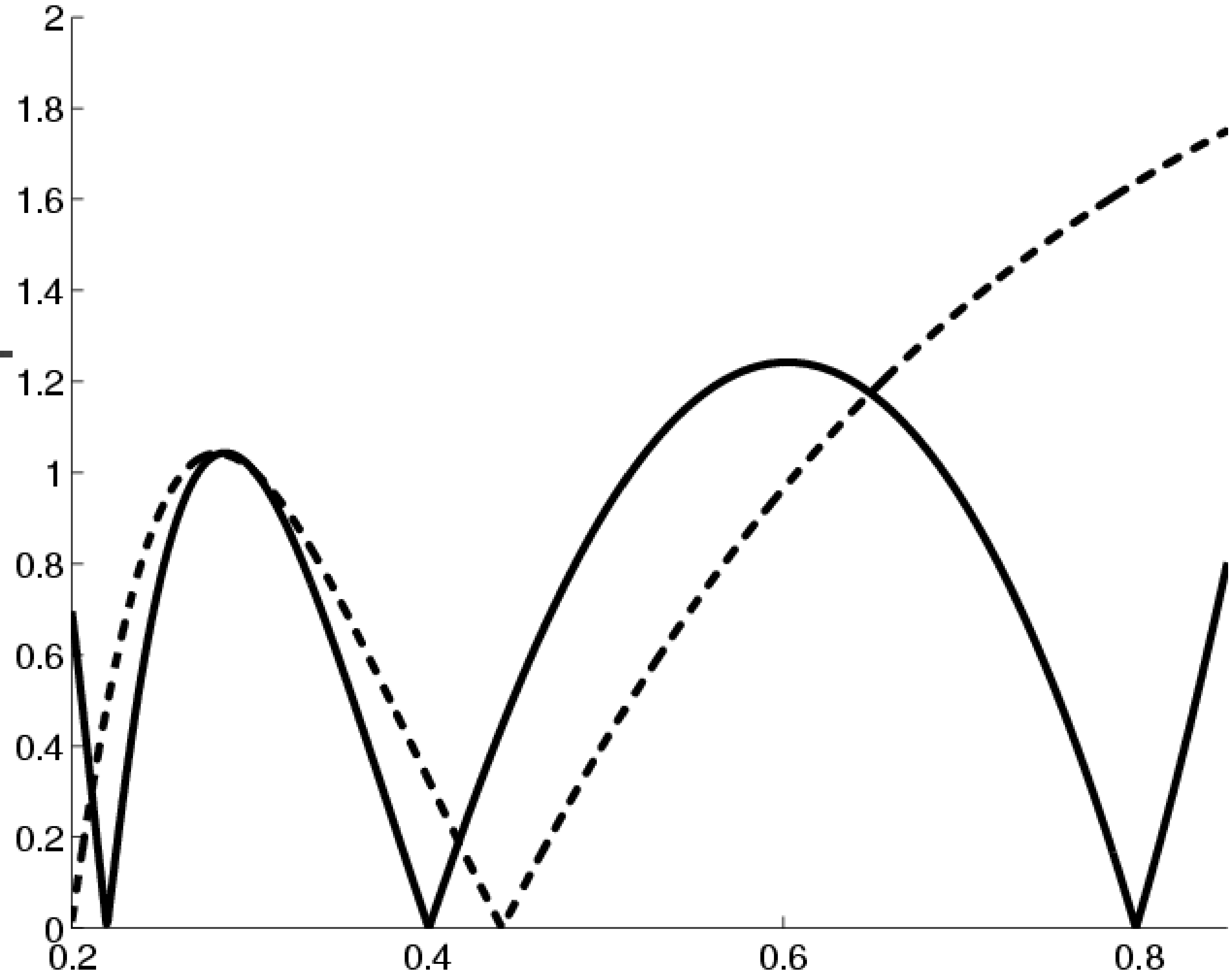
Example

$$W(s) = \frac{1}{s+1},$$

$$P(s) = \frac{e^s - \alpha}{2e^{2s} - 1} \quad (\alpha > 0,\ \alpha \neq 1).$$

$$H_\gamma := \begin{bmatrix} -1 & \gamma^{-1} \\ -\gamma^{-1} & 1 \end{bmatrix}.$$

$$e^{-Ls} \frac{\widehat{p}_2}{\widehat{p}_2^{\sim}} = e^{2s} \cdot \frac{e^{-s} - \alpha}{e^s - \alpha} = e^s \cdot \frac{1 - \alpha e^s}{e^s - \alpha}$$



Plot of $\det \tilde{m}(H_\gamma)$

Dash: stable zeros case ($\alpha < 1$)

Solid: $\alpha = 2$(unstable zeros)

Bezout Identity

Bezout condition

Given (p, q), pseudorational, find a condition under which

$$q * \psi + p * \phi = \delta$$

Or, equivalently,

$$q(s)\psi(s) + p(s)\phi(s) = 1$$

Previous results

- rank$[p(\lambda)\ q(\lambda)]$ = full $\Leftrightarrow$ every eigenspace is controllable (*)
- $\exists \psi_n, \phi_n$ such that $q\psi_n + p\phi_n \to 1$ if the underlying system is approximately controllable. (**)
- But in general this doesn't imply the Bezout condition

Special Case of measures

Theorem p, q: Radon measures.

$\exists x, y$ measures s.t. $p * x + q * y = \delta$

$\Longleftrightarrow$

$$|\widehat{p}(s)| + |\widehat{q}(s)| \geq c > 0$$

Remark

- Looks pretty much like the “Corona” theorem
- But $x, y \in \mathcal{M}$, not in $\mathcal{H}^\infty$

Rough Idea

p, q: measures.

$$p * x + q * y = \delta$$

$\iff$ $p =$ invertible over $M/(q)$.

M: space of Radon measures; (q): ideal generated by q.

$\Rightarrow$ $M/(q)$: Gel'fand algebra.

$\Rightarrow$ invertibility question over $M/(q)$

Maximal ideals of $\mathcal{M}/(q)$

'Finite' maximal ideals

$$J_{\lambda_n} := \{f | f(\lambda_n) = 0\}$$

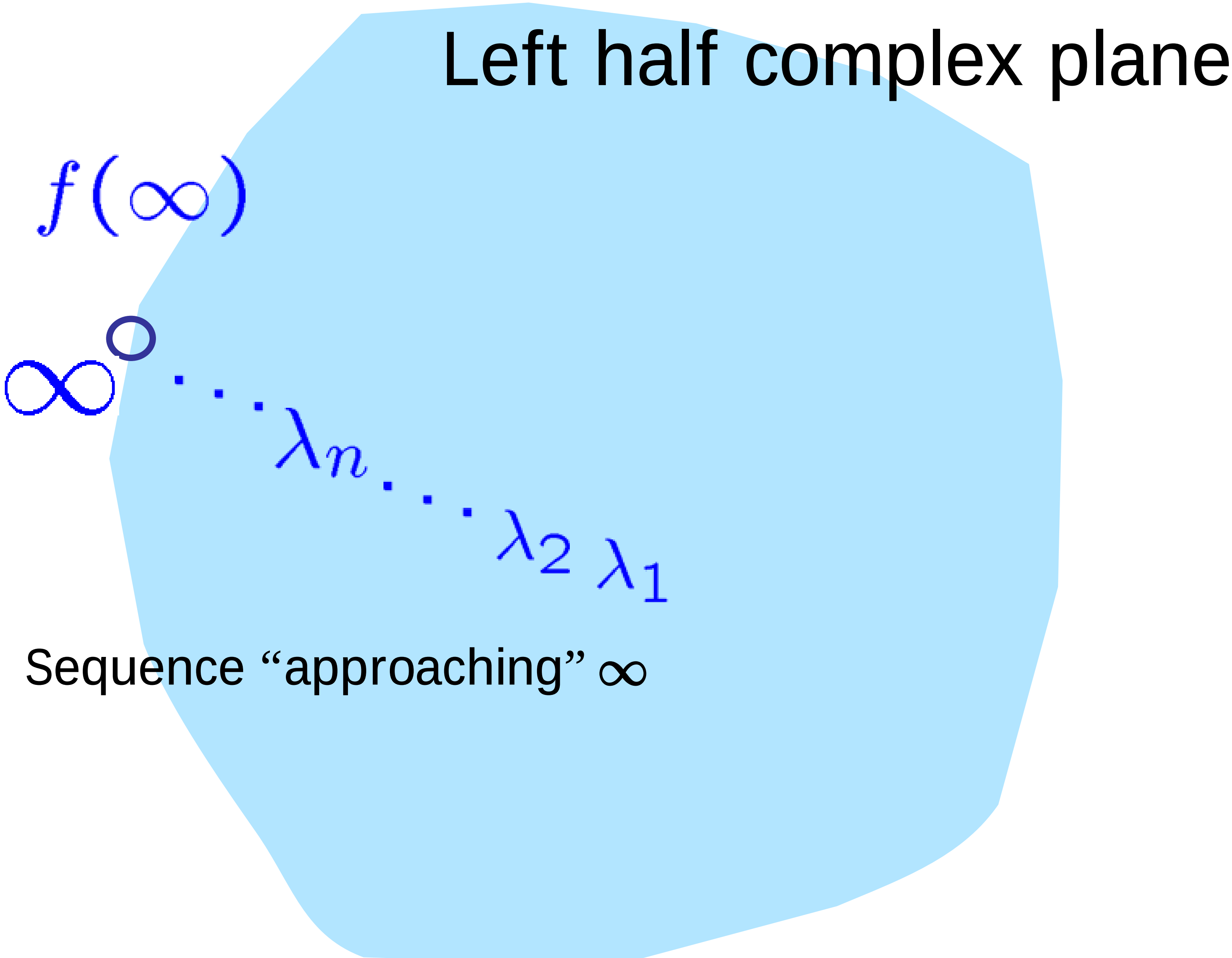
i.e., those vanishing at λ_n.

Consider

$$J := \{f | \lim_{n\to\infty} f(\lambda_n) = 0\}$$

i.e., "vanishing at ∞"

Lots of this kind


Left half complex plane
$f(\infty)$
∞
$\ldots \lambda_n \ldots \lambda_2 \; \lambda_1$
Sequence “approaching” ∞

Hence…

- p is invertible over $\mathcal{M}/(q)$
  - $p(\lambda_n)$ does not vanish $\forall\, n$
  - P does not belong to F_∞, i.e., $p(\lambda_n) \nrightarrow 0$
- I.e., $|p(\lambda_n)| \geq c' > 0$
- Hence
- $|p(s)| + |q(s)| \geq c > 0 \Rightarrow$ Bezout

Examples

- $(e^s - 1, e^s)$: coprime; $e^s = 1$ at $s = 2n\pi i$
- $(e^s - 1, e^{s/2} - 1)$: not coprime
- But

The pair $((se^s - 1)/(s-\alpha), e^s)$, $(\alpha e^\alpha - 1 = 0)$ doesn't satisfy the condition, i.e., $\exists \lambda_n$ such that $\lambda_n e^{\lambda_n} = 1$, and for such $e^{\lambda_n} = 1/\lambda_n \to 0$.

On the other hand,

$$(se^s - 1) \cdot (-1) + se^s = 1!$$

$$(se^s - 1) \cdot (-1) + se^s = 1$$

is puzzling...

- $se^s - 1 \notin \mathcal{M}$ (distribution!)
- To make it a measure (Laplace transform of), divide by $s - \alpha$ with a zero α of $se^s - 1$

But $(se^s - 1/(s - \alpha), e^s)$ still has asymptotic cancellation at ∞ for a sequence such that $\lambda_n = e^{-\lambda_n}$.

Q: "Cancellation condition at ∞" does not work?

What's this all about?

- $se^s - 1$ and e^s has asymptotic cancellation along $\lambda e^\lambda \to 1$
- On the other hand, $se^s - 1$ and se^s do NOT have any common zero, even at "∞"
- Polynomial order cancellation can be "saved" by multiplying a polynomial to $p(s)$
- i.e., $e^\lambda \to 0$, but $\lambda e^\lambda \nrightarrow 0$

Theorem Assume that the multiplicities of the zeros of $q(s)$ are globally bounded. Suppose $\exists r \geq 0$ and $c > 0$ s. t.

$$|\lambda_n^r p(\lambda_n)| \geq c, n = 1, 2, \ldots \qquad (*)$$

Then the pair (p, q) is Bezout, i.e., there exists ψ, ϕ s. t.

$$q(s)\psi(s) + p(s)\phi(s) = 1.$$

Summary

- Class *pseudorational* – ratio of distributions with compact support
- Compact representation for realization
- Eigenfunction completeness
- Reachability and coprimeness
- Some H-infinity solutions
- Bezout conditions

Possible open questions

- Local analyticity?
- Fractional order systems?
- Not confined to real-line concepts?

Thank you for your attention!