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ANR Project HAMECMOCPSYS on "Hamiltonian Methods for the Control of Multidomain Distributed Parameter Systems"

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Beijing, China.
General ideas on port-Hamiltonian systems (pHs)

1. strongly structured mathematical dynamical systems: both linear and non-linear, both finite-dimensional and infinite-dimensional,
2. based on physical grounds, allowing for different modelling levels,
3. all physics permitted: solid mechanics, structural mechanics, fluid mechanics, electromagnetism, electrical circuits, ...
4. comes along with specific numerical methods, which do preserve, at the discrete level, the structure of the continuous equations,
5. allows for open dynamical systems, with interacting ports,
6. modularity: interconnection of sub-systems, and... easy multiphysics modelling, e.g. Fluid-Structure Interaction,
7. physically-based strategy for control and stabilization,
8. extensions to dissipative dynamical systems are available.
1. Infinite-dimensional case: Partial Differential Equations (PDEs)
   - General framework
   - Closed systems
   - More examples in fluid mechanics!
   - A short word on dissipation? Navier-Stokes, at last!

2. Sloshing? A typical Fluid-Structure Interaction in Aeronautics!
   - Fluid-structure excited by piezoelectric actuators
   - Discretization and Preliminary Results
   - Limitations and Further Works
Outline

1. Infinite-dimensional case: Partial Differential Equations (PDEs)
   - General framework
   - Closed systems
   - More examples in fluid mechanics!
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2. Sloshing? a typical Fluid-Structure Interaction in Aeronautics!
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Useful Notions in Infinite Dimension

These notions will be of major help in the following:

- **functions** $u$, instead of vectors $X$,
- an infinite-dimensional **Hilbert functional space** $\mathcal{H}$ for functions, instead of a finite-dimensional Euclian vector space $\mathbb{R}^{2n}$,
- a Hamiltonian **functional** $H$, defined on functions $u$, instead of a Hamiltonian function defined on vectors, e.g.:

$$H : \mathcal{H} \rightarrow \mathbb{R}$$

$$u \mapsto H(u) := \frac{1}{2} \int_{0}^{L} u(z)^2 \, dz.$$
Useful Tools in Infinite Dimension

These tools will be of major help in the following:

- **the variational derivative** of a functional: \( \delta_x H \), in place of the gradient of the function, defined by

\[
H(u + \varepsilon w) = H(u) + \varepsilon (\delta_u H, w)_\mathcal{H} + O(\varepsilon^2)
\]

**N.B.** in the above easy example, \( \delta_u H = u \).

- formally **skew symmetric operators**: \( \mathcal{J}^T = -\mathcal{J} \), w.r.t the scalar product in the Hilbert space \( \mathcal{H} \), i.e.

\[
(u, \mathcal{J} v)_\mathcal{H} = -(\mathcal{J} u, v)_\mathcal{H}
\]

- in the special case of **quadratic Hamiltonian**, hence **linear dynamical systems** \( \dot{X} = AX \): **semigroups**
Definitions for (linear) pHs (1)

Consider the dynamical system

\[
\frac{d}{dt}X(z, t) = \mathcal{J} \delta_X \mathcal{H}(X)
\]

(1)

with (quadratic) Hamiltonian functional :

\[
\mathcal{H}(X) = \frac{1}{2} \int_0^L X(z, t)^T \mathcal{L} X(z, t) \, dz ,
\]

and (linear) variational derivative :

\[
\delta_X \mathcal{H}(X) = \mathcal{L} X(z, t) .
\]

We suppose that operator \( \mathcal{J} \) is formally skew-symmetric.

Indeed in the linear case, we get back to semigroups :

\[
\frac{d}{dt} X(z, t) = (\mathcal{J} \mathcal{L}) X(z, t).
\]
Definitions for pHs (2)

The energy balance associated to this system is:

\[ \frac{d\mathcal{H}}{dt}(t) = \int_0^L \delta_X \mathcal{H}(X) \cdot \frac{dX}{dt} \, dz, \]

\[ = \int_0^L \delta_X \mathcal{H}(X) \cdot \mathcal{J} \cdot \delta_X \mathcal{H}(X) \, dz, \]

\[ = 0, \quad \text{since } \mathcal{J} \text{ is skew-adjoint? Almost!} \]

Be careful, \( \mathcal{J} \) is skew-adjoint, but only... formally.

\[ \Rightarrow \text{Energy flows through the boundary, only!} \]

There is no internal dissipation of any kind in this case.

\[ \Rightarrow \text{Let us compute things more concretely on two examples:} \]

1. Webster horn equation (linear, space varying coefficients)
2. Compressible Euler equation (non-linear, constant coefficients)
Ex 1 : Webster horn equations (1)

Consider an axi-symmetric horn with varying cross-section \( z \mapsto S(z) \).

- **energy variables**: density \( \rho \), and particle velocity \( v \),
- with pressure \( p := c_0^2 \rho \), and energy density \( U(\rho) := \frac{c_0^2}{2 \rho_0^2} \rho^2 \),

Hamiltonian \( H(\rho, v) := \int_0^L \left( \frac{1}{2} \rho_0 v^2 + \rho_0 U(\rho) \right) S(z) \, dz \),

- **co-energy variables**: \( \delta_\rho H = \rho_0 U'(\rho) = \frac{1}{\rho_0} p \) and \( \delta_v H = \rho_0 v \),
- the **dynamical system** reads:

\[
\begin{pmatrix}
\frac{d}{dt} \rho \\
\frac{d}{dt} v
\end{pmatrix} =
\begin{bmatrix}
0 & -\frac{1}{S} \partial_z (S \cdot) \\
-\partial_z & 0
\end{bmatrix}
\begin{bmatrix}
\delta_\rho H \\
\delta_v H
\end{bmatrix}.
\]

(2)

**Proposition**

*Operator \( \mathcal{J} \) is formally skew-symmetric, w.r.t the scalar product*

\[
(e, f)_\mathcal{H} := \int_0^L (e_1 f_1 + e_2 f_2) S(z) \, dz.
\]
Ex 1 : Webster horn equations (2)

Indeed, for the above horn equation, we can compute:

\[
\frac{d\mathcal{H}}{dt}(t) = \int_0^L (e_1 f_1 + e_2 f_2) S(z) \, dz, \quad \text{with } f = J e,
\]

\[
= \int_0^L (-e_1 \partial_z (Se_2) - Se_2 \partial_z e_1) \, dz,
\]

\[
= \int_0^L -\partial_z (e_1 Se_2) \, dz,
\]

\[
= S(0)e_1(0)e_2(0) - S(L)e_1(L)e_2(L),
\]

\[
= p(0).S(0)v(0) - p(L).S(L)v(L).
\]

\[
\Rightarrow \text{Energy flows through the boundary, only!}
\]

Here \( u := S v \) is volume velocity, and \( p.u \) is meaningful.
Ex 2 : compressible Euler equations (1)

Consider an inviscid, irrotational and isentropic fluid, in $\Omega \subset \mathbb{R}^3$:

\[
\begin{align*}
\frac{d}{dt} \rho &= -\text{div}(\rho \mathbf{v}) \tag{3} \\
\frac{d}{dt} \mathbf{v} &= -(\mathbf{v} \cdot \text{grad}) \mathbf{v} - \frac{1}{\rho} \text{grad} p. \tag{4}
\end{align*}
\]

Following e.g. [van der Schaft & Maschke, 2002],

- **Energy variables**: $\rho$, $\mathbf{v}$,
- **Hamiltonian**: $H(\rho, \mathbf{v}) := \int_\Omega \left( \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + \rho U(\rho) \right) \, dV$,
- **Co-energy variables**: $\delta_\rho H = \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + h(\rho)$ and $\delta_\mathbf{v} H = \rho \mathbf{v}$,
- **Dynamical system** in standard form:

\[
\frac{d}{dt} \begin{bmatrix} \rho \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 & -\text{div} \\ -\text{grad} & 0 \end{bmatrix} \begin{bmatrix} \delta_\rho H \\ \delta_\mathbf{v} H \end{bmatrix}.
\]
Ex 2: compressible Euler equations (2)

Proposition

Operator $\mathcal{J}$ is formally skew-symmetric.

$\Rightarrow$ hence, energy flows through the boundary $\partial \Omega$, only!

$$\frac{dH}{dt}(t) = \int_{\Omega} (e_1 f_1 + e_2 f_2) \, dV,$$

$$= \int_{\Omega} (-e_1 \text{div}(e_2) - e_2 \text{grad}(e_1)) \, dV,$$

$$= \int_{\Omega} -\text{div}(e_1 e_2) \, dV,$$

$$= -\int_{\partial \Omega} e_1 e_2 \cdot n \, dS,$$

$$= -\int_{\partial \Omega} \left( \frac{1}{2} \rho |v|^2 + \rho h(\rho) \right) v \cdot n \, dS,$$

$$= -\int_{\partial \Omega} p v \cdot n \, dS,$$

thanks to $p(\rho) := \rho^2 U'(\rho)$. 
More involved examples in fluid mechanics

Some more specific Hamiltonian models for fluid mechanics are available for:

- **potential flow** (irrotational fluid), with the use of the velocity potential (classical and quite easy, but still...)
- **incompressible fluid**, (much more difficult: an infinite-dimensional Differential Algebraic Equation!), with the constraint $\text{div}(v) = 0$, make use of the vorticity vector $\omega := \text{curl}(v)$, together with the stream function $\psi$: see e.g. example 7.10 in [Olver, 1993].
- compressible and rotational flow: see a recent paper by [A. Castro and D. Lannes, 2015]

⇒ Many models for fluid dynamics can also be found in the seminal review paper by [Morrison, 1998].
Definitions for pHs (1)

Consider the dynamical system

\[ \frac{d}{dt}X(z, t) = (\mathcal{J} - \mathcal{R}) \delta_X \mathcal{H}_0(X) \]  

(5)

with (quadratic) Hamiltonian:

\[ \mathcal{H}_0(X) = \frac{1}{2} \int_0^L X(z, t)^T \mathcal{L}X(z, t) \, dz, \]

and (linear) variational derivative:

\[ \delta_X \mathcal{H}_0(X) = \mathcal{L}X(z, t). \]

We suppose that

- operator \( \mathcal{J} \) is formally skew-symmetric,
- operator \( \mathcal{R} \) is positive self-adjoint.
Definitions for pHs (2)

The energy balance associated to this system is:

\[
\frac{d\mathcal{H}_0}{dt}(t) = \int_0^L \delta_X \mathcal{H}_0(X) \frac{dX}{dt} \, dz,
\]

\[
= \int_0^L \delta_X \mathcal{H}_0(X) (\mathcal{J} - \mathcal{R}) \delta_X \mathcal{H}_0(X) \, dz,
\]

\[
= -\int_0^L \delta_X \mathcal{H}_0(X) \mathcal{R} \delta_X \mathcal{H}_0(X) \, dz,
\]

\[
\leq 0. \quad \text{up to the boundary terms!}
\]

But when \( \mathcal{R} \neq 0 \), no underlying Dirac structure is to be found for this damped system, with efforts \( e := \delta_X \mathcal{H}_0(X) \) and flows \( f := \frac{dX}{dt} \), linked by \( f = \mathcal{J} e \).
Ex 3 : compressible Navier-Stokes (1)

Consider an irrotational and isentropic fluid, in $\Omega \subset \mathbb{R}^3$:

$$
\frac{d}{dt} \rho = -\text{div}(\rho \mathbf{v}) \tag{6}
$$

$$
\frac{d}{dt} \mathbf{v} = -(\mathbf{v} \cdot \text{grad})\mathbf{v} - \frac{1}{\rho} \text{grad}p + \frac{1}{Re} \Delta \mathbf{v}. \tag{7}
$$

Still following [van der Schaft & Maschke, 2002],

- Energy variables: $\rho$, $\mathbf{v}$,
- Hamiltonian: $H := \int_{\Omega} \left( \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + \rho U(\rho) \right) dV$,
- Co-energy variables: $\delta_\rho H = \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + h(\rho)$ and $\delta_\mathbf{v} H = \rho \mathbf{v}$,
- Dynamical system in standard form:

$$
\frac{d}{dt} \begin{bmatrix} \rho \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 & -\text{div} \\ -\text{grad} & 0 \end{bmatrix} \begin{bmatrix} \delta_\rho H_0 \\ \delta_\mathbf{v} H_0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_\rho H \\ \delta_\mathbf{v} H \end{bmatrix};
$$
Ex 3 : compressible Navier-Stokes (2)

With \( C = -\frac{1}{Re} \Delta \). It has the desired \((J - R)\) form:

- \( J \) is a skew-symmetric operator, since the formal adjoint of \( \text{div} \) is \(-\text{grad}\),
- \( R \) is a symmetric and positive operator, since \(-\Delta\) is.

More important, the parametrization \( R = GS \) is very easily found to be:

\[
G := \begin{bmatrix} 0 \\ \text{grad} \end{bmatrix}, \quad G^* = \begin{bmatrix} 0 & -\text{div} \end{bmatrix}, \quad \text{and} \quad S := \frac{1}{Re} I.
\]
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Fluid-structure excited by piezoelectric actuators

- Very flexible plate
Fluid-structure excited by piezoelectric actuators

- Very flexible plate
- Partially filled tank

= fluid/structure interactions
Fluid-structure excited by piezoelectric actuators

Proposition:
- Use piezoelectric patches as actuators (or sensors)
- Reduce vibrations and liquid sloshing by feedback

Problems:
- How to model the experiment?
- How to design a *robust* control law?
The problem can be decomposed into several subsystems that exchange energy

Some reasons for using pHs:
- Multi-domain;
- Power-conserving;
- Object-oriented approach (modularity);
The beam can be written in pHs form (1/3)

**Euler-Bernoulli**

\[
\mu \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial z} \left( EI \frac{\partial^2 w}{\partial z^2} \right) = 0,
\]

boundary conditions:

- **Fixed end** \((z = 0)\): \(w(0, t) = 0\) and \(\frac{\partial w}{\partial z}(0, t) = 0\)
- **Free end** \((z = L)\):
  \(EI \frac{\partial^3 w}{\partial z^3} = \) tip force and
  \(EI \frac{\partial^2 w}{\partial z^2} = \) tip moment
The beam can be written in pHs form (2/3)

**Euler-Bernoulli**

\[
\mu \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial z^2} \left( EI \frac{\partial^2 w}{\partial z^2} \right) = 0,
\]

**Port-Hamiltonian version**

Defining: \( x_1 := \mu \frac{\partial w}{\partial t} \) and \( x_2 := \frac{\partial^2 w}{\partial z^2} \):

\[
\frac{\partial x_1}{\partial t} = - \frac{\partial^2}{\partial z^2} (EI x_2),
\]
\[
\frac{\partial x_2}{\partial t} = \frac{\partial^2}{\partial z^2} \left( \frac{x_1}{\mu} \right).
\]

Energy function: \( H = \frac{1}{2} \int_{z=0}^{L} \left( \frac{x_1^2}{\mu} + EI x_2^2 \right) dz \)

- **Kinetic energy**
- **Elastic energy**
The beam can be written in pHs form (2/3)

**Euler-Bernoulli**

\[ \mu \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial z} \left( EI \frac{\partial^2 w}{\partial z^2} \right) = 0, \]

**Port-Hamiltonian version**

Defining: \( x_1 := \mu \frac{\partial w}{\partial t} \) and \( x_2 := \frac{\partial^2 w}{\partial z^2} \):

\[ \frac{\partial x_1}{\partial t} = -\frac{\partial^2}{\partial z^2} (EIx_2), \quad \frac{\partial x_2}{\partial t} = \frac{\partial^2}{\partial z^2} \left( \frac{x_1}{\mu} \right). \]

\[ \left[ \begin{array}{c} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \end{array} \right] = \left[ \begin{array}{cc} 0 & -\frac{\partial^2}{\partial z^2} \\ \frac{\partial^2}{\partial z^2} & 0 \end{array} \right] \left[ \begin{array}{c} \frac{\delta H}{\delta x_1} \\ \frac{\delta H}{\delta x_2} \end{array} \right] - \left[ \begin{array}{c} f_c \\ 0 \end{array} \right]. \]

Energy function: \[ H = \frac{1}{2} \int_{z=0}^{L} \left( \frac{x_1^2}{\mu} + EIx_2^2 \right) dz \]

- Kinetic energy
- Elastic energy
The beam can be written in pHs form (3/3)

Port-Hamiltonian version

\[ x_1 := \mu \frac{\partial w}{\partial t} \quad \text{and} \quad x_2 := \frac{\partial^2 w}{\partial z^2} \]

\[
\begin{bmatrix}
\frac{\partial x_1}{\partial t} \\
\frac{\partial x_2}{\partial t}
\end{bmatrix} =
\begin{bmatrix}
0 & -\frac{\partial^2}{\partial z^2} \\
\frac{\partial^2}{\partial z^2} & 0
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}
\]

\[
-f_c \quad \mathcal{J} \quad e_c
\]

The energy flow becomes:

\[
\dot{H} = \int_{z=0}^{L} e_1 \dot{x}_1 + e_2 \dot{x}_2 \, dz = \int_{z=0}^{L} -e_1 \frac{\partial^2 e_2}{\partial z^2} + e_2 \frac{\partial^2 e_1}{\partial z^2} \, dz
\]

\[
= \int_{z=0}^{L} \frac{\partial e}{\partial z} \left( -e_1 \frac{\partial e_2}{\partial z} + e_2 \frac{\partial e_1}{\partial z} \right) \, dz = \left. \begin{bmatrix}
-e_1 \\
e_1 \\
e_2 \\
e_1
\end{bmatrix}
\right|_{z=0}^{L} \quad \text{moment} \quad \text{angular speed} \quad \text{speed} \quad \text{force}
\]
The fluid can be written in pHs form (1/3)

1D Saint-Venant

**Hypothesis**
- Incompressible fluid;
- Shallow water;
- Only hydrostatic pressure: \( P = \rho gh \)

**Equations**

\[
\begin{align*}
\frac{\partial h}{\partial t} &= - \frac{\partial}{\partial x} (hu) \\
\frac{\partial u}{\partial t} &= - u \frac{\partial u}{\partial x} - g \frac{\partial}{\partial x} h
\end{align*}
\]

- \( u \) : fluid speed;
- \( h \) : fluid height;
The fluid can be written in pHs form (2/3)

\[
\rho \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} \frac{1}{2} = -g \frac{\partial h}{\partial x} \quad \text{(Navier-Stokes)},
\]

\[
\frac{\partial h}{\partial t} = - \frac{\partial}{\partial x} (hu) \quad \text{(Mass conservation)}.
\]

with energy given by:

\[
H(u, h) = \frac{1}{2} \int_{x=-a/2}^{a/2} \left( \rho bu^2 h + \rho bgh^2 \right) dx.
\]

\[\text{kinetic energy} \quad \text{potential energy}\]
The fluid can be written in pHs form (2/3)

\[
\rho \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \frac{u^2}{2} = -g \frac{\partial h}{\partial x} \quad \text{(Navier-Stokes)},
\]

\[
\frac{\partial h}{\partial t} = - \frac{\partial}{\partial x} (hu) \quad \text{(Mass conservation)}.
\]

with energy given by:

\[
H(u, h) = \frac{1}{2} \int_{x=-a/2}^{a/2} \left( \rho bu^2 h + \rho bgh^2 \right) dx.
\]

\[
\frac{\partial h}{\partial t} = - \frac{\partial}{\partial x} (hu) = - \frac{\partial}{\partial x} \left( \frac{u^2}{2} + gh \right) = \frac{\partial}{\partial x} \left( \frac{1}{\rho b} \frac{\delta H}{\delta h} \right)
\]
The fluid can be written in pHs form (3/3)

**Saint-Venant 1D : written in PH form**

Defining: \( \alpha_1 := \rho u \) and \( \alpha_2 := bh \):

\[
\frac{\partial}{\partial t} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{bmatrix}
\]

\[
H(\alpha_1, \alpha_2) = \frac{1}{2} \int_{x=-a/2}^{a/2} \left( \frac{\alpha_1^2 \alpha_2}{\rho} + \rho g \left( \frac{\alpha_2^2}{b} \right) \right) \, dx.
\]

where:

\[
e_1^F = \frac{\delta H}{\delta \alpha_1} = bh u = \text{volumetric flow}, \quad e_2^F = \frac{\delta H}{\delta \alpha_2} = \rho \frac{u^2}{2} + \rho gh = \text{total pressure}
\]

\[
\frac{\partial H^F}{\partial t} = e_1^F(-a/2, t) e_2^F(-a/2, t) - e_1^F(a/2, t) e_2^F(a/2, t)
\]

vol. flow \quad pressure \quad vol. flow \quad pressure
The tank can be written in pHs form

2nd Newton Law: \[ m_{RB} \ddot{w}_B(t) = F_{\text{ext}}, \]

Port-Hamiltonian equations:

\[
\frac{\partial}{\partial t}\begin{bmatrix} p \\ w_B \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_p H^{RB} \\ \partial_{w_B} H^{RB} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{\text{ext}},
\]

\[
\dot{w}_B = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_p H^{RB} \\ \partial_{w_B} H^{RB} \end{bmatrix}
\]

where the Hamiltonian is equal to the kinetic energy: \[ H^{RB}(p) = \frac{1}{2} \frac{p^2}{m_{RB}}, \] and its rate of change is given by:

\[
\dot{H}^{RB} = \dot{w}_B \dot{p}
= \dot{w}_B F_{\text{ext}}.
\]
We can interconnect each component through their ports

Total Hamiltonian: \( H = H^B + H^F + H^{RB} \)

Kinematic constraint (equal speed):

\[
\dot{w}_B = e_2^B(L, t) = e_1^F(-a/2, t) / S = e_1^F(a/2, t) / S
\]

Sum of forces equal to zero:

\[
\mathcal{F} = -\frac{\partial}{\partial z} e_1^B(L, t) + F_{\text{ext}} e_2^F(a/2, t) S + e_2^F(-a/2, t) S = 0
\]

\[
\implies \dot{H} = \dot{H}^B + \dot{H}^F + \dot{H}^{RB} = \dot{w}_B \mathcal{F} = 0
\]
In order to simulate, we have to approximate infinite-dimensional equations

Spatial discretization methods:
- Mixed finite-elements: [Golo, 2003]
- Port-Hamiltonian structure is conserved after discretization;
- No numerical dissipation;
- Boundary conditions \( e_p \) appears as interconnection ports
- Pseudo-spectral spatial symplectic reduction: [Moulla, 2012]
Discretization methods lead to finite-dim. pHs

\[
\begin{bmatrix}
\frac{\partial x_1}{\partial t} \\
\frac{\partial x_2}{\partial t}
\end{bmatrix} = 
\begin{bmatrix}
0 & -\frac{\partial^2}{\partial z^2} \\
\frac{\partial^2}{\partial z^2} & 0
\end{bmatrix}
\begin{bmatrix}
\delta x_1 H \\
\delta x_2 H
\end{bmatrix}
\]

\[
\begin{aligned}
f_c &\quad \mathcal{J} &\quad e_c \\
\end{aligned}
\]

\[
\dot{x} = J \frac{\partial H^d}{\partial x} + Bu,
\]

\[
y = B^T \frac{\partial H^d}{\partial x} + Du.
\]

\[
u = \begin{bmatrix} e_B^1(L) & e_B^1(0) & e_B^2(0) & e_B^2(0) \end{bmatrix}
\]

\[
y = \begin{bmatrix} -e_B^2(L) & e_B^2(L) & -e_B^1(0) & e_B^1(0) \end{bmatrix}
\]

\[
\dot{H} = \dot{H}^d = y^T u
\]
Once we’ve discretized each subsystem, we find a set of DAE

**Final discrete finite-dimensional system**

\[
\dot{x}(t) = J_d \frac{\partial H_d}{\partial x} + Bu, \\
y^i = (B)^T \frac{\partial H_d}{\partial x} + Du,
\]

\[
u = \begin{bmatrix} e_1^B(L) & F_{ext}^{RB} & e_1^F(-a/2) & e_2^F(a/2) \end{bmatrix}^T,
\]

\[
y = \begin{bmatrix} e_2^B(L) & \dot{w}_{RB}^F & e_2^F(-a/2) & e_1^F(a/2) \end{bmatrix}^T.
\]

\[
\mathcal{M}y + \mathcal{N}u = 0,
\]

**Coupled system : Differential Algebraic Equations**

\[
\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} J_dQ & B \\ \mathcal{M}B^TQ & \mathcal{M}D + \mathcal{N} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}
\]

(8)
Time-discretization using energy-conserving methods can be used

Simulation of a beam equation, with initial conditions.
Time-discretization using energy-conserving methods can be used.

Simulation of a beam equation, with initial conditions: Energy vs time.

See e.g. [S. Aoues, 2014] and references therein.
Numerical results show good agreement with experiments

Natural frequencies (in Hz) - 25% filled tank

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>10</th>
<th>50</th>
<th>200</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slosh+bending</td>
<td>0.4318</td>
<td>0.4332</td>
<td>0.4332</td>
<td>0.46</td>
</tr>
<tr>
<td>Slosh+bending</td>
<td>1.1404</td>
<td>1.1436</td>
<td>1.1437</td>
<td>1.15</td>
</tr>
<tr>
<td>Slosh+bending</td>
<td>1.3690</td>
<td>1.4174</td>
<td>1.4194</td>
<td>1.50</td>
</tr>
<tr>
<td>Slosh+bending</td>
<td>2.0544</td>
<td>2.2770</td>
<td>2.2860</td>
<td>2.38</td>
</tr>
<tr>
<td>Slosh+bending</td>
<td>2.5799</td>
<td>3.1637</td>
<td>3.1880</td>
<td>2.94</td>
</tr>
<tr>
<td>1st Torsion</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>2nd bending</td>
<td>8.4273</td>
<td>8.4268</td>
<td>8.4267</td>
<td>8.01</td>
</tr>
<tr>
<td></td>
<td>8.8955</td>
<td>8.7469</td>
<td>8.7515</td>
<td>9.61</td>
</tr>
</tbody>
</table>
More information ? on the WEB !

On the following Website,

http://github.com/flavioluiz/port-hamiltonian

you will see the comparison between :

- the numerically obtained modes,
- the experimental results obtained by exciting the system at the frequencies of these specific modes.

The animated GIFs show the first 3 modal shapes, which represent coupled fluid+bending modes.

Moreover, the code can be used for a more general time-domain simulation : the beam starts from a deformed initial condition, and it is then released.
Some results of pH-based control

Damping assignment with state-observer:
Some results of pH-based control

Damping assignment with state-observer:

![Graph showing frequency response for different damping assignments]
Nonlinear sloshing appears with high amplitudes

Harmonic excitation near 8.7 Hz (2nd bending mode)
Nonlinear sloshing appears with high amplitudes

Harmonic excitation near 8.7 Hz (2nd bending mode)
Nonlinear sloshing appears with high amplitudes

Harmonic excitation near 8.7 Hz (2nd bending mode)
Nonlinear sloshing appears with high amplitudes

Harmonic excitation near 8.7 Hz (2nd bending mode)
Saint-Venant sloshing model works only for small filling ratios

\[ G(s) = \frac{F(s)}{\ddot{W}(s)} \]

Bode plot of:

- Incompressible Euler
- Saint–Venant

Frequency (Hz):

- Force (N)/Lateral acceleration (m/s²)
Saint-Venant sloshing model works only for small filling ratios

\[ G(s) = \frac{F(s)}{\ddot{W}(s)} \]

Bode plot of:

- Incompressible Euler
- Saint–Venant

\( F(s) \) is fluid force on walls
\( \ddot{W}(s) \) is tank acceleration

10% filled
Saint-Venant sloshing model works only for small filling ratios.
Summary

Port-Hamiltonian formulation provides a:
- modular,
- physically motivated (energy-based),
- multi-domain
framework for analyzing, simulating and controlling (complex) systems.

We used this formulation to:
- Model and simulate a fluid-structure interaction problem,
- Perform some control by damping injection.
What comes next?

Further work (short term):
- Model piezoelectric material;
- More accurate sloshing model;
- Control of the full system.

Further work (long term):
- Distributed fluid-structure problems;
- ...
- Energy-based control of aeroelastic systems?
Some general references


Some specific references


