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Stabilization of the pendulum system by coupling with a heat equation

Dong-Xia Zhao[1,2] and Jun-Min Wang[2]

Abstract

In this work, we study the stabilization of a pendulum system by coupling with a heat equation through the left boundary observation of the heat equation, while a velocity feedback of the pendulum system is designed to affect the same side heat flux of the heat equation. Based on the semigroup approach and Riesz basis method, the well-posedness and exponential stability of the system is deduced. Finally, some numerical simulations are presented to show the effectiveness of this feedback control design.

Keywords

Pendulum system, heat equation, spectrum, Riesz basis, exponential stability

1. Introduction

The control and dynamics of the pendulum system have been studied in the past decades, and many effective control methods have been proposed. In order to stabilize a damped-elastic-jointed inverted pendulum subjected to a time-periodic follower force, Bui et al. (2012) design an optimal fuzzy control using hedge algebras (OFCHA) controller, and compare the control effect of the three controllers, i.e. conventional fuzzy control (CFC), fuzzy control using hedge algebras (FCHA) and OFCHA. Suh and Bien (1979) introduced a proportional-minus-delay (PMD) controller for the stabilization of the pendulum system, which is very popular until now. Masoud et al. (2003), based on the concept of delayed position feedback, developed a control strategy to reduce pendulations of hoisted payloads on rotary cranes. The stability results of the pendulum system under a PMD controller have been demonstrated both numerically by Suh and Bien (1979, 1980) and through a detailed analysis by Atay (1998, 1999, 2002), Wang et al. (2011) and Zhao and Wang (2012).

Recently, Wang et al. (2012) used a collocated boundary feedback compensator in the form of a heat equation, a set-up that has attracted increasing attention, which was applied to stabilize a Schrödinger equation by a collocated input/output pair of the heat equation, and the exponentially stability and Gevrey regularity for the closed-loop system were obtained. In this paper, we shall use the heat-compensator design to stabilize the pendulum system, and give a detailed spectral analysis. The method adopted in this article can be further applied to the analysis of the coupled wave–pendulum system, where the wave equation with Kelvin–Voigt damping is considered as a compensator.

In the field of robotics and intelligent vehicles, a basic problem is controlling the position of a single-link rotational joint by using a motor placed at the pivot. Mathematically, it can be considered as a planar pendulum to which an external torque is applied, whose motion can be formulated as the following second-order nonlinear differential equation:

$$\ddot{\theta}(t) + \frac{g}{l}\sin\theta(t) = u(t), \tag{1}$$

where g is the acceleration due to gravity, θ the angular displacement measured from the natural rest position, ℓ

[1]Department of Mathematics, North University of China, Taiyuan, PR China
[2]Department of Mathematics, Beijing Institute of Technology, Beijing, PR China

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Corresponding author:
Dong-Xia Zhao, Department of Mathematics, North University of China, Taiyuan 030051, PR China.
Email: zhaodongxia6@sina.com

the length of the pendulum, and $u(t)$ the value of the external torque at time t. For simplicity, we assume that the friction is negligible, so that all of the mass of the rod is concentrated at the end. Then the stability of the two equilibria is determined by the linearized equation

$$\ddot{y}(t) \pm \frac{g}{\ell} y(t) = u(t),$$

where y denotes the deviation from the equilibrium, and the "+" and "−" signs correspond to the natural ($\theta = 0$) and the inverted ($\theta = \pi$) equilibrium positions, respectively. By the so-called PMD controller

$$u(t) = \hat{a}y(t) + \hat{b}y(t-\tau), \tag{2}$$

and rescaling the time $t \to \frac{t}{\tau}$ to normalize the delay to 1, the pendulum system with PMD controller (1) and (2) become

$$\ddot{y}(t) + ky(t) = by(t-1),$$

where $k = -\tau^2\hat{a} \pm \tau^2 g/l,\ b = \tau^2\hat{b}$. Since the time delay itself is a dynamical system, by a new variable $v(x,t) = y(t+x-1)$, the system can be written as follows:

$$\begin{cases} \ddot{y}(t) + ky(t) = bv(0,t), & t > 0,\ k, b \in \mathbb{R}, \\ v_t(x,t) = v_x(x,t), & x \in (0,1),\ t \geq 0, \\ v(1,t) = y(t), \end{cases} \tag{3}$$

which is a coupled system consisting of a transport partial differential equation (PDE) and a pendulum system. According to Atay (1999) or Wang et al. (2011), the system (3) is exponentially stable if k,b satisfy

$$\min\{k - n^2\pi^2,\ (n+1)^2\pi^2 - k\} > (-1)^n b > 0 \tag{4}$$

for some nonnegative integer n. On the other hand, Susto and Krstic (2010), Tang and Xie (2011), and Wang et al. (2012) applied the heat equation as a compensator to successfully stabilize an ordinary differential equation (ODE) system and a Schrödinger equation. Hence, with some inspiration, it is natural to raise a question of whether the heat equation can be regarded as a compensator to stabilize the pendulum system, so we study the following coupled heat–pendulum system (see Figure 1):

$$\begin{cases} \ddot{y}(t) + ky(t) = bv(0,t), & t > 0, \\ v_t(x,t) = v_{xx}(x,t), & x \in (0,1), t \geq 0 \\ v_x(0,t) = b\dot{y}(t), \\ v(1,t) = 0, \end{cases} \tag{5}$$

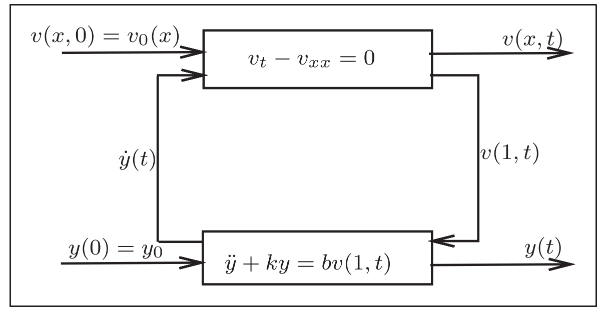

Figure 1. Block diagram for an interconnected heat–ODE system.

where the PDE in (3) is replaced by a heat equation, $v(x,t)$ is the temperature of a heat equation, and $k > 0$ and $b \neq 0$ are two unknown constants. The boundary observation of the heat equation enters the state equation of the ODE through the left side $v(0,t)$ in (5), the right side of the heat equation is kept at zero temperature. While, the derivative output $\dot{y}(t)$ of the ODE is fed into the boundary heat flux of the heat equation. In system (5), the heat equation is considered as a dynamic controller to stabilize the pendulum system with unknown constants. It will be shown (in Section 3) that no matter how much $k > 0$ and $b \neq 0$, system (5) will be exponentially stable under a heat PDE compensator. The aim of this work is to present the well-posedness and exponential stability of the system (5).

The paper is organized as follows. In Section 2, we formulate the system (5) into an abstract evolution equation and set up the well-posedness of the system. Section 3 is devoted to the asymptotic expressions of eigenvalues and eigenfunctions. We show that there is a sequence of generalized eigenfunctions which forms a Riesz basis for the state space, and then we establish the spectrum-determined growth condition and the exponential stability of (5). In Section 4, these analytic results are confirmed by numerical simulations.

2. Well-posedness of the system (5)

We consider system (5) in the energy space

$$\mathcal{H} = \mathbb{C} \times \mathbb{C} \times L^2(0,1) \tag{6}$$

equipped with the usual inner product:

$$\langle Z_1, Z_2 \rangle = k\langle f_1, f_2\rangle_{\mathbb{C}} + \langle g_1, g_2\rangle_{\mathbb{C}} + \int_0^1 h_1(x)\overline{h_2(x)}\mathrm{d}x, \tag{7}$$

where $Z_1 = (f_1, g_1, h_1), Z_2 = (f_2, g_2, h_2) \in \mathcal{H}$, and $\langle f_1, f_2\rangle_{\mathbb{C}} = f_1\overline{f_2}$.

Define a linear operator $\mathcal{A}$: $D(\mathcal{A})(\subseteq \mathcal{H}) \to \mathcal{H}$ by

$$\mathcal{A}(f, g, h) = (g, -kf + bh(0), h''), \tag{8}$$

with

$$D(\mathcal{A}) = \left\{ (f, g, h) \in \mathcal{H} \,\middle|\, \begin{array}{l} h \in H^2(0,1), \\ h'(0) = bg, \\ h(1) = 0 \end{array} \right\}. \tag{9}$$

Then (5) can be written as an evolution equation in $\mathcal{H}$:

$$\begin{cases} \dot{Z}(t) = \mathcal{A}Z(t), & t > 0, \\ Z(0) = Z_0, \end{cases} \tag{10}$$

where $Z(t) = (y(t), \dot{y}(t), v(\cdot, t))$.

Now we have the following result on the properties of $\mathcal{A}$.

Lemma 2.1 *Let $\mathcal{A}$ be given by (8) and (9). Then $\mathcal{A}^{-1}$ exists and is compact. Hence, $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues of finite algebraic multiplicity only. Moreover, $\mathcal{A}$ is dissipative in $\mathcal{H}$ and $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions on $\mathcal{H}$.*

Proof For any given $\tilde{Z} = (\tilde{f}, \tilde{g}, \tilde{h}) \in \mathcal{H}$, solve

$$\begin{aligned} \mathcal{A}(f, g, h) &= (g, -kf + bh(0), h'') \\ &= \left(\tilde{f}, \tilde{g}, \tilde{h}\right), \quad (f, g, h) \in D(\mathcal{A}) \end{aligned}$$

to get

$$\begin{cases} g = \tilde{f}, \\ -kf + bh(0) = \tilde{g}, \\ h'' = \tilde{h}, \\ h'(0) = bg, \\ h(1) = 0. \end{cases}$$

Then we have

$$\begin{cases} f = \dfrac{-1}{k}\left(\tilde{g} + b^2\tilde{f} + b\displaystyle\int_0^1 (1-s)\tilde{h}(s)\mathrm{d}s\right), \\ g = \tilde{f}, \\ h(x) = -b(1-x)\tilde{f} - (1-x)\displaystyle\int_0^x \tilde{h}(s)\mathrm{d}s - \int_x^1 (1-s)\tilde{h}(s)\mathrm{d}s. \end{cases} \tag{11}$$

Hence, $\mathcal{A}^{-1}$ exists and is compact on $\mathcal{H}$ by the Sobolev embedding theorem. Therefore, $\sigma(\mathcal{A})$ consists of isolated eigenvalues of finite algebraic multiplicity only.

Now we show that $\mathcal{A}$ is dissipative in $\mathcal{H}$. For each $Z = (f, g, h) \in D(\mathcal{A})$, we have

$$\begin{aligned} \langle \mathcal{A}Z, Z\rangle &= \big\langle \big(g, -kf + bh(0), h', (f, g, h)'\big)\big\rangle \\ &= k\langle g, f\rangle_{\mathbb{C}} + \langle -kf + bh(0), g\rangle_{\mathbb{C}} + \int_0^1 h''(x)\overline{h(x)}\mathrm{d}x, \\ &= k\langle g, f\rangle_{\mathbb{C}} - k\langle f, g\rangle_{\mathbb{C}} + b\langle h(0), g\rangle_{\mathbb{C}} + \overline{h(x)}h'(x)|_0^1 \\ &\quad - \int_0^1 |h'(x)|^2\mathrm{d}x \\ &= kg\bar{f} - kf\bar{g} + bh(0)\bar{g} - \overline{h(0)}bg - \int_0^1 |h'(x)|^2\mathrm{d}x, \end{aligned}$$

so,

$$\mathrm{Re}\langle \mathcal{A}Z, Z\rangle = -\int_0^1 |h'(x)|^2\mathrm{d}x \le 0. \tag{12}$$

Hence, $\mathcal{A}$ is dissipative in $\mathcal{H}$, and $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ of contractions on $\mathcal{H}$ by the Lumer–Philips theorem of Pazy (1983). The proof is complete. □

3. Stability of system (10)

In this section, we consider the eigenvalue problem of (10). By $\mathcal{A}Z = \lambda Z$, where $Z = (f, g, h) \in D(\mathcal{A})$, we have

$$\begin{cases} g = \lambda f, \\ -kf + bh(0) = \lambda g, \\ h'' = \lambda h, \\ h'(0) = bg, \\ h(1) = 0. \end{cases} \tag{13}$$

That is

$$\begin{cases} g = \lambda f, \\ h = c\sinh\sqrt{\lambda}(1-x), \\ -kf + b \cdot c\sinh\sqrt{\lambda} = \lambda^2 f, \\ -c \cdot \sqrt{\lambda}\cosh\sqrt{\lambda} = b\lambda f, \end{cases} \tag{14}$$

i.e.

$$\begin{cases} g = \lambda f, \\ h = c\sinh\sqrt{\lambda}(1-x), \\ f = \dfrac{bc\sinh\sqrt{\lambda}}{\lambda^2 + k}, \\ \left(b^2 \cdot \sqrt{\lambda}\sinh\sqrt{\lambda} + \left(\lambda^2 + k\right)\cosh\sqrt{\lambda}\right)c = 0. \end{cases} \tag{15}$$

Then (13) has a nontrivial solution if and only if

$$(\lambda^2 + k)\cosh\sqrt{\lambda} + b^2\sqrt{\lambda}\sinh\sqrt{\lambda} = 0$$

have solutions. Hence, we get the following lemma immediately.

Lemma 3.1 Let $\mathcal{A}$ be given by (8) and (9), and let

$$\Delta(\lambda) = (\lambda^2 + k)\cosh\sqrt{\lambda} + b^2\sqrt{\lambda}\sinh\sqrt{\lambda}. \tag{16}$$

Then

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} | \Delta(\lambda) = 0\}, \tag{17}$$

and each $\lambda \in \sigma(\mathcal{A})$ is geometrically simple.

Lemma 3.2 *Let $\mathcal{A}$ be given by (8) and (9). Then for each $\lambda \in \sigma(\mathcal{A})$, we have* $\mathrm{Re}\,\lambda < 0$.

Proof By Lemma 2.1, $\mathcal{A}$ is dissipative, then we have $\mathrm{Re}\,\lambda \leq 0$, $\forall\lambda \in \sigma(\mathcal{A})$. So we only need to show there is no eigenvalues on the imaginary axis. Let $\lambda = ia \in \sigma(A)$, $a \in \mathbb{R}$ and let $Z = (f, g, h) \in D(\mathcal{A})$ be its associated eigenfunction of $\mathcal{A}$. Then by (12), we have

$$\mathrm{Re}\langle \mathcal{A}Z, Z\rangle = -\int_0^1 |h'(x)|^2 \mathrm{d}x = 0, \tag{18}$$

and, hence, $h'(x) = 0$. By $h(1) = 0$, we have $h \equiv 0$. Moreover, it follows from (13) that $f = g = 0$. Therefore, there are no eigenvalues on the imaginary axis. The proof is complete. □

Proposition 3.1 *Let $\mathcal{A}$ be given by (8) and (9), and let $\Delta(\lambda)$ be given by (16). The eigenvalues of $\mathcal{A}$ have the following asymptotic expressions:*

$$\lambda_n = -\left(n - \frac{1}{2}\right)^2 \pi^2 + \mathcal{O}(n^{-2}), \quad n > N, \tag{19}$$

where N is a positive number. Therefore, $\mathrm{Re}\,\lambda_n \to -\infty$, as $n \to \infty$.

Proof By $\Delta(\lambda) = 0$, we get

$$(\lambda^2 + k)\left(e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}\right) + b^2\sqrt{\lambda}\left(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}\right) = 0,$$

or

$$\left(\lambda^2 + k + b^2\sqrt{\lambda}\right)e^{\sqrt{\lambda}} + \left(\lambda^2 + k - b^2\sqrt{\lambda}\right)e^{-\sqrt{\lambda}} = 0,$$

which yields

$$\begin{aligned} e^{2\sqrt{\lambda}} &= \frac{-\lambda^2 - k + b^2\sqrt{\lambda}}{\lambda^2 + k + b^2\sqrt{\lambda}} \\ &= -1 + \frac{2b^2}{\lambda^{\frac{3}{2}} + k\lambda^{-\frac{1}{2}} + b^2} \\ &= -1 + \mathcal{O}\left(\lambda^{-\frac{3}{2}}\right). \end{aligned} \tag{20}$$

A direct computation gives

$$\sqrt{\lambda_n} = i\left(n - \frac{1}{2}\right)\pi + i\mathcal{O}(n^{-3}), \quad n > N, \tag{21}$$

where N is a positive number. Therefore,

$$\lambda_n = -\left(n - \frac{1}{2}\right)^2 \pi^2 + \mathcal{O}(n^{-2}), \quad n > N, \tag{22}$$

which is (19). The proof is complete. □

Proposition 3.2 *Let $\{\lambda_n, n \in \mathbb{N}\}$ be the eigenvalues of $\mathcal{A}$ with λ_n being given by (19). Then the corresponding eigenfunctions $\{(f_n, g_n, h_n), n \in \mathbb{N}\}$ have the following asymptotic expressions:*

$$\begin{cases} f_n = \mathcal{O}(n^{-4}), \\ g_n = \mathcal{O}(n^{-2}), \\ h_n = \sin\left(n - \frac{1}{2}\right)\pi x + \mathcal{O}(n^{-3}), \end{cases} \quad n > N, \tag{23}$$

where N is a positive number.

Proof It is found from (14) and (21) that for λ_n, its corresponding eigenfunction has the asymptotic form:

$$\begin{aligned} h_n(x) &= -i\sinh\sqrt{\lambda_n}(1 - x) \\ &= \sin\left(n - \frac{1}{2}\right)\pi(1 - x) + \mathcal{O}(n^{-3}), \quad n > N. \end{aligned} \tag{24}$$

By the fourth equality of (13), we get that

$$g_n = \mathcal{O}(n^{-2}). \tag{25}$$

Moreover, by (19) and the first equality of (13), we have

$$f_n = \mathcal{O}(n^{-4}). \tag{26}$$

The proof is complete. □

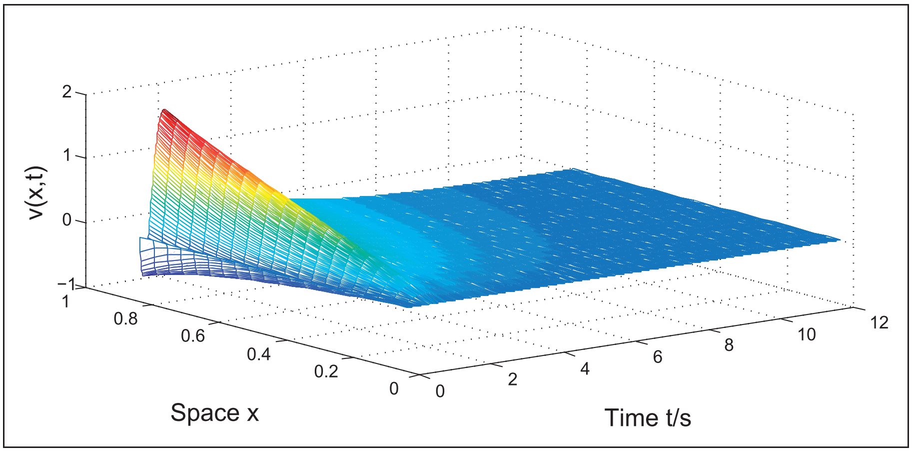

Figure 2. The stability convergence of $v(x,t)$ in (5).

Now we show the Riesz basis property and exponential stability for system (5).

Theorem 3.1 *Let $\mathcal{A}$ be given by (8) and (9). Then there is a set of generalized eigenfunctions of $\mathcal{A}$ which forms a Riesz basis for $\mathcal{H}$. Moreover, $e^{\mathcal{A}t}$, generated by $\mathcal{A}$, is an analytic semigroup for $\mathcal{H}$.*

Proof Let $e_1 = (1,0,0), e_2 = (0,1,0)$, $F_n = (0, 0, \sin\left(n - \frac{1}{2}\right)\pi(1-x))$, $n \in \mathbb{N}$. Then $\{e_1, e_2, F_n, n \in \mathbb{N}\}$ form an orthogonal basis for $\mathcal{H}$. Let $G_n = (f_n, g_n, h_n)$, $n \in \mathbb{N}$, where f_n, g_n, h_n are given by (23), then

$$\sum_{n=N}^{\infty} ||F_n - G_n||^2_{\mathcal{H}} = \sum_{n=N}^{\infty} \left(|f_n|^2 + |g_n|^2\right) = \sum_{n=N}^{\infty} \mathcal{O}\left(n^{-4}\right) < \infty. \tag{27}$$

By Theorem 6.3 of Guo (2001), a modified classical Bari's theorem, there is a set of generalized eigenfunctions of $\mathcal{A}$ which forms a Riesz basis for $\mathcal{H}$. Finally, by Theorem 13 of Opmeer (2008), $\mathcal{A}$ generates an analytic semigroup $e^{\mathcal{A}t}$. The proof is complete. □

Theorem 3.2 *Let $\mathcal{A}$ be given by (8) and (9). Then the spectrum-determined growth condition holds true for $e^{\mathcal{A}t}$, that is, $s(\mathcal{A}) = \omega(\mathcal{A})$, where*

$$s(\mathcal{A}) := \sup\{\mathrm{Re}\,\lambda | \lambda \in \sigma(\mathcal{A})\}$$

is the spectral bound of $\mathcal{A}$, and

$$\omega(\mathcal{A}) := inf\{\omega | \exists M > 0 \text{ such that } ||e^{\mathcal{A}t}|| \leq Me^{\omega t}\}$$

denotes the growth bound of $e^{\mathcal{A}t}$. Moreover, the system (10) is exponentially stable, that is, there exist two positive constants M and σ such that the C_0-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ satisfies

$$||e^{\mathcal{A}t}|| \leq Me^{-\sigma t}. \tag{28}$$

Proof The spectrum-determined growth condition is a direct conclusion of an analytical semigroup. This together with Lemma 3.2 and Proposition 3.1 allows us to deduce that $e^{\mathcal{A}t}$ is exponentially stable. The proof is complete. □

4. Numerical applications

In this section, numerical simulations are carried out for the system (5) with Matlab software. By using the finite difference method, we can obtain the approximate solution of the system (5). In Figures 2 and 3, we show the simulation results for (5) with $k = 5$, $b = 3$, and with the initial conditions $y(0) = 1$, $\dot{y}(0) = 1$, and $v(x, 0) = 0$. Figure 4 presents the stability convergence of the system (3) with the same feedback constants and initial conditions.

5. Conclusions

In this paper, we show that the pendulum system is stabilized by compensating with a heat equation. This kind of control design is quite different from the previous PMD controller and the latest control design based on a backstepping method. First, compared with the system (3) with condition (4), the system (5) only requires $k > 0$, $b \neq 0$, which largely relaxes the restrictions on the parameters k, b. Second, by the Riesz basis approach, not using the traditional Lyapunov function, we show that there is a sequence of generalized eigenfunctions which forms a Riesz

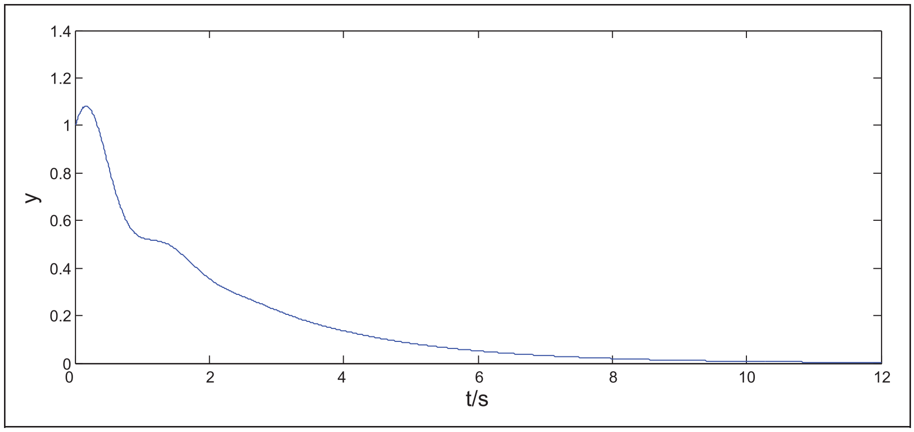

Figure 3. The stability convergence of $y(t)$ in (5).

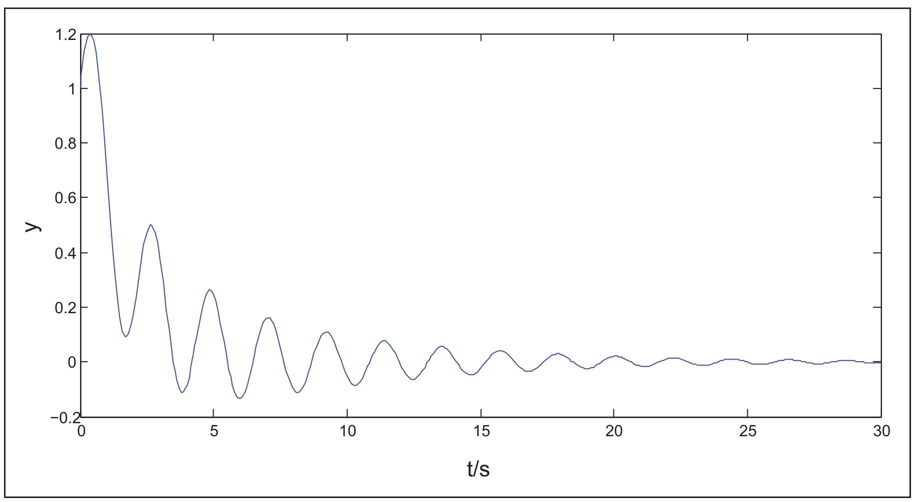

Figure 4. The stability convergence of $y(t)$ in (3).

basis for the state space of the closed-loop system, and then the spectrum-determined growth condition and the exponential stability are established. Moreover, by comparison with (3), we get from the simulations in Section 4 that the convergence rate of $y(t)$ in (5) is improved remarkably.

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