Passive LTI systems with a time-varying perturbation

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Summary.

We study a time-varying well-posed system resulting from the additive perturbation of the generator of a time-invariant well-posed system. The associated generator family has the form $A + G(t)$, where $G(t)$ is a bounded operator on the state space and $G(\cdot)$ is strongly continuous. We show that the resulting time-varying system (the perturbed system) is well-posed.
In the particular case when the unperturbed system is scattering passive, we derive an energy balance inequality for the perturbed system. If the operators $G(t)$ are dissipative, then the perturbed system is again scattering passive. We illustrate this theory by using it to formulate the system corresponding to a conductor moving in an electromagnetic field described by Maxwell's equations.

Assumptions

Let U, X and Y be Hilbert spaces. Assume that the operators A, B, $\overline{C}$ and D determine a well-posed linear system Σ (time-invariant) with input space U, state space X and output space Y, via the state equations

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad 0 \leq t < \infty, \\ y(t) = \overline{C}x(t) + Du(t). \end{cases} \tag{1}$$

In particular, this implies that A is the generator of a strongly continuous semigroup of operators $\mathbb{T}$ on X. We refer, for instance, to the book of Staffans on well-posed systems for the background on such systems. In this research we investigate the time-varying linear system corresponding to the perturbed state equations

$$\begin{cases} \dot{x}(t) = [A + G(t)]x(t) + Bu(t), \quad 0 \leq t < \infty, \\ y(t) = \overline{C}x(t) + Du(t). \end{cases} \tag{2}$$

Assumptions - continued

Here, $G : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous, that is, $t \to G(t)x$ is continuous on $[0, \infty)$ for each $x \in X$. (We may also work on a closed time interval $J \subset [0, \infty)$.)

We need the concept of an evolution family, which is the natural generalization of the concept of a strongly continuous semigroup, motivated by Cauchy problems of the type

$$\dot{x}(t) = A(t)x(t), \quad t \in J, \quad t \geq s, \quad x(s) = x_0, \qquad (3)$$

where $A(\cdot)$ is a family of operators on X, usually unbounded, defined on the closed time interval J. The theory of evolution families has received much attention since it was developed by T. Kato in the 1970s. We refer to the monographs Engel and Nagel (2000), Pazy (1983) or Tanabe (1979) for more background.

Evolution families

Definition. Let J be a closed interval and let $\mathbb{U} = \{\mathbb{U}(t,s) \in \mathcal{L}(X) : s,\ t \in J \text{ and } s \leq t\}$ be a family of operators. $\mathbb{U}$ is called an *evolution family* on X with time interval J if:

(i) $\mathbb{U}(t,t) = I$ for all $t \in J$.

(ii) $\mathbb{U}(t,s) = \mathbb{U}(t,\tau)\mathbb{U}(\tau,s)$ for all s, τ, $t \in J$ with $s \leq \tau \leq t$.

(iii) For each $x \in X$ and s, $t \in J$ with $s \leq t$, the function $(t,s) \to \mathbb{U}(t,s)x$ is continuous.

Exponentially bounded evolution families

An evolution family $\mathbb{U}$ is called *exponentially bounded* if there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|\mathbb{U}(t,s)\| \leq M e^{\omega(t-s)} \qquad \forall\, s,\, t \in J, \quad s \leq t\,. \tag{4}$$

We need an extrapolation space denoted by X_{-1}: this is the completion of X with repect to the norm

$$\|x\|_{-1} = \|(\beta I - A)^{-1} x\|\,,$$

where β is some fixed number in $\rho(A)$. We may think of this space as the dual of $\mathcal{D}(A^*)$, with repect to the pivot space X.

Some perturbation theory

In the paper related to this talk we prove that the differential equation in (2) has a solution in X_{-1}, for any initial state $x(s) \in X$ and every input function $u \in L^2_{\mathrm{loc}}([0,\infty);U)$. In fact, the restriction of the state trajectory $x(\cdot)$ to an interval $[s,t] \subset [0,\infty)$ depends only on the restriction of u to the interval $[s,t]$, which is denoted by $\mathbf{P}(t,s)u$ and (obviously) is an element of $L^2([s,t];U)$. We mention that this result is new even for $u=0$. Indeed, it is known (see Engel and Nagel (2000)) that there is an evolution family associated with $A+G(\cdot)$ in the sense of an integral equation, but to our knowledge, it was not known that the state trajectories of this evolution family actually satisfy the equation $\dot{x}(t) = [A+G(t)]x(t)$ in the space X_{-1}.

Well-posed time-varying systems

The concept of well-posedness has been generalized to time-varying systems in several references, reaching various degrees of generality. We adopt the definition in the MCSS paper by Schnaubelt and Weiss (2010):

Definition. A *time-varying well-posed system* Σ^v on a time interval $J \subset \mathbb{R}$ with input space U, state space X and output space Y is a quadruple of operator families

$$\mathbb{U}(t,s) : X \to X, \quad \Phi(t,s) : L^2(J;U) \to X, \tag{5}$$

$$\Psi(t,s) : X \to L^2(J;Y), \quad \mathbb{F}(t,s) : L^2(J;U) \to L^2(J;Y) \tag{6}$$

indexed by s, $t \in J$ with $s \leq t$ which satisfy:

(i) $\mathbb{U}$ is an evolution family on X with time interval J.

(ii) $\mathbb{U}(t,s)$, $\Phi(t,s)$, $\Psi(t,s)$, $\mathbb{F}(t,s)$ are locally uniformly bounded.

Well-posed time-varying systems - continued

(iii) The causality conditions: for all s, $t \in J$ with $s \leq t$,

$$\Phi(t,s) = \Phi(t,s)\mathbf{P}(t,s), \quad \Psi(t,s) = \mathbf{P}(t,s)\Psi(t,s) \tag{7}$$

$$\mathbb{F}(t,s) = \mathbf{P}(t,s)\mathbb{F}(t,s) = \mathbb{F}(t,s)\mathbf{P}(t,s) \tag{8}$$

(iv) The composition conditions: for all t, τ, $s \in J$ with $s \leq \tau \leq t$ and all $u \in L^2(J;U)$, $x_0 \in X$,

$$\Phi(t,s)u = \Phi(t,\tau)u + \mathbb{U}(t,\tau)\Phi(\tau,s)u, \tag{9}$$

$$\Psi(t,s)x_0 = \Psi(t,\tau)\mathbb{U}(\tau,s)x_0 + \Psi(\tau,s)x_0, \tag{10}$$

$$\mathbb{F}(t,s)u = \mathbb{F}(t,\tau)u + \mathbb{F}(\tau,s)u + \Psi(t,\tau)\Phi(\tau,s)u. \tag{11}$$

The first main result

Our first main result is that the equations (2) determine a time-varying well-posed system:

Theorem. Assume that:

(i) The time-invariant system (1) is well-posed.

(ii) The function $G : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous.

Then the time-varying system (2) is well-posed and its constituent evolution family $\mathbb{U}$ is the one associated with $A + G(\cdot)$.
If G is strongly continuously differentiable, then for every $t, s \in [0, \infty)$ with $t \geq s$, $\mathbb{U}(t, s)$ has an extension to an operator in $\mathcal{L}(X_{-1})$. For every such t, s we have

$$x(t) = \mathbb{U}(t, s)x(s) + \int_s^t \mathbb{U}(t, \tau)Bu(\tau)\,\mathrm{d}\tau\,,$$

for all $x(s) \in X$ and for all $u \in L^2([0, \infty); U)$.

Comments on the theorem

The proof of the above theorem is rather technical. It is given in the paper associated to this talk, that has just appeared in MCSS. The main idea of the proof is to use the Lax-Phillips semigroup associated to the system Σ, as introduced in Staffans and Weiss (2002), and to show that the perturbed system corresponds to a bounded (time-varying) perturbation of the generator of the Lax-Phillips semigroup. Similar ideas were used also in Schnaubelt and Weiss (2010) for two classes of time-varying multiplicative perturbations that lead to time-varying well-posed systems.

Scattering passive systems

It is of particular interest to consider the case when the system Σ is scattering passive with certain weight operators, as defined below.

Definition. Let Σ be a well-posed time-invariant system with state space X, input and output spaces $U = Y$. Let $P = P^* \in \mathcal{L}(X)$ and $R = R^* \in \mathcal{L}(U)$. Σ is called *scattering passive with respect to the storage operator P and the supply operator R* if for each pair

$$(x_0, u) \in X \times \mathcal{H}^1((0,\infty); U)$$

with $Ax_0 + Bu(0) \in X$, the corresponding classical solution $x(\cdot)$ of the differential quation in (1) satisfies, for all $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Px(t), x(t)\rangle \leq \langle Ru(t), u(t)\rangle - \langle Ry(t), y(t)\rangle \,. \qquad (12)$$

The system Σ is called *scattering energy preserving with respect to the storage operator P and the supply operator R* if we have equality in (12).

The second main result

The balance equation (12) is actually true for any input, state and output trajectory of the system (not only for classical solutions), if we rewrite it in the integral form (i.e., integrated on a finite time interval). Scattering passive systems often originate from a physical model where $\langle Px(t), x(t)\rangle$ is twice the energy stored in the system Σ, $\langle Ru(t), u(t)\rangle$ is twice the power flowing into Σ while $\langle Ry(t), y(t)\rangle$ is twice the power flowing out of Σ.

Our main result in this context is that if the unperturbed system is scattering passive in the sense of (12), then all the trajectories of the perturbed system (2) satisfy the power balance inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Px(t), x(t)\rangle \leq 2\mathrm{Re}\,\langle Px(t), G(t)x(t)\rangle + \langle Ru(t), u(t)\rangle - \langle Ry(t), y(t)\rangle,$$

which should again be understood in the integral sense. Equality holds above if Σ is scattering energy preserving with storage operator P and supply operator R.

Maxwell's equations with a moving conductor

As an example illustrating our previous results, we consider the electromagnetic field in and around a moving conductor. This model is based on the example in Secttion 5 of Weiss and Staffans (SICON, 2013), which describes Maxwell's equations (without moving objects) as a scattering passive linear system.

Let Ω be a domain filled with vacuum or air and consider a conducting rigid body occupying the domain $\Omega_1(t) \subset \Omega$ moving inside Ω with the velocity field $\mathbf{v}$. The electromagnetic field evolves according to *Maxwell's equations*:

$$\begin{cases} \frac{\partial \mathbf{B}}{\partial t} = -\operatorname{rot} \mathbf{E}, \\ \frac{\partial \mathbf{D}}{\partial t} = -\mathbf{J} + \operatorname{rot} \mathbf{H}, \\ \operatorname{div} \mathbf{B} = 0, \qquad \operatorname{div} \mathbf{D} = \rho. \end{cases} \tag{13}$$

The physical meaning of the vector fields $\mathbf{B}, \mathbf{D}, \mathbf{H}$ and $\mathbf{E}$ is standard, see for instance Jackson (2001), Orfanidis (2010), and it will not be explained here.

Maxwell's equations - continued

While the formulas (13) are exact, the *constitutive relations* shown below are only approximations of reality:

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}. \tag{14}$$

It is not easy to express the current density $\mathbf{J}$ in terms of the electric and magnetic fields, since this depends on a variety of physical effects at the microscopic scale, that must be averaged in order to obtain the macroscopic equation, see for instance Section 6.6 in the book of Jackson (2001). Any such formula is only an approximation of reality. We adopt the following formula:

$$\mathbf{J} = g\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) + \rho \mathbf{v}. \tag{15}$$

The reasoning behind this is the following: According to *Ohm's law*, in a non-moving environment, $\mathbf{J} = g\mathbf{E}$, where the scalar function $g \geq 0$ is called the *conductivity*.

Maxwell's equations - change of reference frame

If we regard the electromagnetic field, the charge density and the current density in a reference frame attached to the moving rigid body (the conductor), then the electric field at a certain position and time is

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B},$$

where $\mathbf{E}$ and $\mathbf{B}$ are expressed in the fixed reference frame, at the same position and time, see for instance p. 211 in Jackson (2001). (This is the non-relativistic formula, valid for $\|\mathbf{v}\| \ll c$.) Thus, applying Ohm's law in the moving reference frame, we get the following current in the moving reference frame:

$$\mathbf{J}' = g\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

Coming back to the fixed reference frame, using that $\mathbf{J} = \mathbf{J}' + \rho\mathbf{v}$, we obtain (15). (Note that $\rho\mathbf{v}$ is the current due to charges moving together with the rigid body.)

Maxwell's equations - an approximation

There are many electro-mechanical systems where $\rho\mathbf{v}$ is very small compared to the other two terms on the right side of (15). For example, in an induction motor, we estimate that the term $\rho\mathbf{v}$ is about 10^{10} times less than the other terms. Of course, there may be other systems where the term $\rho\mathbf{v}$ is significant. We shall analyze the system with the term $\rho\mathbf{v}$ removed from (15). Thus, the formula for $\mathbf{J}$ in our approximate model is

$$\mathbf{J} = g\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

Maxwell's equations - the moving conductor

The rigid conductor moves with the velocity field

$$\mathbf{v}(t, \mathbf{x}(t)) = \mathbf{v}_0(t) + \omega(t) \times [\mathbf{x}(t) - \mathbf{x}_0(t)] \,, \tag{16}$$

where $\mathbf{x}_0(t)$ is a fixed point in the moving reference frame, moving with velocity $\mathbf{v}_0(t)$, and $\omega(t)$ is the instantaneous angular velocity vector. The functions $\mathbf{v}_0$ and ω are of class C^1. Consider the (non-linear) transport evolution family $T(t, \tau)$ on $\mathbb{R}^3$ induced by the velocity field $\mathbf{v}$ via the differential equation

$$\dot{\xi}(t) = \mathbf{v}(t, \xi(t))$$

whose solution on an interval $[\tau, t]$ is $\xi(t) = T(t, \tau)\xi(\tau)$. Then $\Omega_1(t) = T(t, 0)\Omega_1(0)$. The rigid body in the region $\Omega_1(t)$ has the same μ and ε as the surrounding vacuum. The conductivity of the rigid body is the constant $g_1 > 0$. Hence, $g(t, \mathbf{x}) = g_1$ for $\mathbf{x} \in \Omega_1(t)$, and $g(t, \mathbf{x}) = 0$ otherwise.

Maxwell's equations - boundary conditions

We assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary Γ which is composed of two relatively open sets Γ_0 and Γ_1 satisfying

$$\Gamma_0 \cap \Gamma_1 = \emptyset, \quad \overline{\Gamma_0} \cup \overline{\Gamma_1} = \Gamma.$$

The surface measures of the boundaries $\partial\Gamma_0$ and $\partial\Gamma_1$ are zero.

The surface of Γ_0 is assumed to be superconductive. Denote by $\pi_\tau \mathbf{E}$ the tangential component trace of $\mathbf{E}$ on Γ. Then we have one boundary condition $\pi_\tau \mathbf{E} = 0$ on Γ_0. Let r be a scalar function on the active boundary Γ_1 such that $r > 0$, r, $r^{-1} \in L^\infty(\Gamma_1)$. The input and output of our system satisfy

$$u = \frac{1}{\sqrt{2}}\left(r(\nu \times \mathbf{H}) + \pi_\tau \mathbf{E}\right), \quad \text{on} \ \ \Gamma_1, \tag{17}$$

$$y = \frac{1}{\sqrt{2}}\left(r(\nu \times \mathbf{H}) - \pi_\tau \mathbf{E}\right), \quad \text{on} \ \ \Gamma_1. \tag{18}$$

Here, $\nu \in L^\infty(\Gamma, \mathbb{R}^3)$ represents the unit outward normal vector field on Γ.

Maxwell's equations - formulation as a passive system

Using the theory from the first section, the above system can be formulated as a time-varying well-posed system with the state space $X = L^2(\Omega;\mathbb{R}^3) \oplus L^2(\Omega,\mathbb{R}^3)$, the state $\begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix}$ and the input and output spaces

$$U = Y = \left\{ u \in L^2(\Gamma_1;\mathbb{R}^3) \mid u \cdot \nu = 0 \right\} .$$

For the proof see the recent MCSS paper by Chen and Weiss. We note that $\langle \mathbf{H}, \mathbf{B} \rangle + \langle \mathbf{E}, \mathbf{D} \rangle$ is twice the physical energy.

This system becomes passive if we add a mechanical port to it, representing forces and torques acting on the moving conductor on one hand, and velocities and angular velocities, on the other hand.