Stability Analysis of a Wave Equation with Local Kelvin-Voigt Damping

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BIT, Beijing, 29 June 2015

Overview

Background

Consider a linear evolution equation on Hilbert space $\mathcal{H}$

$$\begin{cases} \dfrac{dx}{dt} = \mathcal{A}x(t) \\ x(0) = x_0. \end{cases}$$

Assume that $\mathcal{A}$ generates a C_0 semigroup of contractions on $\mathcal{H}$.
General interest is on

- Stability – such as exponential stability, polynomial stability, strong stability
- Rugularity – such as analyticity, Gevrey class, differentiability

Background

Definition

$e^{\mathcal{A}t}$ is

- polynomially stable of order δ if

$$\|e^{\mathcal{A}t}X\| \leq Ct^{-\delta}\|X\|_{D(\mathcal{A})}, \qquad \forall X \in D(\mathcal{A}), \quad t > 1.$$

- exponentially stable if there are constants $M, \omega > 0$ such that

$$\|e^{\mathcal{A}t}\| \leq Me^{-\omega t}, \qquad t \geq 0.$$

Background

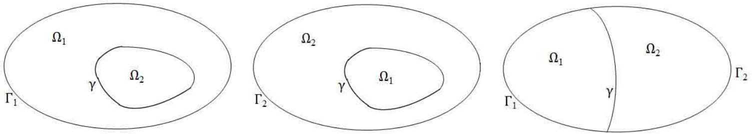
Wave Equation with local Kelvin-Voigt damping

$$\begin{cases} u_{tt} - \text{div}\,[\nabla u + a(x)\nabla u_t] = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ u = 0 & \text{on} \quad (0,\infty)\times\Gamma, \end{cases} \tag{1}$$

where

$$a(x) = \begin{cases} \beta(x) \geq 0, & x \in \Omega_1, \\ 0, & x \in \Omega_2, \end{cases}$$

$\Omega \subset R^N$ is a bounded domain with C^2 boundary Γ, $\Omega_1 \subset \Omega$ and $\Omega_2 = \Omega\backslash\overline{\Omega}_1$. Denote by γ the interface, $\Gamma_i = \partial\Omega_i\backslash\gamma$ $(i = 1, 2)$, and ν_i by the unit outward normal vector of Γ_i $(i = 1, 2)$.



Previous Work

The semigroup of (1) is

- analytic when K-V damping is global (1-d[Russell, Huang, 1988], n-d[Chen, Liu and Liu, SICON, 1998]).
- non-exponentially stable when $\beta(x) \equiv 1$ ([Chen, Liu and Liu, SICON, 1998; Q.Z.]).
- exponentially stable when K-V damping is local, $a(\cdot) \in C^2$, $a \geq 0$ and proper geometry assumptions are satisfied ([Liu and Rao, CRASP,2004]).

Previous work

Wave equation with velocity damping

$$\begin{cases} u_{tt} - \Delta u - a(x)u_t = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ u = 0 & \text{on} \quad (0,\infty)\times\Gamma, \end{cases}$$

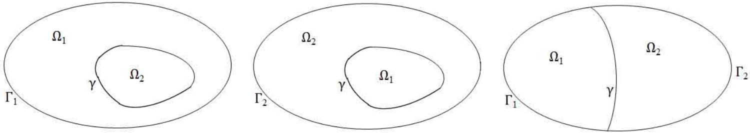
where $a(x) \geq 0$ is p.w. continuous.

- The semigroup decays exponentially when the velocity damping is global. [Chen, Fulling. etc, 1991]
- For the local damping, the system is exponentially stable under Geometrical Optics condition. [Bardos, Lebeau and Rauch, 1993]

Previous Work

The semigroup is

- analytic when K-V damping is global (1-d [Russell, Huang, 1988], n-d [Chen, Liu and Liu, SICON, 1998]).
- non-exponentially stable when $\beta(x) \equiv 1$ (1-d [Chen, Liu and Liu, 1998], n-d [Q.Z.]).
- exponentially stable when K-V damping is local, $a(\cdot) \in C^2$, $a \geq 0$ and proper geometry assumptions are satisfied ([Liu and Rao, 2004]).



Problem:

- Polynomial stability if there is no exponential stability?
- How about the case $a(\cdot) \in C(\overline{\Omega}) \backslash C^2(\overline{\Omega})$? The relationship between $a(\cdot)$ and the decay rate?

Polynomial stability of n-d system

Wave equation with local Kelvin-Voigt damping and non-continuous coefficient.

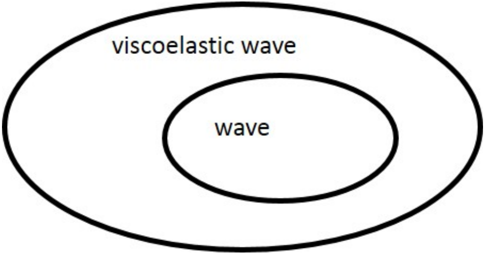
$$\begin{cases} y_{tt} - \Delta(y + y_t) = 0 & \text{in} \quad \Omega_1, \\ z_{tt} - \Delta z = 0 & \text{in} \quad \Omega_2, \\ y = 0 & \text{on} \quad \Gamma_1, \\ z = 0 & \text{on} \quad \Gamma_2, \\ y = z, \quad \partial_{\nu_1} y + \partial_{\nu_1} y_t = -\partial_{\nu_2} z & \text{on} \quad \gamma, \\ y(0) = y^0, \quad y_t(0) = y^1 \\ z(0) = z^0, \quad z_t(0) = z^1 \end{cases} \tag{2}$$

Polynomial stability

Z. Liu, Q. Zhang (2015)

Theorem

Let $\Omega \subset R^N$ be a bounded convex connected domain with partition $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. Assume that $\Gamma_2 = \emptyset$, Γ_1 and γ are C^2 smooth. Then, the energy of system (2) decays polynomially with order 1.



Tool of Proof

Introduce Hilbert space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega),$$

and unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A}X = \Big((y_2, z_2),\ (\Delta(y_1 + y_2), \Delta z_1)\Big), \quad \forall\, X = \Big((y_1, z_1), (y_2, z_2)\Big) \in D(\mathcal{A}),$$

$$\mathcal{D}(\mathcal{A}) = \Big\{\Big((y_1, z_1), (y_2, z_2)\Big) \in \mathcal{H}\ \Big|\ (y_2, z_2) \in H_0^1(\Omega), \\ \big(\Delta(y_1 + y_2), \Delta z_1\big) \in L^2(\Omega), \\ \partial_{\nu_1} y_1 + \partial_{\nu_1} y_2 = -\partial_{\nu_2} z_1 \ \text{ on } \gamma\Big\}.$$

Then system (2) can be written as

$$\frac{d}{dt}\Big((y, z),\ (y_t, z_t)\Big) = \mathcal{A}\Big((y, z),\ (y_t, z_t)\Big), \quad \forall\, t \geq 0$$

Tool of Proof

A.Borichev, Y.Tomilov(2010)

Lemma

Let e^{tA} be a bounded C_0-semigroup on a Hilbert space H with generator A such that $iR \subset \rho(A)$. Then for a fixed $\delta > 0$ the following conditions are equivalent:

(i)

$$\limsup_{|\lambda|\to\infty} \left\| \frac{1}{\lambda^\delta}(i\lambda - A)^{-1} \right\|_H < \infty,$$

(ii)

$$\|e^{tA}X_0\|_H \leq C\left(\frac{1}{t}\right)^{\frac{1}{\delta}} \|X_0\|_{D(A)}, \quad \forall X_0 \in D(A),\ \forall t > 1.$$

Sketch of Proof

- Aim: $\limsup_{|\lambda|\to\infty} \left\| \frac{1}{\lambda^\delta}(i\lambda - \mathcal{A})^{-1} \right\|_{\mathcal{H}} < \infty$

Suppose it is false. Then, there exist $\lambda_n \neq 0$ with $\lambda_n \to \infty$, and $\{X^n\}_{n=1}^\infty = \left\{ \left((y_1^n, z_1^n),\ (y_2^n, z_2^n) \right) \right\}_{n=1}^\infty \subset D(\mathcal{A})$ such that

$$\lim_{n\to\infty} \|\lambda_n^\delta(\mathrm{i}\lambda_n I - \mathcal{A})X^n\|_{\mathcal{H}} = 0. \tag{3}$$

Suppose $\|X^n\|_{\mathcal{H}} = 1$. It follows from (3) that

$$f_{1,n} \doteq \lambda_n^\delta(\mathrm{i}\lambda_n y_1^n - y_2^n) \to 0 \quad \text{in } H^1(\Omega_1), \tag{4}$$

$$f_{2,n} \doteq \lambda_n^\delta[\mathrm{i}\lambda_n y_2^n - \Delta(y_1^n + y_2^n)] \to 0 \quad \text{in } L^2(\Omega_1), \tag{5}$$

$$f_{3,n} \doteq \lambda_n^\delta(\mathrm{i}\lambda_n z_1^n - z_2^n) \to 0 \quad \text{in } H^1(\Omega_2), \tag{6}$$

$$f_{4,n} \doteq \lambda_n^\delta(\mathrm{i}\lambda_n z_2^n - \Delta z_1^n) \to 0 \quad \text{in } L^2(\Omega_2). \tag{7}$$

- Introduce a function:

$$v^n = z_1^n - \mathcal{D}\big(z_1^n|_\gamma\big),$$

where $\mathcal{D} \in L\big(H^s(\gamma), H^{s+\frac{1}{2}}(\Omega_2)\big)$, $(s \in [0,1])$ is Dirichlet map

$$\mathcal{D}f = g \;\Leftrightarrow \begin{cases} \Delta g = 0 & \text{in} \quad \Omega_2, \\ g = f & \text{on} \quad \partial\Omega_2. \end{cases}$$

v^n satisfies

$$\begin{cases} \Delta v^n + \lambda_n^2 v^n = V^n \doteq -\lambda_n^2 \mathcal{D}\big(z_1^n|_\gamma\big) - \mathrm{i}\lambda_n^{1-\delta} f_{3,n} - \lambda_n^{-\delta} f_{4,n} & \text{in} \quad \Omega_2, \\ v^n = 0 & \text{on} \quad \gamma. \end{cases} \tag{8}$$

Lemma

Assume $m(x) = x - x_0$ where $x_0 \in \mathbb{R}^N$. By multiplying (8) by $m \cdot \nabla v^n + \frac{N}{2} v^n$, we obtain

$$\int_{\Omega_2} |\nabla v^n|^2 dx = \frac{1}{2}\int_\gamma (m\cdot\nu_2)|\partial_{\nu_2} v^n|^2 d\Gamma - \mathcal{R}e \int_{\Omega_2} V^n \left(m \cdot \nabla v^n + \frac{N}{2} v^n\right) dx. \tag{9}$$

- Estimation of the right side hand of (9)

By the dissipativeness, $\lambda_n^{\delta}\mathrm{Re}\Big((\lambda_n I - \mathcal{A})U_n, U_n\Big)_{\mathcal{H}} \to 0$. Thus

$$\|y_2^n\|_{H^1_{\Gamma_1}(\Omega_1)} = o\Big(\lambda_n^{-\frac{\delta}{2}}\Big). \quad \|y_1^n\|_{H^1_{\Gamma_1}(\Omega_1)} = o\Big(\lambda_n^{-\frac{\delta}{2}-1}\Big). \tag{10}$$

Note that

$$\|\mathcal{D}\big(z_1^n|_\gamma\big)\|_{H^1(\Omega_2)} \le C\|z_1^n\|_{H^{\frac{1}{2}}(\gamma)} = C\|y_1^n\|_{H^{\frac{1}{2}}(\gamma)} \le C\|y_1^n\|_{H^1(\Omega_1)},$$

$$\|\partial_{\nu_2}\mathcal{D}\big(z_1^n|_\gamma\big)\|_{H^{-\frac{1}{2}}(\gamma)} \le C\|\mathcal{D}\big(z_1^n|_\gamma\big)\|_{H^{\frac{1}{2}}(\gamma)} = C\|z_1^n\|_{H^{\frac{1}{2}}(\gamma)} \le C\|y_1^n\|_{H^1(\Omega_1)}.$$

Then,

$$\|\mathcal{D}\big(z_1^n|_\gamma\big)\|_{H^1(\Omega_2)}, \quad \|\partial_{\nu_2}\mathcal{D}\big(z_1^n|_\gamma\big)\|_{H^{-\frac{1}{2}}(\gamma)} = o\Big(\lambda_n^{-1-\frac{\delta}{2}}\Big). \tag{11}$$

Consequently,

$$\lim_{n\to\infty}\left|\mathcal{R}e\int_{\Omega_2} V^n\left(m\cdot\nabla v^n + \frac{N}{2}v^n\right)dx\right| \le C\lim_{n\to\infty}\lambda_n^{-\delta}\|\nabla v^n\|^2_{[L^2(\Omega_2)]^N}. \tag{12}$$

Lemma

$$\lim_{n\to\infty} \|\nabla v^n\|^2_{L^2(\Omega_2)} \leq C \lim_{n\to\infty} \int_\gamma (m\cdot\nu_2)|\partial_{\nu_2} v^n|^2 d\Gamma. \tag{13}$$

Notice that

$$\partial_{\nu_2} v^n = -\partial_{\nu_1}(y_1^n + y_2^n) - \partial_{\nu_2}\mathcal{D}\left(z_1^n|_\gamma\right) \quad \text{on} \quad \gamma$$

We introduce three sequences of variables $\{Y_1^n\}_{n=1}^{\infty}$, $\{Y_2^n\}_{n=1}^{\infty}$, $\{Y_3^n\}_{n=1}^{\infty}$ such that

$$\begin{cases} \Delta Y_1^n = -\lambda_n^2 y_1^n - \mathrm{i}\lambda_n^{1-\delta} f_{1,n} & \text{in} \quad \Omega_1, \\ Y_1^n = 0 & \text{on} \quad \Gamma_1 \cup \gamma, \end{cases} \tag{14}$$

$$\begin{cases} \Delta Y_2^n = -\lambda_n^{-\delta} f_{2,n} & \text{in} \quad \Omega_1, \\ Y_2^n = 0 & \text{on} \quad \Gamma_1 \cup \gamma, \end{cases} \tag{15}$$

$$\begin{cases} \Delta Y_3^n = 0 & \text{in} \quad \Omega_1, \\ Y_3^n = y_1^n + y_2^n & \text{on} \quad \Gamma_1 \cup \gamma. \end{cases} \tag{16}$$

It is clear that

$$y_1^n + y_2^n = Y_1^n + Y_2^n + Y_3^n.$$

Lemma

$$\|\partial_{\nu_1} Y_1^n\|_{H^{\frac{1}{2}}(\partial\Omega_1)} = o\left(\lambda_n^{\frac{1}{2}-\frac{\delta}{2}}\right), \tag{17}$$

$$\|\partial_{\nu_1} Y_1^n\|_{L^2(\partial\Omega_1)} = o\left(\lambda_n^{\frac{1}{4}-\frac{\delta}{2}}\right), \tag{18}$$

$$\|\partial_{\nu_1} Y_2^n\|_{H^{\frac{1}{2}}(\partial\Omega_1)} = o\left(\lambda^{-\delta}\right), \tag{19}$$

$$\|\partial_{\nu_1} Y_3^n\|_{H^{-\frac{1}{2}}(\partial\Omega_1)} = o\left(\lambda^{-\frac{\delta}{2}}\right). \tag{20}$$

Consequently, when $\delta \geq 2$,

$$\lim_{n\to\infty} \|\partial_{\nu_2} v^n\|^2_{L^2(\gamma)} = 0, \tag{21}$$

and

$$\lim_{n\to\infty} \|\nabla v^n\|_{L^2(\Omega_2)} = 0. \tag{22}$$

Therefore, we have that

$$\|\nabla z_1^n\|_{L^2(\Omega_2)},\ \|z_1^n\|_{L^2(\Omega_2)},\ \|y_2^n\|_{L^2(\Omega_1)},\ \|y_1^n\|_{H^1(\Omega_1)} \to 0,$$

i.e., $\|X^n\|_{\mathcal{H}} \to 0$, which contradicts to the assumption.

String with local K-V damping

One-dimensional wave equation with LOCAL Kelvin-Voigt damping

$$\begin{cases} u_{tt}(t,x) = [u_x(t,x) + a(x)u_{t,x}(t,x)]_x, \quad x \in (-1,1) \\ u(t,-1) = u(t,1) = 0, \qquad t \in R^+ \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), \end{cases} \tag{23}$$

where the damping coefficient function $a(x)$ satisfies

$$a(x) = 0, \quad x \in [-1,0), \qquad a(x) \geq 0, \quad x \in [0,1].$$

Previous Works

- If $a(x) = 0$ for $x \in [-1, 0]$ and $a(x) = 1$ for $x \in (0, 1)$, then the energy of system (23) is polynomially stable with order 4 ([Z. Liu and B. Rao, 2005])
- If $a(x) = 0$ for $x \in [-1, 0]$, $a(x) > 0$ for $x \in (0, 1]$, and for $\alpha > 1$,

$$\lim_{x \to 0^+} \frac{a(x)}{x^\alpha} = k > 0,$$

then the real part of the eigenvalues of system (23) is unbounded below (M. Renardy, 2004).

Exponential Stability

Q. Zhang, 2010

Theorem

If the damping coefficient function $a(x) \in C^1[-1,1]$ *satisfies*

$$a(x) = \begin{cases} 0, & x \in [-1,0] \\ \beta(x) > 0, & x \in [0,1], \end{cases}$$

where

$$\int_0^x \frac{|\beta'(s)|^2}{\beta(s)}\, ds \le \beta_0 |\beta'(x)| \quad \text{for all} \quad x \in [0,1].$$

then, system (23) is exponentially stable.

Eventual Differentiability

Since the wave started from the undamped region need time to reach the damped region, the semigroup associated with system (23) will neither be analytic nor be of Gevrey class. The best we can expect is differentiability for $t \geq t_0 > 0$.

K. Liu, Z. Liu, and Q. Zhang (2014)

Theorem

If the damping coefficient function $a(x) \in C^1[-1,1]$ satisfies

$$a(x) = 0, \quad x \in [-1,0], \qquad a(x) > 0, \quad x \in (0,1]$$

and for $\alpha > 1$,

$$\lim_{x \to 0^+} \frac{a(x)}{x^\alpha} = k > 0.$$

then the semigroup associated with system (23) is eventually differentiable.

Polynomial Stability

Z. Liu and Q. Zhang (2015)

Theorem

If the damping coefficient function $a(x) \in C[-1,1]$ *satisfies*

$$a(x) = 0, \quad x \in [-1,0], \qquad a(x) > 0, \quad x \in (0,1],$$

and for $0 < \alpha < 1$,

$$\lim_{x \to 0^+} \frac{a(x)}{x^\alpha} = k > 0.$$

Then,

(i) *For* $\alpha = 1$, *system (23) is exponentially stable.*

(ii) *For* $0 < \alpha < 1$, *the energy of system (23) decays at a rate of* $\dfrac{2}{1-\alpha}$.

More general assumption:

(A1) $$\beta_0 \doteq \sup_{x\in(0,1)} \left[\beta(x)\int_x^1 \frac{ds}{\beta(s)}\right] < \infty.$$

(A2) $$\tilde{\beta}_0 \doteq \sup_{x\in(0,1)} \left(\int_0^x |\beta'(s)|^2 ds\right)\left(\int_x^1 \frac{ds}{\beta(s)}\right) < \infty.$$

Theorem

If the damping coefficient function $a(x) \in C[-1,1]$ satisfies

$$a(x) = \begin{cases} 0, & x \in [-1,0) \\ \beta(x) > 0, & x \in [0,1], \end{cases}$$

Then,

(i) *the energy of the system (23) decays polynomially with order 2 if function $\beta(\cdot)$ satisfies (A1).*

(ii) *the energy of the system (23) decays polynomially with order 3 if function $\beta(\cdot)$ satisfies (A2) and $\beta(0) = 0$.*

- It is clear that assumption (A2) is stronger than (A1). Especially, if we take $a(x) = x^\alpha$, a direct computation gives that (A1) is satisfied for any $\alpha \geq 0$, and (A2) is true when $\alpha > \frac{1}{2}$.

Theorem

If the damping coefficient function $a(x) \in C[-1,1]$ satisfies

$$a(x) = \begin{cases} 0, & x \in [-1,0) \\ \beta(x) > 0, & x \in [0,1], \end{cases}$$

Then,

(i) *the energy of the system (23) decays polynomially with order 2 if function $\beta(\cdot)$ satisfies (A1).*

(ii) *the energy of the system (23) decays polynomially with order 3 if function $\beta(\cdot)$ satisfies (A2) and $\beta(0) = 0$.*

In Summary

- **1-d system** $(\beta(x) \sim x^{\alpha})$
  - $\alpha > 1$: eventually differentiable, exponentially stable
  - $\alpha = 1$: exponentially stable
  - $0 < \alpha < 1$: polynomially stable with order $\frac{2}{1-\alpha}$
  - $\alpha = 0$: polynomially stable with order 4 (Rao,Liu)
- **N-d system** $(\beta(x) \equiv 1)$
  - non-exponentially stable
  - polynomially stable with order 1

Open problems

- What is the best decay rate of system (23) when $0 < \alpha < 1$?
- Eventual differentiability in high dimensional case?
- Stability for more general geometry?
- Beam, plate equation with local K-V damping

Thanks For Your Attention!