Boundary Port Hamiltonian systems with a moving interface: examples and properties

B. Maschke

Université Lyon 1, Université de Lyon

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Introduction

Introduction

Systems with moving interface : motivation

Systems with moving interface occur in numerous processes when the system is heterogeneous in the considered spatial domain, leading to consider several phases.
The most simple example (which we shall also consider here) consists in two fluids which are separated by some moving wall.
The interfaces arize in models of different chemical processes:

- polymerization in emulsion for elaboration of nanoparticules
- evaporation processes where an interface separates the domain of existence of liquid, vapor phase or their mixture
- screw-extruders with fully filled or partially filled zones
- phase changes in solid batteries

May the port Hamiltonian formulation be extended to systems of conservation laws coupled by some moving interface ?
First thoughts based on M. Diagne's Ph.D. thesis.

Sketch of talk

Boundary Port Hamiltonian systems of 2 conservation laws

Boundary port Hamiltonian systems of 2 conservation laws

Hamiltonian system of two conservation laws

Consider a system of two conservation laws with spatial domain $Z = [a, b] \subset \mathbb{R}$

$$\partial_t x + \partial_z \mathscr{N}(x) = 0 \tag{1}$$

with the 2-dimensional state vector $x(z,t) = \begin{pmatrix} x_1(z,t) \\ x_2(z,t) \end{pmatrix}$ and the flux variables

$$\mathscr{N}(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta_{x_1}\mathscr{H} \\ \delta_{x_2}\mathscr{H} \end{pmatrix} \tag{2}$$

generated by the functional $\mathscr{H}(x) = \int_a^b H(x)\, dz$ with potential density function .

Hamiltonian system of two conservation laws

The system of conservation laws (1) with the closure relations (2) may be rewritten as the Hamiltonian system

$$\partial_t x = \mathscr{J}\, \delta_x \mathscr{H} \tag{3}$$

generated by the Hamiltonian functional $\mathscr{H}(x)$ and defined with respect to the Hamiltonian differential operator

$$\mathscr{J} = \begin{pmatrix} 0 & -\partial_z \\ \partial_z^* & 0 \end{pmatrix} \tag{4}$$

where ∂_z^* is the formal adjoint of the operator $\partial_z$with boundary conditions $\delta_{x_1}\mathscr{H}(a) = \delta_{x_1}\mathscr{H}(b) = \delta_{x_2}\mathscr{H}(a) = \delta_{x_2}\mathscr{H}(b) = 0$,

Extension to a port-Hamiltonian system

In order to allow for energy exchange of the system with its environment, the Hamiltonian system (3) is augmented with the boundary port variables

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \delta_{x_2}\mathscr{H} \\ \delta_{x_1}\mathscr{H} \end{pmatrix}\Big|_{a,b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta_{x_1}\mathscr{H} \\ \delta_{x_2}\mathscr{H} \end{pmatrix}\Big|_{a,b} \qquad (5)$$

and is thereby extended to a boundary port Hamiltonian system defined with respect to
the Stokes-Dirac structure which extends the Hamiltonian operator.

Stokes-Dirac structure

Stokes Dirac structures

Dirac structures

Let $\mathscr{F}$ and $\mathscr{E}$ be linear spaces, together with a pairing, that is, a bilinear form:

$$\begin{array}{ccc} \mathscr{F} \times \mathscr{E} & \rightarrow & \mathbb{R} \\ (f, e) & \mapsto & < e|f > \end{array}$$

Symmetrize this pairing to $\ll, \gg$ on $\mathscr{F} \times \mathscr{E}$:

$$\ll (f_1, e_1), (f_2, e_2) \gg := < e_1|f_2 > + < e_2|f_1 >, \quad (f_i, e_i) \in \mathscr{F} \times \mathscr{E}$$

Definition

A Dirac structure is a linear subspace $D \subset \mathscr{F} \times \mathscr{E}$ such that

$$D = D^{\perp}$$

, with $\perp$ denoting the orthogonal complement with respect to the bilinear form $\ll, \gg$

Dirac structures are the graphs of skew-symetric linear maps.

Stokes-Dirac structure extending the Hamiltonian operator $\mathscr{J}$

Definition

The linear subset $\mathscr{D} \subset \mathscr{F} \times \mathscr{E}$ defined by:

$$\mathscr{D} = \left\{ \left(\begin{pmatrix} f_1 \\ f_2 \\ f_\partial \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \\ e_\partial \end{pmatrix} \right) \in \mathscr{F} \times \mathscr{E} \,/\, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix}}_{\mathscr{J}} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \right.$$

$$\left. \text{and} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \Big|_{a,b} \right\}$$

is a Dirac structure with respect to the symmetric pairing:

$$\left\langle\left\langle \begin{pmatrix} f \\ e \end{pmatrix}, \begin{pmatrix} f' \\ e' \end{pmatrix} \right\rangle\right\rangle = \int_a^b (e_1\, f_1' + e_2\, f_2')\, dz + e_\partial(b)\, f_\partial(b)' - e_\partial(a)\, f_\partial(a)'$$
$$+ \int_a^b (e_1'\, f_1 + e_2'\, f_2)\, dz + e_\partial(b)'\, f_\partial(b) - e_\partial(a)'\, f_\partial(a)$$

Boundary port Hamiltonian system

Definition

A *boundary port Hamiltonian system* with

state variables $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in L_2([a,b]) \times L_2([a,b])$

port variables $\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \in \mathbb{R}^{\{a,b\}} \times \mathbb{R}^{\{a,b\}}$

generated by the Hamiltonian functional $H[x] = \int_a^b \mathscr{H}(z,x)\,dz$ with respect to the Stokes-Dirac structure $\mathscr{D}$
is defined by:

$$\left(\begin{pmatrix} \partial_t x_1 \\ \partial_t x_2 \\ f_\partial \end{pmatrix}, \begin{pmatrix} \frac{\delta H}{\delta x_1} \\ \frac{\delta H}{\delta x_2} \\ e_\partial \end{pmatrix} \right) \in \mathscr{D}$$

The Hamiltonian function satisfies the balance equation $\frac{d}{dt} H = -e_\partial^\top \Sigma f_\partial$ with $\Sigma = \mathrm{diag}(-1, 1)$.

Example: the p-system

The p-system is a model of an isentropic gas with state variables

- the specific volume $x_1(t,z) = \mathfrak{v}(t,z)$
- the velocity $x_2(t,z) = v(t,z)$

and Hamiltonian is the sum of the internal energy density $\mathscr{U}(\mathfrak{v})$ and the kinetic energy density :

$$\mathscr{H}(\mathfrak{v}, v) = \mathscr{U}(\mathfrak{v}) + \frac{v^2}{2}$$

The variational derivative of the Hamiltonian is

$$\begin{pmatrix} \delta_{\mathfrak{v}} H \\ \delta_v H \end{pmatrix} = \begin{pmatrix} \delta_{\mathfrak{v}} U(\mathfrak{v}) \\ \delta_v H \end{pmatrix} = \begin{pmatrix} -p(\mathfrak{v}) \\ v \end{pmatrix} \begin{matrix} -\text{pressure} \\ \text{velocity} \end{matrix}$$

Example: the p-system as port Hamiltonian system

The p-system may be expressed as a Hamiltonian system

$$\partial_t \begin{pmatrix} \mathfrak{v} \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -\partial_z \\ \partial_z^* & 0 \end{pmatrix}}_{=\mathscr{J}} \underbrace{\begin{pmatrix} -p(\mathfrak{v}) \\ v \end{pmatrix}}_{=\delta_x \mathscr{H}} = \begin{pmatrix} -\partial_z v \\ \partial_z p(\mathfrak{v}) \end{pmatrix} \begin{matrix} \text{spatial balance} \\ \text{momentum balance} \end{matrix}$$

augmented with port boundary variables

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \delta_{x_2} \mathscr{H} \\ \delta_{x_1} \mathscr{H} \end{pmatrix} \Big|_{a,b} = \begin{pmatrix} v \\ -p(\mathfrak{v}) \end{pmatrix} \begin{matrix} \text{velocity} \\ \text{-pressure} \end{matrix}$$

Example open surface fluid flow: the shallow water equation

One-dimensional fluid flow: the shallow water equation

- state variables: $x(t) = \begin{pmatrix} p \\ q \end{pmatrix}$ $\begin{matrix} \text{momentum} \\ \text{section area of the water} \end{matrix}$
- total energy : $H_0(p, q) = \frac{1}{2} \int_a^b \frac{\rho g}{W} q^2 + \frac{1}{\rho} q p^2 dz$
- co-energy variables :
$$\frac{\delta H_0}{\delta x} = \begin{pmatrix} \frac{qp}{\rho} \\ \frac{p^2}{2\rho} + \frac{\rho g}{W} q \end{pmatrix} \quad \begin{matrix} \text{volumic flow} \\ \text{hydrodynamic pressure} \end{matrix}$$

Interconnection of port Hamiltonian systems through an interface

Interconnection of port Hamiltonian systems through an interface

Interface relations

Consider two port Hamiltonian systems indexed by $i \in \{-, +\}$ defined in two spatial domains $[a, 0[\subset \mathbb{R}^-$ and $]0, b] \subset \mathbb{R}^+$ with the interface at $\{0\}$.

Consider the pair interface relations where the pair of interface port variables (f_I, e_I)

$$f_I = \delta_{x_2^+} \mathscr{H}^+ = \delta_{x_2^-} \mathscr{H}^- \qquad \text{continuity equation} \qquad (6)$$

$$0 = \delta_{x_1^+} \mathscr{H}^+ + \delta_{x_1^-} \mathscr{H}^- + e_I \qquad \text{balance equation} \qquad (7)$$

This are commonly considered interface relations [Godlewski et al. 2008] .

- a continuity equation of one of the flux variable (then called *privileged variable*)
- a balance equation of the other flux variable with the introduction of a source term

Two port Hamiltonian systems connected through a fixed interface

Denoting $e_i^+ = \delta_{x_i^+}\mathscr{H}^+$ and $e_i^- = \delta_{x_i^-}\mathscr{H}^-$ with $i = 1, 2$, the interface relations (6) (7) define the linear relations between the conjugated power variables

$$\begin{pmatrix} e_2^- \\ e_1^+ \\ f_I \end{pmatrix} \Big| (0^+) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1^- \\ e_2^+ \\ e_I \end{pmatrix} \Big| (0^-) \qquad (8)$$

with respect to a skew-symmetric matrix and therefore define a Dirac structure.

Then by composition of Dirac structures, a dissipative port Hamiltonian system is obtained on the product space of the state space of the subsystems [Cervera and van der Schaft 2007].

Preparing moving interfaces Interface and color functions

For moving interfaces, time-varying integration domains have to be considered:which do not appear explicitely as *variables* in the definition of boundary port Hamiltonian systems.

In the sequel:

- we consider an interface within the integration domain
- augment the Hamiltonian system with the characteristic functions of the subdomains separated by this interface

Port Hamiltonian systems coupled through a fixed interface

Port Hamiltonian systems coupled through a fixed interface

Interface and color functions

We start with considering a fixed interface at the point $z = 0$.
The interface at $z = 0$ becomes then an interior point of the spatial domain $Z = [a, b]$,
however some external variables are still associated with the interface.

We augment the state varibales with the characteristic functions of the domains (or color functions) of the two systems

$$c_0(z,t) = \begin{cases} 1 & \forall z \in [a, 0[\\ 0 & \forall z \in [0, b] \end{cases} \quad \text{and} \quad \bar{c}_0(z,t) = \begin{cases} 1 & \forall z \in]0, b] \\ 0 & \forall z \in [a, 0] \end{cases} \tag{9}$$

Color function and variables

The state variables of the coupled system are expressed as the sum of prolongations of the variables of each subsystem to the total spatial domain $Z = [a, b]$ by

$$x(z,t) = x^-(z,t) + x^+(z,t) \quad (10)$$

$$x^-(z,t) = c_0(z,t)x(z,t) \quad x^+(z,t) = \bar{c}_0(z,t)x(z,t) \quad (11)$$

And the flux variables of the two conservation laws, becomes

$$\mathcal{N}(x, c_0, \bar{c}_0) = c_0\,\mathcal{N}^-(x) + \bar{c}_0\,\mathcal{N}^+(x) \quad (12)$$

with

$$c_0\mathcal{N}(x, c_0, \bar{c}_0) = c_0\,\mathcal{N}^-(x) \quad (13)$$

$$\bar{c}_0\mathcal{N}(x, c_0, \bar{c}_0) = \bar{c}_0\,\mathcal{N}^+(x) \quad (14)$$

where it should be noticed that $\mathcal{N}^-(x)$ and $\mathcal{N}^+(x)$ in (12), (13), (14), are different flux functions in general.

Conservation law of the state variable x_1

The conservation law of the state variable x_1

$$\begin{aligned} \partial_t x_1 &= -\partial_z \left(c_0 \mathcal{N}_1^-(x) + \bar{c}_0 \mathcal{N}_1^+(x) \right) \\ &= -\underbrace{[\partial_z c_0 . + \partial_z \bar{c}_0 .]}_{\mathbf{d}_0} \mathcal{N}_1(x, c_0, \bar{c}_0) \end{aligned} \tag{15}$$

where the operator

$$\mathbf{d}_0 = -[\partial_z c_0 . + \partial_z \bar{c}_0 .] \tag{16}$$

acts as the differential operator $-\partial_z$ on each sub-domain (according to the system (3) (4)).
It is the local formulation of the conservation laws, including the continuity equation at the interface

$$\frac{d}{dt} \int_{a'}^{b'} x_1(z,t) = -\mathcal{N}_1(a',t) + \mathcal{N}_1(b',t)$$

Balance equation of the state variable x_2

The balance equation of the state variablex_2 becomes

$$\partial_t x_2 = -\mathbf{d}_0^* \mathcal{N}_2 - e_I \delta(z) \tag{17}$$

where the dual operator is defined by

$$\begin{aligned} \mathbf{d}_0^* &= [\partial_z c_0. + \partial_z \bar{c}_0.] - [(\partial_z c_0) + (\partial_z \bar{c}_0)] \\ &= -\mathbf{d}_0 + [(\partial_z c_0) + (\partial_z \bar{c}_0)] \end{aligned} \tag{18}$$

and $\delta(z)$ denote the Dirac mass.

It is the local formulation of the conservation law including the balance equation (7) of the interface relation.

The dual operator $\mathbf{d}_0^*$

$$
\begin{aligned}
\int_a^b e_1\,(\mathbf{d}_0\, e_2)\, dz &= -\int_a^b \left(e_1 \left[\partial_z\, c_0 . + \partial_z\, \bar{c}_0 .\right] e_2\right) dz \\
&= -\int_a^b \left(e_1 \left[\partial_z\,(c_0 e_2) + \partial_z\,(\bar{c}_0 e_2)\right]\right) dz \\
&= -\left[(c_0 + \bar{c}_0) e_1 e_2\right]_a^b + \int_a^b (c_0 e_2 + \bar{c}_0 e_2)\,(\partial_z e_1)\, dz \\
&= -\left[(c_0 + \bar{c}_0) e_1 e_2\right]_a^b \\
&\quad + \int_a^b e_2 \left[\partial_z\, c_0 . + \partial_z\, \bar{c}_0 .\right] e_1\, dz - \int_a^b e_2 \left[(\partial_z\, c_0) + (\partial_z\, \bar{c}_0)
\end{aligned}
$$

Hence the dual operator is defined by

$$
\begin{aligned}
\mathbf{d}_0^* &= \left[\partial_z\, c_0 . + \partial_z\, \bar{c}_0 .\right] - \left[(\partial_z\, c_0) + (\partial_z\, \bar{c}_0)\right] \\
&= -\mathbf{d}_0 + \left[(\partial_z\, c_0) + (\partial_z\, \bar{c}_0)\right]
\end{aligned}
\tag{19}
$$

Balance equations for the color functions

Noticing that the spatial domains separated by the *fixed* interface are *constant*, hence also their characteristic functions c_0 and $\bar{c}_0$ are constant,: they satisfy the trivial conservation laws

$$\partial_t c = \partial_t \bar{c} = 0 \tag{20}$$

with initial conditions being precisely c_0 and $\bar{c}_0$ and compatible boundary conditions.

Augmenting the Hamiltonian functional

Define the Hamiltonian functional $\mathscr{H}(x, c, \bar{c}) = \int_a^b H(x, c, \bar{c})\, dz$ with density

$$H(x, c, \bar{c}) = c\, H^-(x) + \bar{c}\, H^+(x) \tag{21}$$

Denoting the *extended state variable* by

$$\tilde{x} = \left(x^T, c, \bar{c}\right)^T \tag{22}$$

one computes the variational derivatives

$$\delta_{\tilde{x}} \mathscr{H}(\tilde{x}) = \begin{pmatrix} \delta_x \mathscr{H}(x, c, \bar{c}) \\ \delta_c \mathscr{H}(x, c, \bar{c}) \\ \delta_{\bar{c}} \mathscr{H}(x, c, \bar{c}) \end{pmatrix} = \begin{pmatrix} c\, \delta_x \mathscr{H}^-(x) + \bar{c}\, \delta_x \mathscr{H}^+(x) \\ \mathscr{H}^-(x) \\ \mathscr{H}^+(x) \end{pmatrix} \tag{23}$$

Augmented Hamiltonian systems

The Hamiltonian system augmented with the color variables is

$$\partial_t \tilde{x} = \mathscr{J}_a \, \delta_{\tilde{x}} \mathscr{H}(\tilde{x}) + l e_I \tag{24}$$

$$l^T = \begin{pmatrix} 0 & -1 & 0 & 0 \end{pmatrix} \tag{25}$$

with respect to the operator

$$\mathscr{J}_a = \begin{pmatrix} \begin{matrix} 0 & \mathbf{d} \\ -\mathbf{d}^* & 0 \end{matrix} & 0_2 \\ 0_2 & 0_2 \end{pmatrix} \tag{26}$$

where operator $\mathbf{d}$ is the differential operator, modulated by the color functions defined by

$$\mathbf{d} = -\left[\partial_z \, c. + \partial \, \bar{c}.\right] \tag{27}$$

and its formal dual $\mathbf{d}^* = -\mathbf{d} + \left[(\partial_z \, c) - (\partial_z \, \bar{c})\right]$

Dirac structure for systems with an inerface

Theorem

The set $\mathscr{D}_I$

$$\mathscr{D}_I = \left\{ \left(\left(\begin{array}{c} \tilde{f} \\ f_I \\ f_\partial \end{array} \right), \left(\begin{array}{c} \tilde{e} \\ e_I \\ e_\partial \end{array} \right) \right) \in \mathscr{F} \times \mathscr{E} \, / \right.$$

$$\left(\begin{array}{c} \tilde{f} \\ f_I \end{array} \right) = \left(\begin{array}{cc} \mathscr{J}_a & I \\ -I^T & 0 \end{array} \right) \left(\begin{array}{c} \tilde{e} \\ e_I \end{array} \right) \tag{28}$$

$$\left. \text{and} \left(\begin{array}{c} f_\partial \\ e_\partial \end{array} \right) = \left(\begin{array}{cc} 0 & 1 \\ (c + \bar{c}) & 0 \end{array} \right) \left(\begin{array}{c} e_1 \\ e_2 \end{array} \right)_{a,b} \right\}$$

defines a Dirac structure on the bond space $\mathscr{B} = \mathscr{F} \times \mathscr{E}$ *with* $\mathscr{F} = \mathscr{E} = L^2\left((a, b), \mathbb{R}\right)^5 \times \mathbb{R}^2$ *endowed with the pairing*

$$\left\langle \left(\begin{array}{c} \tilde{f} \\ f_I \\ f_\partial \end{array} \right), \left(\begin{array}{c} \tilde{e} \\ e_I \\ e_\partial \end{array} \right) \right\rangle = \int_a^b \tilde{e}^T \tilde{f} \, dz + e_\partial^T \Sigma f_\partial + \int_a^b e_I{}^T f_I \, dz \tag{29}$$

Some properties

The augmented Hamiltonian system (24) satisfies the following power balance equation:

$$\frac{d}{dt}H(x) \quad = \quad e_\partial^T \Sigma f_\partial + \int_a^b e_I^T f_I \, dz \tag{30}$$

If the Hamiltonians $H^+(x)$ and $H^+(x)$ are bounded from

below,the augmented system restricted to submanifold with constant color functions is passive.

For color functions being characteristic functions, then the power balance equation is local at the interface

$$\int_a^b e_I^T f_I \, dz = -e_1\left(0^-\right) e_2\left(0^+\right) + e_1\left(0^-\right) e_2\left(0^+\right)$$

Port Hamiltonian systems coupled through a moving interface

Port Hamiltonian systems coupled through a moving interface

Balance equations for the color functions

For a time-varying position $l(t)$ of the interface the spatial domains of the two subsystems are the intervals $[a, l(t)[$ and $]l(t), b]$.
The the characteristic functions become

$$c_{l(t)}(z,t) = \begin{cases} 1 & \forall z \in [a, l(t)[\\ 0 & \forall z \in [l(t), b] \end{cases} \qquad \bar{c}_{l(t)}(z,t) = \begin{cases} 1 & \forall z \in]l(t), b] \\ 0 & \forall z \in [a, l(t)] \end{cases} \tag{31}$$

These color functions are the solutions of the transport equations depending on the velocity $\dot{l}(t)$ of the interface

$$\partial_t c(z,t) = -\dot{l}(t)\,\partial_z c(z,t) \;\text{ and }\; \partial_t \bar{c}(z,t) = -\dot{l}(t)\,\partial_z \bar{c}(z,t) \tag{32}$$

with initial conditions

$$c(z,0) = c_{l(0)}(z,t) \;\text{ and }\; \bar{c}(z,0) = \bar{c}_{l(0)}(z,t) \tag{33}$$

and compatible boundary conditions.

Balance of the state variable x_1

The balance equation of the variable x_1 becomes

$$\begin{aligned}
\frac{d}{dt}\int_{a'}^{b'} x_1(z,t)\,dz &= \frac{d}{dt}\int_{a'}^{l(t)} x_1(z,t)\,dz + \frac{d}{dt}\int_{l(t)}^{b'} x_1(z,t)\,dz \\
&= \int_{a'}^{l(t)} \partial_t x_1(z,t)\,dz + \int_{l(t)}^{b'} \partial_t x_1(z,t)\,dz \\
&\quad + \dot{l}(t)\,[x_1^-(l(t),t) - x_1^+(l(t),t)] \\
&= \int_{a'}^{b'} \mathbf{d}_0\,\mathcal{N}_1(x, c_l, \bar{c}_l)\,dz + \dot{l}(t)\,[x_1^-(l(t),t) - x_1^+(l(t),t)] \\
&= -\mathcal{N}_1(a',t) + \mathcal{N}_1(b',t) + \dot{l}(t)\,[x_1^-(l(t),t) - x_1^+(l(t),t)] \\
&= -e_2(a',t) + e_2(b',t) + \dot{l}(t)\,[x_1^-(l(t),t) - x_1^+(l(t),t)]
\end{aligned} \tag{34}$$

and its local formulation becomes

$$\partial_t x_1 = \mathbf{d}_0\,\mathcal{N}_1(x, c_l, \bar{c}_l) + \dot{l}(t)\,[c\,x_1 \partial_z c + \bar{c}\,x_1 \partial_z \bar{c}] \tag{35}$$

Balance equation of the state variable x_2

The balance equation of the state variablex_2

$$\begin{aligned}
\frac{d}{dt}\int_{a'}^{b'} x_2(z,t) &= \int_{a'}^{b'} \{-\mathbf{d}_0^* \mathscr{N}_2(x, c_I, \bar{c}_I) - e_I\}\, dz + \dot{l}(t)\,[x_1^-(l(t),t) - x_1^+(l(t),t)] \\
&= \int_{a'}^{b'} \{(\mathbf{d}_0 - [(\partial_z c_I) - (\partial_z \bar{c}_I)])\, \mathscr{N}_2(x, c_I, \bar{c}_I) - e_I\}\, dz \\
&\quad + \dot{l}(t)\,[x_1^-(l(t),t) - x_1^+(l(t),t)] \\
&= -\mathscr{N}_2^-(a',t) + \mathscr{N}_2^+(b',t) + e_I(l(t)) + \dot{l}(t)\,[x_1^-(l(t),t) - x_1^+(l(t),t)] \\
&= -e_1(a',t) + e_1(b',t) - e_I(l(t)) + \dot{l}(t)\,[x_2^-(l(t),t) - x_2^+(l(t),t)]
\end{aligned} \tag{36}$$

and its local formulation becomes

$$\partial_t x_1 = -\mathbf{d}_0^* \mathscr{N}_2(x, c_I, \bar{c}_I) - e_I + \dot{l}(t)\,[c\, x_1 \partial_z c + \bar{c}\, x_1 \partial_z \bar{c}] \tag{37}$$

Augmented Hamiltonian systems

The augmented Hamiltonian formulation becomes

$$\partial_t \begin{pmatrix} x \\ c \\ \bar{c} \end{pmatrix} = \mathscr{J}_a \begin{pmatrix} \delta_x \mathscr{H}(x, c, \bar{c}) \\ \delta_c \mathscr{H}(x, c, \bar{c}) \\ \delta_{\bar{c}} \mathscr{H}(x, c, \bar{c}) \end{pmatrix} + le_l + \dot{l}(t) \begin{pmatrix} c\,x & \bar{c}\,x \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \partial_z \begin{pmatrix} c \\ \bar{c} \end{pmatrix} \tag{38}$$

This defines an input map associated with the input $\dot{l}(t)$, velocity of the interface, as follows

$$G(x, c, \bar{c}) = \begin{pmatrix} c\,x & \bar{c}\,x \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \partial_z \begin{pmatrix} c \\ \bar{c} \end{pmatrix} \tag{39}$$

One may define then the *conjugated output e_l* by

$$e_l = \left\langle G\left({}^T x, c, \bar{c}\right) \Big| , \delta_{\tilde{x}} \mathscr{H}(\tilde{x}) \right\rangle = \int_a^b \delta_{\tilde{x}} \mathscr{H}(\tilde{x})^T \; G(x, c, \bar{c})\, dz \tag{40}$$

Dirac structure for systems with a moving interface

The set $\mathscr{D}_M$

$$\mathscr{D}_M = \left\{ \left(\begin{pmatrix} \tilde{f} \\ f_I \\ e_I \\ f_\partial \end{pmatrix}, \begin{pmatrix} \tilde{e} \\ e_I \\ \dot{l} \\ e_\partial \end{pmatrix} \right) \in \mathscr{F} \times \mathscr{E} \, / \right.$$

$$\begin{pmatrix} \tilde{f} \\ f_I \\ -e_I \end{pmatrix} = \begin{pmatrix} \mathscr{J}_a & I & G(x, c, \bar{c}) \\ -I^T & 0 & 0 \\ -\langle G^T(x, c, \bar{c})| & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{e} \\ e_I \\ \dot{l} \end{pmatrix} \tag{41}$$

$$\left. \text{and} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (c+\bar{c}) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}_{a,b} \right\}$$

defines a Dirac structure. on the bond space $\mathscr{B} = \mathscr{F} \times \mathscr{E}$ endowed with the pairing

$$\left\langle \begin{pmatrix} \tilde{f} \\ f_I \\ e_I \\ f_\partial \end{pmatrix}, \begin{pmatrix} \tilde{e} \\ e_I \\ \dot{l} \\ e_\partial \end{pmatrix} \right\rangle = \int_a^b \tilde{e}^T \tilde{f} \, dz + \int_a^b e_I{}^T f_I \, dz + e_\partial^T \Sigma f_\partial - e_I \dot{l} \tag{42}$$

Example: the gas compartement separated by a piston

Consider a compartement separated by a moving piston, each filled with gas modeled as a p-system, indexed by $i \in \{-, +\}$

$$\partial_t \begin{pmatrix} \mathfrak{v} \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -\partial_z \\ \partial_z^* & 0 \end{pmatrix}}_{=\mathscr{J}} \underbrace{\begin{pmatrix} -p(\mathfrak{v}) \\ v \end{pmatrix}}_{=\delta_x \mathscr{H}}$$

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \delta_{x_2} \mathscr{H} \\ \delta_{x_1} \mathscr{H} \end{pmatrix} \Big|_{a,b} = \begin{pmatrix} v \\ -p(\mathfrak{v}) \end{pmatrix}$$

with $\mathfrak{v}(t,z)$ the specific volume and $v(t,z)$ the velocity and the total energy $\mathscr{H}(\mathfrak{v}, v) = \mathscr{U}(\mathfrak{v}) + \frac{v^2}{2}$

Example: the model of the piston

Assume the piston has no mass but is subject to friction with coefficient ν and an linear elastic force with stiffness k.
hence its dynamics is a simple integrator with input being its velocity v

$$\frac{dl}{dt} = \phi_l = v$$

and the conjugated output port variable is the sum of all forces applying on the piston

$$\varepsilon_i = -k\,l - \nu\,\phi_l$$

It is a finite-dimensional port Hamiltonian system with state variable l , structure matrix being zero, Hamiltonian function $\frac{1}{2}k\,l^2$ port-variables (ϕ_l, ε_l) and dissipative feedthrough term.

Example: coupling the fluids' interface and piston port variables

The first relation expresses that the velocity of the piston and of the interface are equal

$$\dot{l} = \phi_l = v$$

The second one has been chosen as simple as possible, and expresses the continuity of the Hamiltonian density

$$e_l = \left(-\mathscr{H}^-(l) + \mathscr{H}^+(l)\right) = 0$$

The total energy of the conservation laws and the piston model

$$\mathrm{H}_{tot}(\mathfrak{v}, v, l) = \int_a^b \left(\mathscr{U}(\mathfrak{v}) + \frac{v^2}{2}\right) dz + \frac{1}{2} k\, l^2$$

satisfies the power balance equation

$$\frac{d\mathrm{H}_{tot}}{dt} = -\nu\, v^2 - v^-(a) p^-(\mathfrak{v})(a) + v^+(b) p^+(\mathfrak{v})(b)$$

Conclusion

In this paper we have suggested port Hamiltonian formulation of a system of two conservation laws (on a 1-dimensional spatial domain) coupled by a moving interface by

- augmenting the system of conservation laws with two transport equations of the characteristic functions of the subdomains defined by the interface
- deriving a the port Hamiltonian formulation of this augmented system with,
  - in addition to the boudary port variables at the boundary of the total domain
  - two pairs of port-variables associated with the interface
    - The first pair corresponds to a particular choice of interface relation corresponding to a continuity and a balance equation on the flux variables at the interface
    - the second pair is defined by the velocity of interface and its conjugated variable.