Solvability of Forward-Backward Evolution Equations

Jiongmin Yong

(University of Central Florida/Fudan University)

June 29, 2015

Outline

1. Introduction
2. Decoupling Approach
3. Method of Continuation
4. Open Questions

1. Introduction

Consider controlled system in Hilbert space X:

$$(1.1) \qquad \begin{cases} \dot{y}(t) = Ay(t) + f(t, y(t), u(t)), & t \in [0, T], \\ y(0) = x, \end{cases}$$

with cost functional:

$$(1.2) \qquad J(x; u(\cdot)) = \int_0^T f^0(s, y(s), u(s))ds + f^1(y(T)).$$

e^{At} — a C_0-semigroup on X

$f : [0, T] \times X \times U \to X$, $f^0 : [0, T] \times X \times U \to \mathbb{R}$,

$f^1 : X \to \mathbb{R}$, U a separable metric space.

$y(\cdot)$ — state trajectory, $u(\cdot)$ — control, $x \in X$ — initial state.

$$\mathcal{U} = \{u : [0, T] \to U \mid u(\cdot) \text{ is measurable}\}.$$

Under proper conditions, $\forall x \in X$, $u(\cdot) \in \mathcal{U}$, *state equation* (1.1) admits a unique *mild solution* $y(\cdot) \equiv y(\cdot\,; x, u(\cdot))$, i.e., the solution to the following integral equation:

$$y(t) = e^{At}x + \int_0^t e^{A(t-s)} f(s, y(s), u(s))ds, \qquad t \in [0, T], \tag{1.3}$$

and $J(x; u(\cdot))$ is well-defined.

Problem (C). For any $x \in X$, find a $\bar{u}(\cdot) \in \mathcal{U}$ such that

$$J(x; \bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}} J(x; u(\cdot)). \tag{1.4}$$

$\bar{u}(\cdot) \in \mathcal{U}$ — *optimal control*

$\bar{y}(\cdot) \equiv y(\cdot\,; x, \bar{u}(\cdot))$ — *optimal state trajectory*

$(\bar{y}(\cdot), \bar{u}(\cdot))$ — *optimal pair*

Proposition 1.1. (Pontryagin's Minimum Principle) *Let $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (C). Then the following minimum condition holds:*

$$
(1.5) \quad \begin{aligned} &\langle \psi(t), f(t, y(t), u(t)) + f^0(t, y(t), u(t)) \\ &= \min_{u \in U} \big[\langle \psi(t), f(t, y(t), u) + f^0(t, y(t), u) \big], \quad t \in [0, T], \end{aligned}
$$

where $\psi(\cdot)$ is the mild solution to the following adjoint equation:

$$
(1.6) \quad \begin{cases} \dot{\psi}(t) = -A^* \psi(t) - f_y(t, \bar{y}(t), \bar{u}(t))^* \psi(t) \\ \qquad\qquad -f_y^0(t, \bar{y}(t), \bar{u}(t)), \quad t \in [0, T], \\ \psi(T) = f_y^1(\bar{y}(T)), \end{cases}
$$

i.e., the following holds:

$$
\begin{aligned} \psi(t) = e^{A^*(T-t)} f_y^1(\bar{y}(T)) + \int_t^T e^{A^*(s-t)} \Big[& f_y(s, \bar{y}(s), \bar{u}(s))^* \psi(s) \\ & + f_y^0(s, \bar{y}(s), \bar{u}(s)) \Big] ds, \quad t \in [0, T]. \end{aligned}
$$

Suppose there exists a map $\varphi : [0, T] \times X \times X \to U$ such that

$$\begin{aligned} \langle\, \psi, f(t, y, \varphi(t, y, \psi))\,\rangle + & f^0(t, y, \varphi(t, y, \psi)) \\ & = \min_{u \in U} \Big[\langle\, \psi, f(t, y, u) + f^0(t, y, u) \Big]. \end{aligned}$$

Then we obtain the **optimality system** of Problem (C):

$$(1.7) \quad \begin{cases} \dot{y}(t) = Ay(t) + f(t, y(t), \varphi(t, y(t), \psi(t))), \quad t \in [0, T], \\ \dot{\psi}(t) = -A^*\psi(t) - f_y(t, y(t), \varphi(t, y(t), \psi(t)))^*\psi(t) \\ \qquad\qquad -f_y^0(t, y(t), \varphi(t, y(t), \psi(t)), \quad t \in [0, T], \\ y(0) = x, \qquad \psi(T) = f_y^1(y(T)). \end{cases}$$

If $(y(\cdot), \psi(\cdot))$ is a mild solution to the above, then $y(\cdot)$ will be a candidate of optimal trajectory and $\varphi(\cdot\,, y(\cdot), \psi(\cdot))$ will be a candidate of optimal control.

Usually, except some special cases (say, LQ problem), the story ends with some conclusion like:

When $T > 0$ is small enough, the optimality system has a unique solution, which gives a candidate of optimal state trajectory and an optimal control.

Question: Is it possible to determine the well-posedness of the optimality system, without restricting $T > 0$ to be small?

The above suggests to consider:

$$(1.8) \qquad \begin{cases} \dot{y}(t) = Ay(t) + b(t, y(t), \psi(t)), \\ \dot{\psi}(t) = -A^*\psi(t) - g(t, y(t), \psi(t)), \\ y(0) = x, \qquad \psi(T) = h(y(T)), \end{cases} \qquad t \in [0, T],$$

$A : \mathcal{D}(A) \subseteq X \to X$ generates a C_0-semigroup e^{At}

X — a Hilbert space

$b, g : [0, T] \times X \times X \to X$, $h : X \to X$ — given maps

It could be called an **infinite-dimensional two-point boundary value problem**.

Some Relevant Situations:

(i) Sturm-Louiville Problem:

$$\begin{cases} \dfrac{d}{dx}\left[p(x)\dfrac{dy}{dx}\right] + q(x)y = -\lambda w(x)y, \qquad x \in (a,b), \\ \alpha_1 y(a) + \alpha_2 y'(a) = 0, \\ \beta_1 y(b) + \beta_2 y'(b) = 0, \end{cases}$$

with $p(x), w(x) > 0$, $(\alpha_1, \alpha_2) \neq 0$, $(\beta_1, \beta_2) \neq 0$.

Interested in the existence of eigenvalues $\lambda_1 < \lambda_2 < \cdots$ and corresponding eigenfunctions $y_1(\cdot), y_2(\cdot), \cdots$

A special case:

$$\begin{cases} y'' = -\lambda y, \qquad x \in (a,b), \\ y(a) = y(b) = 0. \end{cases}$$

(ii) [Kong–Wu 2013], [Dower–McEneaney 2015]

$$\begin{cases} y_{tt} = y_{xx}, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = y(t,L) = 0, & t \in (0,T). \end{cases}$$

Given $\xi(\cdot)$ and $\eta(\cdot)$, find initial velocity $y_t(0,x)$ (if it exists) such that $y(\cdot\,,\cdot)$ exists satisfying the above and

$$y(0,x) = \xi(x), \quad y(T,x) = \eta(x), \qquad x \in (0,L).$$

A kind of controllability problem (May let $y_t(0,x) = u(x)$).

Two-point boundary value problem is relevant to the "controllability" problem.

Recall

$$
(1.8) \qquad \begin{cases} \dot{y}(t) = Ay(t) + b(t, y(t), \psi(t)), \\ \dot{\psi}(t) = -A^*\psi(t) - g(t, y(t), \psi(t)), \\ y(0) = x, \qquad \psi(T) = h(y(T)), \end{cases} \qquad t \in [0, T],
$$

Our problem has a special structure:

One initial value problem and one terminal value problem.

We prefer to call it a **forward-backward evolution equation**,

FBEE, for short.

One may regard our FBEE as a **special case** of infinite-dimensional two-point boundary value problem.

FBEE is comparable with FBSDE.

2. Decoupling

The idea is called **Invariant Imbedding**: Transform a two-point boundary valued problem into an initial value problem (for ODEs).

Ambarzumyan (1943), Chandrasekhar (1950), Bellman–Kalaba–Wing (1950s).

Consider

$$\begin{cases} \dot{y}(t) = b(t, y(t), \psi(t)), \\ \dot{\psi}(t) = g(t, y(t), \psi(t)), \\ y(0) = x, \qquad \psi(T) = h(y(T)). \end{cases}$$

Want to seek a $\mathbb{K} : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that

$$\psi(t) = \mathbb{K}(t, y(t)), \qquad t \in [0, T].$$

If such a $\mathbb{K}(\cdot\,,\cdot)$ exists, then $y(\cdot)$ solves initial value problem:

$$\begin{cases} \dot{y}(t) = b(t, y(t), \mathbb{K}(t, y(t))), \qquad t \in [0, T], \\ y(0) = x. \end{cases}$$

Formally, one should require

$$
\begin{aligned}
g(t, y(t), \mathbb{K}(t, y(t))) &= \dot{\psi}(t) = \frac{d}{dt}\mathbb{K}(t, y(t)) \\
&= \mathbb{K}_t(t, y(t)) + \mathbb{K}_y(t, y(t))b(t, y(t), \mathbb{K}(t, y(t))).
\end{aligned}
$$

Hence, $\mathbb{K}(\cdot\,,\cdot)$ should be the solution to the following PDE:

$$
\begin{cases}
\mathbb{K}_t(t, y) + \mathbb{K}_y(t, y)b(t, y, \mathbb{K}(t, y)) - g(t, y, \mathbb{K}(t, y)) = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t, y) \in [0, T] \times \mathbb{R}, \\
\mathbb{K}(T, y) = h(y), \qquad y \in \mathbb{R}.
\end{cases}
$$

$\mathbb{K}(\cdot\,,\cdot)$ is called a **decoupling field** of FBODE.

The idea was used in the study of FBSDEs (**Four-step scheme**) [Ma–Protter–Yong 1994], [Ma–Wu–Zhang–Zhang 2014].

Consider linear FBEE:

$$(2.1)\qquad \begin{cases} \dot{y}(t) = Ay(t) + B_{11}(t)y(t) + B_{12}(t)\psi(t) + b_0(t), \\ \dot{\psi}(t) = -A^*\psi(t) - B_{21}(t)y(t) - B_{22}(t)\psi(t) - g_0(t), \\ y(0) = x, \qquad \psi(T) = Hy(T) + h_0. \end{cases}$$

Proposition 2.1. *Let $A^* = -A$ and the coefficients are bounded. Let*

$$\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \mathbb{B}(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ -B_{21}(t) & -B_{22}(t) \end{pmatrix},$$

and let $\widehat{\Phi}(\cdot\,,\cdot)$ be the evolution operator generated by $\mathbb{A} + \mathbb{B}(\cdot)$. Then linear FBEE (2.1) admits a unique mild solution $(y(\cdot), \psi(\cdot))$ for any $x, h_0 \in X$ if and only if

$$(2.2)\qquad \left[(-H, I)\widehat{\Phi}(T,0)\begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1} \in \mathcal{L}(X \times X).$$

Recall linear FBEE:

$$
\begin{cases}
\dot{y}(t) = Ay(t) + B_{11}(t)y(t) + B_{12}(t)\psi(t) + b_0(t), \\
\dot{\psi}(t) = -A^*\psi(t) - B_{21}(t)y(t) - B_{22}(t)\psi(t) - g_0(t), \\
y(0) = x, \qquad \psi(T) = Hy(T) + h_0.
\end{cases} \tag{2.1}
$$

Proposition 2.2. *If* $A^* = -A$ *and*

$$B_{22}(t) = B_{11}(t)^*, \quad t \in [0, T],$$

$$H \in \mathbb{S}^+(X), \qquad -B_{12}(\cdot), B_{21}(\cdot) \in L^\infty(0, T; \mathbb{S}^+(X)),$$

where $\mathbb{S}^+(X)$ *is the set of all non-negative bounded operators on* X*. Then a decoupling field exists for the linear FBEE* (2.1)*.*

- This corresponds to a standard LQ problem.

Proposition 2.3. *Let $\mathbb{K} : [0, T] \times X \to X$ satisfy the following:*

$$
\begin{cases}
\mathbb{K}_t(t, y) + \mathbb{K}_y(t, y)\big[Ay + b(t, y, \mathbb{K}(t, y))\big] \\
\qquad + A^*\mathbb{K}(t, y) + g(t, y, \mathbb{K}(t, y)) = 0, \\
\qquad\qquad\qquad\qquad (t, y) \in [0, T] \times \mathcal{D}(A), \\
\mathbb{K}(T, y) = h(y), \qquad y \in X.
\end{cases} \tag{2.3}
$$

Let $y(\cdot)$ be a classical solution to the following:

$$
\begin{cases}
\dot{y}(t) = Ay(t) + b(t, y(t), \mathbb{K}(t, y(t))), \quad t \in [0, T], \\
y(0) = x,
\end{cases}
$$

and

$$
\psi(t) = \mathbb{K}(t, y(t)), \qquad t \in [0, T].
$$

Then $(y(\cdot), \psi(\cdot))$ is a solution to FBEE (1.8).

How to solve (2.3)?

3. Method of Continuation

Recall

$$
\text{(1.8)} \qquad \begin{cases} \dot{y}(t) = Ay(t) + b(t, y(t), \psi(t)), \\ \dot{\psi}(t) = -A^*\psi(t) - g(t, y(t), \psi(t)), \\ y(0) = x, \qquad \psi(T) = h(y(T)). \end{cases}
$$

We only consider two cases: either $A^* = A$, or $A^* = -A$. Let

$$
\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix},
$$

$$
\mathbb{B}(t, y, \psi) = \begin{pmatrix} b_y(t, y, \psi) & b_\psi(t, y, \psi) \\ -g_y(t, y, \psi) & -g_\psi(t, y, \psi) \end{pmatrix}.
$$

$$
(3.1) \qquad \begin{cases} \dot{y}^\rho(t) = Ay^\rho(t) + \rho b(t, y^\rho(t), \psi^\rho(t)) + b_0(t), \\ \dot{\psi}^\rho(t) = -A^*\psi^\rho(t) - \rho g(t, y^\rho(t), \psi^\rho(t)) - g_0(t), \\ y^\rho(0) = x, \qquad \psi(T) = \rho h(y^\rho(T)) + h_0. \end{cases}
$$

Clearly,

$$
\begin{cases} \dot{y}^0(t) = Ay^0(t) + b_0(t), \\ \dot{\psi}^0(t) = -A^*\psi^0(t) - g_0(t), \\ y^0(0) = x, \qquad \psi^0(T) = h_0, \end{cases}
$$

$$
\begin{cases} \dot{y}^1(t) = Ay^1(t) + b(t, y^1(t), \psi^1(t)) + b_0(t), \\ \dot{\psi}^1(t) = -A^*\psi^1(t) - g(t, y^1(t), \psi^1(t)) - g_0(t), \\ y^1(0) = x, \qquad \psi^1(T) = h(y^1(T)) + h_0. \end{cases}
$$

Want to show:

$$(y^{\rho}(\cdot), \psi^{\rho}(\cdot)) \text{ exists} \quad \Rightarrow \quad (y^{\rho+\varepsilon}(\cdot), \psi^{\rho+\varepsilon}(\cdot)) \text{ exists.}$$

Therefore,

$$(y^{0}(\cdot), \psi^{0}(\cdot)) \quad \text{exists} \quad \Rightarrow \quad (y^{1}(\cdot), \psi^{1}(\cdot)) \quad \text{exists}$$

Definition 3.1. A continuous function

$$\Pi(\cdot) \equiv \begin{pmatrix} P(\cdot) & \Gamma(\cdot)^* \\ \Gamma(\cdot) & \bar{P}(\cdot) \end{pmatrix} : [0, T] \to \mathbb{S}(X \times X)$$

is called a **Lyapunov operator** of the generator (b, g, h) if there exist $\mathbb{Q} : [0, T] \to \mathbb{S}(X \times X)$ and $\mathbb{M} : [0, T] \to \mathcal{L}(X \times X)$ with

$$\mathbb{Q}(t) = \begin{pmatrix} Q_0(t) & \Theta(t)^* \\ \Theta(t) & \bar{Q}_0(t) \end{pmatrix}, \quad \mathbb{M}(t) = \begin{pmatrix} M(t) & 0 \\ 0 & -\bar{M}(t)^* \end{pmatrix},$$

such that $\Pi(\cdot)$ is a mild solution to the **Lyapunov equation**

$$\dot{\Pi}(t) + \Pi(t)[\mathbb{A} - \mathbb{M}(t)] + [A - \mathbb{M}(t)]^* \Pi(t) + \mathbb{Q}(t) = 0,$$

and for some constants $\mu, K > 0$, the following are satisfied:

$$\begin{cases} \Pi(0) + \begin{pmatrix} -K & 0 \\ 0 & 0 \end{pmatrix} \leq 0, \\ \begin{pmatrix} I & \rho h_y(y)^* \\ 0 & I \end{pmatrix} \Pi(T) \begin{pmatrix} I & 0 \\ \rho h_y(y) & I \end{pmatrix} + \begin{pmatrix} -\mu h_y(y)^* h_y(y) & 0 \\ 0 & K \end{pmatrix} \geq 0, \\ \qquad \forall y \in X, \ \rho \in [0,1], \\ \begin{pmatrix} \mathbb{H}^\rho - \mathbb{Q}(t) + \mu \begin{pmatrix} g_y(t,y,\psi)^* g_y(t,y,\psi) & 0 \\ 0 & 0 \end{pmatrix} & \Pi(t) \\ \Pi(t) & -K \end{pmatrix} \leq 0, \\ \qquad \forall (t,y,\psi) \in [0,T] \times X \times X, \ \rho \in [0,1], \end{cases}$$

$$\mathbb{H}^\rho = \rho\big[\Pi\mathbb{B}(t,y,\psi) + \mathbb{B}(t,y,\psi)^*\Pi\big] + \Pi\mathbb{M}(t) + \mathbb{M}(t)^*\Pi,$$

$$\forall (t,\Pi,y,\psi) \in [0,T] \times \mathbb{S}(X) \times X \times X, \ \rho \in [0,1],$$

or,

$$\begin{cases} \Pi(0) + \begin{pmatrix} -K & 0 \\ 0 & 0 \end{pmatrix} \leq 0, \\ \begin{pmatrix} I & \rho h_y(y)^* \\ 0 & I \end{pmatrix} \Pi(T) \begin{pmatrix} I & 0 \\ \rho h_y(y) & I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \geq 0, \\ \qquad\qquad \forall y \in X, \ \rho \in [0,1], \\ \begin{pmatrix} \mathbb{H}^\rho - \mathbb{Q}(t) + \mu \begin{pmatrix} 0 & 0 \\ 0 & b_\psi(t,y,\psi)^* b_\psi(t,y,\psi) \end{pmatrix} & \Pi(t) \\ \Pi(t) & -K \end{pmatrix} \leq 0, \\ \qquad\qquad \forall (t,y,\psi) \in [0,T] \times X \times X, \ \rho \in [0,1], \end{cases}$$

Theorem 3.2. If (b, g, h) admits a Lyapunov operator, then the FBEE is well-posed.

The idea behind: Consider linear FBEE:

$$(2.1)\quad \begin{cases} \dot{y}(t) = Ay(t) + B_{11}(t)y(t) + B_{12}(t)\psi(t) + b_0(t), \\ \dot{\psi}(t) = -A^*\psi(t) - B_{21}(t)y(t) - B_{22}(t)\psi(t) - g_0(t), \\ y(0) = x \qquad \psi(T) = Hy(T) + h_0, \end{cases}$$

with $B_{ij} : [0, T] \to \mathcal{L}(X)$. Let

$$\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}, \quad \mathbb{B}(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ -B_{21}(t) & -B_{22}(t) \end{pmatrix}.$$

Then the above FBEE can be written as

$$\begin{cases} \begin{pmatrix} \dot{y}(t) \\ \dot{\psi}(t) \end{pmatrix} = \big[\mathbb{A} + \mathbb{B}(t)\big] \begin{pmatrix} y(t) \\ \psi(t) \end{pmatrix} + \begin{pmatrix} b_0(t) \\ -g_0(t) \end{pmatrix}, \quad t \in [0, T], \\ y(0) = x, \qquad \psi(T) = Hy(T) + h_0. \end{cases}$$

Proposition 3.3. Let $(y(\cdot), \psi(\cdot))$ be a mild solution of linear FBEE and $\Pi(\cdot)$ be a mild solution of **Lyapunov equation**. Then

$$\langle \Pi(T) \begin{pmatrix} y(T) \\ \psi(T) \end{pmatrix}, \begin{pmatrix} y(T) \\ \psi(T) \end{pmatrix} \rangle - \langle \Pi(0) \begin{pmatrix} y(0) \\ \psi(0) \end{pmatrix}, \begin{pmatrix} y(0) \\ \psi(0) \end{pmatrix} \rangle$$
$$= \int_0^T \Big[\langle \Big(\Pi[\mathbb{B} + \mathbb{M}] + [\mathbb{B} + \mathbb{M}]^* \Pi - \mathbb{Q} \Big) \begin{pmatrix} y \\ \psi \end{pmatrix}, \begin{pmatrix} y \\ \psi \end{pmatrix} \rangle$$
$$+ 2 \langle \Pi \begin{pmatrix} b_0 \\ -g_0 \end{pmatrix}, \begin{pmatrix} y \\ \psi \end{pmatrix} \rangle \Big] dt.$$

When $\Pi(\cdot)$ is a **Lyapunov operator**, we will have

$$\text{Left} \geq \mu \|Hy(T)\|^2 - K\big(\|x\| + h_0\|\big).$$

$$\text{Right} \leq K \int_0^T \Big(\|b_0(s)\|^2 + \|g_0(s)\|^2\Big) ds.$$

This will lead to a **uniform a priori estimate**.

Questions:

- **When a Lyapunov operator exists?**
- **Generators from Lyapunov operators?**

Corollary 3.4. *Let* $p_1, \bar{p}_0, q_0, \bar{q}_0, \delta, \bar{\delta}, \varepsilon > 0$*,* $\gamma \in \mathbb{R}$ *such that*

$$p_1 + \gamma[h_y(y) + h_y(y)^*] - (\bar{p}_0 + \bar{q}_0 T) h_y(y)^* h_y(y) \geq \delta,$$

and for $t \in [0, T]$*,*

$$\begin{pmatrix} q_0[1 + 2\sigma_0(T - t)] + 2\sigma_0 p_1 & \theta \\ \theta & \bar{q}_0[1 + 2\sigma_0 t] + 2\sigma_0 \bar{p}_0 \end{pmatrix} \geq \bar{\delta} + \varepsilon$$

Then the FBEE is well-posed if $A^* = -A$*, and*

$$\begin{pmatrix} [p_1 + q_0(T - t)]I & [\gamma + \theta(T - t)]I \\ [\gamma + \theta(T - t)]I & -[\bar{p}_0 + \bar{q}_0 t]I \end{pmatrix} \mathbb{B}(t, y, \psi)$$

$$+\mathbb{B}(t, y, \psi)^* \begin{pmatrix} [p_1 + q_0(T - t)]I & [\gamma + \theta(T - t)]I \\ [\gamma + \theta(T - t)]I & -[\bar{p}_0 + \bar{q}_0 t]I \end{pmatrix} \leq \varepsilon.$$

Recall

$$\mathbb{B}(t, y, \psi) = \begin{pmatrix} b_y(t, y, \psi) & b_\psi(t, y, \psi) \\ -g_y(t, y, \psi) & -g_\psi(t, y, \psi) \end{pmatrix}.$$

Corollary 3.5. Suppose

$$h_y(y) + h_y(y)^* \geq 0, \qquad y \in X,$$

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \mathbb{B}(t, y, \psi) + \mathbb{B}(t, y, \psi)^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \leq -\delta,$$

$$\forall (t, y, \psi) \in [0, T] \times X \times X,$$

for some $\delta > 0$. Then the corresponding FBEE is well-posed.

The above is equivalent to the **uniform monotonicity** of the map

$$\begin{pmatrix} y \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} g(t, y, \psi) \\ -b(t, y, \psi) \end{pmatrix},$$

in the sense that for some $\delta > 0$,

$$\left\langle \begin{pmatrix} g(t, y, \psi) - g(t, \bar{y}, \bar{\psi}) \\ -b(t, y, \psi) + b(t, \bar{y}, \bar{\psi}) \end{pmatrix}, \begin{pmatrix} y - \bar{y} \\ \psi - \bar{\psi} \end{pmatrix} \right\rangle \geq \delta\big(\|y - \bar{y}\|^2 + \|\psi - \bar{\psi}\|^2\big),$$

$$\forall t \in [0, T],\ y, \bar{y}, \psi, \bar{\psi} \in X.$$

4. An Illustrative Example

Example 4.1. Consider the controlled state equation:

$$\begin{cases} y_t = \Delta y - (\lambda + u)y + f, & \text{in } (0,T)\times\Omega, \\ y\big|_{\partial\Omega} = 0, \\ y(0,x) = y_0(x), & x \in \Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ is bounded with $\partial\Omega$ smooth. The cost functional is:

$$J(u(\cdot)) = \frac{1}{2}\int_0^T \int_\Omega \Big(L|y - y_d|^2 + Nu^2\Big)dxdt \\ + \frac{1}{2}\int_\Omega M|y(T,x) - z(x)|^2 dx.$$

Assume that

$$\begin{cases} f(t,x) \geq 0, \quad y_d(t,x) \leq 0, & (t,x) \in (0,T)\times\Omega, \\ y_0(x) \geq 0, \quad z(x) \leq 0, & x \in \Omega. \end{cases}$$

The optimality system reads:

$$\begin{cases} y_t = \Delta y - \lambda y - \dfrac{1}{N}\psi y^2 + f, & \text{in } (0,T)\times\Omega, \\ \psi_t = -\Delta\psi + \lambda\psi + \dfrac{1}{N} y\psi^2 - L(y - y_d), & \text{in } (0,T)\times\Omega, \\ y\big|_{\partial\Omega} = \psi\big|_{\partial\Omega} = 0, & \\ y(0,x) = y_0(x), \quad \psi(T,x) = M\big(y(T,x) - z(x)\big), & x \in \Omega, \end{cases}$$

Then

$$\begin{cases} b(s,y,\psi) = -\lambda y - \dfrac{1}{N}\psi y^2 + f, \\ g(s,y,\psi) = -\lambda\psi - \dfrac{1}{N} y\psi^2 + L(y - y_d), \end{cases}$$

$$\begin{cases} b_y = -\lambda - \dfrac{2}{N} y\psi, & b_\psi = -\dfrac{1}{N} y^2, \\ g_y = L - \dfrac{1}{N}\psi^2, & g_\psi = -\lambda - \dfrac{2}{N} y\psi = b_y. \end{cases}$$

Hence,

$$\mathbb{B}(t,y,\psi) = \begin{pmatrix} -\lambda - \frac{2}{N}y\psi & -\frac{1}{N}y^2 \\ -L + \frac{1}{N}\psi^2 & \lambda + \frac{2}{N}y\psi \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \mathbb{B}(t,y,\psi) + \mathbb{B}(t,y,\psi) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = 2 \begin{pmatrix} -L + \frac{1}{N}\psi^2 & 0 \\ 0 & -\frac{1}{N}y^2 \end{pmatrix} \leq 0,$$

provided ψ is bounded (which can be proved) and N is large enough.

5. Open Questions

- Decoupling approach for nonlinear cases.

(Involving studying semilinear first order PDEs in infinite-dimensional spaces)

- Method of continuation:

(i)The case beyond $A^* = A$ and $A^* = -A$.

(ii) The case that b, g containing unbounded operators.

Thank You!