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Memory-Type Null Controllability of Evolution Equations

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This talk is based on the following two paper:

[1] F. W. Chaves-Silva, X. Zhang and E. Zuazua, *Controllability of evolution equations with memory*, In submission.

[2] Qi Lü, X. Zhang and E. Zuazua, *Null Controllability for Wave Equations with Memory*, Preprint.

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4. Memory-type null controllability of parabolic equations

5. Memory-type null controllability of hyperbolic equations

1. Introduction

We begin with the following controlled system:

$$\begin{cases} \frac{d}{dt}y = Ay + Bu, & t \in (0,T), \\ y(0) = y_0. \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $T > 0$. System (1) is said to be controllable on $(0,T)$ if for any $y_0, y_1 \in \mathbb{R}^n$, there exists a $u \in L^1(0,T;\mathbb{R}^m)$ such that $y(T) = y_1$.

Theorem： System (1) is controllable on $(0,T)$ $\Leftrightarrow$

$$rank(B, AB, A^2B, \cdots, A^{n-1}B) = n.$$

The above controllability theory for finite dimensional linear systems was introduced by R.E. Kalman (1960).

Stimulated by Kalman's work, many mathematicians devoted to extend it to more general systems including infinite dimensional systems, and its nonlinear and stochastic counterparts.

Consider the following general controlled (deterministic) evolution system

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) + B(t)u(t), \quad t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{2}$$

where $y(t) \in Y$ is the state variable, $u(t) \in U$ is the control variable, Y and U are called respectively the state space and control space, both of which are suitable Hilbert spaces. A generates an C_0-semigroup on Y, while the control operator $B(\cdot)$ maps U into Y.

Exact controllability: For any $y_0, y_1 \in Y$, find (if possible) a $u \in L^2(0,T;U)$ such that the solution of (2) satisfies

$$y(T) = y_1? \tag{3}$$

• From the equation point of view, this is typically an ill-posed problem.

• When $\dim Y = \infty$, one has to relax the exact controllability requirement (3) in many cases. This leads to the notions of approximate controllability, null controllability, partial controllability, etc. In the sequel, we focus on the null controllability.

• There exists numerous studies on the controllability of (deterministic) PDEs. The story began at 1960s.

Classical works: D. L. Russell (SIAM Rev., 1978); J. L. Lions (SIAM Rev., 1988).

Recent book/survey: T.-T. Li (2010); J. M. Coron (2007); E. Zuazua (2006).

• One needs to develop various new tools to solve the controllability of concrete PDEs. In some case, one can solve easily the PDE problem; while the corresponding control problem is challenging, say the null controllability of the heat equation with non-local terms.

• In many control problems, the non-local effect is indispensable. For example, realistic models in many practice should contain suitable memory terms.

To see this, we fix a bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$) and a (small) nonempty subset ω of Ω. Let us recall the controllability for the classical heat equation:

$$\begin{cases} y_t - \Delta y = u\chi_\omega(x) & \text{in } Q \equiv (0,T) \times \Omega, \\ y = 0 & \text{on } \Sigma \equiv (0,T) \times \partial\Omega, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{4}$$

It is well-known (A.V. Fursikov and O.Yu. I-manuvilov, 1996) that for any given $T > 0$ and non-empty open subset ω of Ω, the equation (4) is null controllable in $L^2(\Omega)$, i.e., for any given $y_0 \in L^2(\Omega)$, one can find a control $u \in L^2((0,T) \times \omega)$ such that the weak solution $y(\cdot)$ of (4) satisfies $y(T) = 0$.

Here, it is notable that the controllability time T and the control region ω can be chosen as small as one likes.

This is due to the fact that the classical heat equation admits an infinite speed of propagation for a finite heat pulse.

However, it has been known for quite a long time that the property of instantaneous propagation for the heat equation is not really physical!

To eliminate this paradox, a modified Fourier's law was introduced, which results in a heat equation with memory. We refer to J. Yong and X. Zhang (2011) for an updated analysis on the well-posedness and the propagation speed of the heat equation with memory, derived from a general modified Fourier's law.

In view of J. Yong and X. Zhang (2011), instead of (4), we consider the following controlled heat e-quaiton with a memory kernel $b(\cdot,\cdot)$:

$$\begin{cases} y_t - \Delta \left[\int_0^t b(t-s,x) y(s,x) ds \right] = u\chi_\omega(x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{5}$$

As shown in X. Fu, J. Yong and X. Zhang (2009), the hyperbolic nature of (5) allows to show its (instantaneous) exact controllability (under suitable conditions on T and ω), which means that, for any given $y_0,\ y_1 \in L^2(\Omega)$, there is a control $u \in L^2((0,T) \times \omega)$ such that the corresponding solution $y \in C([0,T];\ L^2(\Omega))$ satisfies $y(T) = y_1$ in Ω.

• On the other hand, the memory effect may appear as a state/control constraint in some control problems. To see this, fix a metric space Z and $\Gamma \subset Z$. Suppose $F : C([0,T];Y) \times L^2(0,T;U) \to Z$ is a given map. We introduce the following concept of (F,Γ)-controllability.

Definition 1. *The equation* (2) *is called* (F,Γ)*-controllable if for any* $y_0 \in Y$, *there is a control* $u(\cdot) \in L^2(0,T;U)$ *such that the corresponding solution* $y(\cdot)$ *satisfies*

$$F(y(\cdot),u(\cdot)) \in \Gamma. \tag{6}$$

The usual null controllability is a special case of (F, Γ)-controllability (by choosiing $Z = Y$, $\Gamma = \{0\}$ and $F(y(\cdot), u(\cdot)) = y(T)$). If we choose

$$Z = Y \times Y, \quad \Gamma = \{0\} \times \{0\} \tag{7}$$

and

$$F(y(\cdot), u(\cdot)) = \left(y(T), \quad \int_0^T y(s)ds \right)^\top, \tag{8}$$

then the (F, Γ)-controllability is equivalent to the usual null controllability with the following extra requirement:

$$\int_0^T y(s)ds = 0. \tag{9}$$

Clearly, (9) can be regarded as a memory-type (or integral-type) constraint on the state of (2).

One can introduce a similar (F,Γ)-controllability concept for following memory counterpart of (2):

$$\begin{cases} y_t = Ay + \int_0^t M(t-s)y(s)ds + B(t)u, \ t \in (0,T], \\ y(0) = y_0, \end{cases} \tag{10}$$

where $M(\cdot) \in L^1(0,T;\mathcal{L}(Y))$. We also fix a memory kernel $\widetilde{M}(\cdot) \in L^1(0,T;\mathcal{L}(Y))$. In the sequel, unless other stated, for the (F,Γ)-controllability of (10), we choose Z and Γ as that in (7), and

$$F(y(\cdot),u(\cdot)) = \begin{pmatrix} y(T) \\ \int_0^T \widetilde{M}(T-s)y(s)ds \end{pmatrix}. \tag{11}$$

Definition 2. *The equation* (10) *is called memory-type null controllable (with the memory kernel* $\widetilde{M}(\cdot)$*) if for any* $y_0 \in Y$*, there is a control* $u(\cdot) \in L^2(0, T; U)$ *such that the corresponding solution* $y(\cdot)$ *satisfies*

$$y(T) = 0 \quad \textit{and} \quad \int_0^T \widetilde{M}(T - s)y(s)ds = 0. \tag{12}$$

Clearly, the null controllability of (10) is a special case of memory-type null controllability of (10) with the memory kernel $\widetilde{M}(\cdot) \equiv 0$.

For the null controllability of (10), it is very natural to impose an extra requirement:

$$\int_0^T M(T-s)y(s)ds = 0, \tag{13}$$

which means the memory-type null controllability of (10) with the memory kernel $\widetilde{M}(\cdot) = M(\cdot)$.

Indeed, unlike the case for the system (2), without (13), the solution of (10) would not be kept to be zero after T even if one could find a control $u(\cdot) \in L^2(0,T;U)$ such that the corresponding solution $y(\cdot)$ satisfies $y(T) = 0$ while letting $u(t) = 0$ for a.e. $t > T$.

2. Some general consideration

We consider the problem of memory-type null controllability (with the memory kernel $\widetilde{M}(\cdot)$) of (10). For this, we introduce the following adjoint system:

$$\begin{cases} w_t = -A^*w - \int_t^T M(s-t)^* w(s)ds + \widetilde{M}(T-t)^* z_T, \ t \in [0,T), \\ w(T) = w_T, \end{cases} \tag{14}$$

where $w_T, z_T \in Y$.

By means of the duality argument, we have the following result.

Theorem 1. *Equation* (10) *is memory-type null controllable (with the memory kernel* $\widetilde{M}(\cdot)$*) if and only if there is a constant* $C > 0$ *such that solutions of* (14) *satisfy*

$$|w(0)|_{Y}^2 \leq C \int_0^T |B(s)^* w(s)|_U^2 ds, \ \ \forall\, w_T, z_T \in Y. \tag{15}$$

By Theorem 1, it is easy to see that for the special case that $M(\cdot) = 0$ and $B(\cdot) \equiv B$ for some $B \in \mathcal{L}(U, Y)$, the equation (10) is memory-type null controllable with the memory kernel $\widetilde{M}(\cdot) \equiv 1$ if and only if the operator B (from U to Y) is onto.

This indicates that the memory-type null controllability is a very strong requirement for the equation (10).

Indeed, for the case of PDE control, this means that one should control everywhere if the controller ω is fixed!

3. The finite dimensional case

Consider the following controlled ordinary differential equation with a memory term:

$$\begin{cases} y_t = Ay + \int_0^t M(t-s)y(s)ds + Bu, \quad t \in (0,T], \\ y(0) = y_0. \end{cases} \tag{16}$$

Here, $y = y(t)$ is the state variable which takes values in $\mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $M(\cdot) \in L^1(0,T;\mathbb{R}^{n\times n})$, u denotes the control variable taking values in $\mathbb{R}^m$ ($m \in \mathbb{N}$), and $B \in \mathbb{R}^{n\times m}$.

Theorem 2. *(i) Assume that* $M(\cdot), \widetilde{M}(\cdot) \in L^1(0,T;\mathbb{R}^{n\times n}) \cap C^\infty([0,T);\mathbb{R}^{n\times n})$, *and define* A_i, $M_i(\cdot)$ *and* $\widetilde{M}_i(\cdot)$ *(*$i=1,2,\cdots$*) inductively by*

$$\begin{cases} A_{i+1} = AA_i + M_i(0), \quad M_{i+1}(\cdot) = M(\cdot)A_i + M_i'(\cdot), \\ A_1 = A, \qquad M_1(\cdot) = M(\cdot), \qquad \widetilde{M}_1(\cdot) = \widetilde{M}(\cdot). \end{cases} \tag{17}$$

If

$$rank\begin{pmatrix} B & A_1B & A_2B & \cdots & A_iB & A_{i+1}B & \cdots \\ 0 & \widetilde{M}_1(0)B & \widetilde{M}_2(0)B & \cdots & \widetilde{M}_i(0)B & \widetilde{M}_{i+1}(0)B & \cdots \end{pmatrix} = 2n, \tag{18}$$

then the equation (16) *is memory-type null controllable;*

(ii) Assume that both $M(\cdot)$ and $\widetilde{M}(\cdot)$ are analytic in $[0, T]$, and define A_i as that in (17) *($i = 1, 2 \cdots$). Assume that $\frac{d^i \widetilde{M}(t)}{dt^i}\Big|_{t=0} = \widetilde{M}(0) G_i$ for some $G_i \in \mathbb{R}^{n \times n}$, and define*

$$F_i = A_i + G_1 A_{i-1} + \cdots + G_{i-1} A_1 + G_i. \tag{19}$$

If

$$rank \begin{pmatrix} B & A_1 B & A_2 B & \cdots & A_i B & A_{i+1} B & \cdots \\ 0 & B & F_1 B & \cdots & F_{i-1} B & F_i B & \cdots \end{pmatrix} = 2n, \tag{20}$$

then the equation (16) *is memory-type null controllable;*

(iii) Assume that $M(\cdot) \equiv M \in \mathbb{R}^{n\times n}$ *and* $\widetilde{M}(\cdot) \equiv \widetilde{M} \in \mathbb{R}^{n\times n}$*, and define* A_i *(*$i = 1, 2 \cdots$*) inductively by*

$$\begin{cases} A_{i+1} = AA_i + M_i, & M_{i+1} = MA_i \\ A_1 = A, & M_1 = M. \end{cases} \tag{21}$$

If

$$rank \begin{pmatrix} B & A_1B & A_2B & \cdots & A_{2n+1}B \\ 0 & B & A_1B & \cdots & A_{2n}B \end{pmatrix} = 2n, \tag{22}$$

then the equation (16) *is memory-type null controllable. If, additionally,* $\det \widetilde{M} \neq 0$*, then the condition* (22) *is also necessary for* (16) *to be memory-type null controllable.*

4. Memory-type null controllability of parabolic equations

We begin with the following heat equation with a memory term and a fixed controller:

$$\begin{cases} y_t - \Delta y + a \displaystyle\int_0^t y(s)ds = u\chi_\omega(x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{23}$$

where $a \in \mathbb{R}$. Clearly, when $\omega = \Omega$, the control u can absorb the memory term "$a \int_0^t y(s)ds$", and therefore, one can easily obtain the null controllability of (23) for this special case.

However, when ω is a proper subset of Ω, by S. Guerrero and O. Yu. Imanuvilov (2013) and X. Zhou and H. Gao (2014), the equation (23) is null controllable if and only if $a = 0$, i. e. in the absence of memory terms.

This indicates that (23) is not null controllable (needless to say memory-type null controllable) whenever $a \neq 0$ and $\omega \subsetneq \Omega$.

Because of this, in order to obtain the memory-type null controllability for parabolic equations, we need to make the controller to move so that its support covers the whole domain Ω during the control time horizon $[0, T]$.

Given a (space-independent) memory kernel $M(\cdot) \in L^1(0,T)$, we consider the following heat equation with memory, and with a moving control region $\omega(\cdot)(\subset \Omega)$:

$$\begin{cases} y_t - \Delta y + \int_0^t M(t-s)y(s)ds = u\chi_{\omega(t)}(x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{24}$$

Theorem 3. *Suppose that*

$$M(t) = e^{at}\sum_{k=0}^{K} a_k t^k, \quad \widetilde{M}(t) = e^{at}\sum_{k=0}^{K} b_k t^k,$$

where $K \in \mathbb{N}$, *and* $a, a_0, \cdots, a_K$, $b_0, \cdots, b_K$ *are real constants. Under some assumptions on* $\omega(\cdot)$, *for any* $y_0 \in L^2(\Omega)$, *there is a control* $u \in L^2(Q)$ *such that the corresponding solution* $y(\cdot)$ *to* (24), *satisfies*

$$y(T) = \int_0^T \widetilde{M}(T-s)y(s)ds = 0 \quad \textit{in } \Omega.$$

5. Memory-type null controllability of hyperbolic equations

We now analyze the controllability properties of the following wave propagation involving memory terms:

$$
\begin{cases}
y_{tt} - \Delta y + \displaystyle\int_0^t M(t,s)y(s)ds = \chi_{O(t)}(x)u & \text{in } Q,\\
y = 0 & \text{on } \Sigma,\\
y(0) = y_0,\ y_t(0) = y_1 & \text{in } \Omega.
\end{cases}
\tag{25}
$$

Here $O(t) \in \Omega$. Write

$$O = \{(t,x) \in Q \mid x \in O(t)\}.$$

Definition 3. *System (25) is said to be memory-type null controllable at time T if for any $(y_0, y_1) \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H_0^1(\Omega)$, there is a control $u \in L^2(O)$ such that the corresponding solution y satisfies that*

$$y(T) = y_t(T) = 0 \textit{ and } \int_0^T M(T,s)y(s)ds = 0 \textit{ in } \Omega. \tag{26}$$

Definition 4. *We say that an open set $U \subset Q$ satisfies the Memory Geometric Control Condition (MGC-C for short), if*

1. *All rays of geometric optics of the wave equation enter into U before the time T;*
2. *For all $x_0 \in \Omega$, the vertical line $\{(s, x_0) \mid s \in \mathbb{R}\}$ enters into U before the time T and*

$$L_U \equiv \inf_{x \in \Omega} \sup_{(t_1,t_2)\times\{x\}\subset U} (t_2 - t_1) > 0. \tag{27}$$

• The above Condition 2 needs that vertical rays, which do not propagate in space, also reach the control set and stay in it for some time. Hence, the cross section $U(t)$ of U has to move as time t evolves covering the whole domain Ω.

Theorem 4. *Suppose that O fulfills the MGCC and that the memory kernel $M(\cdot,\cdot) \in C^3([0,T] \times [0,T])$ satisfies*

$$M(t,0)M(T,t) \neq 0, \ \forall\, t \in [0,T].$$

Then the system (25) *is memory-type null controllable.*

Thank You