SIAM J. CONTROL OPTIM.
Vol. 52, No. 5, pp. 3287–3311

NONDISSIPATIVE TORQUE AND SHEAR FORCE CONTROLS OF A ROTATING FLEXIBLE STRUCTURE*

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Abstract. This paper is concerned with the feedback stabilization problem of a rotating disk with a flexible beam attached to its center. We propose a torque control applied on the disk and a shear force control exerted at the free end of the beam which lead to a nondissipative closed-loop system. Despite this situation, we prove that the system can be nonuniformly exponentially stabilized provided that the angular velocity of the disk is less than the square root of the first eigenvalue of the self-adjoint positive operator of the open-loop system. This result is illustrated by a set of numerical simulations.

Key words. rotating flexible structure, torque control, nondissipative boundary shear force control, exponential stability

AMS subject classifications. 35B40, 35P20, 74K10, 93D15

DOI. 10.1137/130928406

1. Introduction. There exists a rich literature about systems arising in the study of large-scale flexible space structures. One of those systems has been introduced in [1] and it consists of a disk where an elastic beam is attached to its center. The disk rotates freely around its axis with a time-varying angular velocity and the motion of the beam is confined to a plane perpendicular to the disk.

Assuming that a torque control is exerted on the disk and a boundary force control is applied at the free end of the beam, the global system is governed by the following nonlinear system [1]:

$$\tag{1.1} \begin{cases} \rho y_{tt} + EIy_{xxxx} = \rho\omega^2(t)y, & (x,t) \in (0,1) \times (0,\infty), \\ y(0,t) = y_x(0,t) = y_{xx}(\ell,t) = 0, & t > 0, \\ EIy_{xxx}(\ell,t) = \alpha\Theta(t), & t > 0, \\ \dfrac{d}{dt}\left\{\omega(t)\left(I_d + \rho\displaystyle\int_0^1 y^2(x,t)dx\right)\right\} = T(t), & t > 0, \end{cases}$$

where the positive constants ℓ, EI, ρ, and I_d are respectively the length of the beam, the flexural rigidity, the mass per unit length of the beam, and the disk's moment of inertia. Furthermore, y is the beam's displacement in the rotating plane and ω is the angular velocity of the disk. Finally, α is a positive feedback gain and $\Theta(t)$ is the force control acting on the free end of the beam, whereas $T(t)$ is the torque control to be applied on the disk.

The stability and stabilization problem of system (1.1) has been the object of considerable mathematical endeavors (see [3, 24, 25, 17, 18, 19, 14, 9, 5, 6] and the

*Received by the editors July 10, 2013; accepted for publication (in revised form) August 18, 2014; published electronically October 21, 2014. This work was presented at the 21st International Symposium on Mathematical Theory of Networks and Systems (MTNS 2014).
http://www.siam.org/journals/sicon/52-5/92840.html

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references therein). In turn, the main and common feature in these works is the dissipativity property of the closed-loop system. In other words, all the systems involved in the articles cited above have a decreasing energy. As the reader may know, this property relatively minimizes the amount of work when dealing with such problems.

The main contribution of this paper is to show that our rotating flexible system can be exponentially stabilized by means of a control torque exerted on the disk, combined with only one shear force control applied at the free end of the beam, that is,

$$\begin{cases} \Theta(t) = y_{xt}(\ell, t), \\ T(t) = -\gamma(\omega(t) - \varpi) \text{ for each } \varpi \in \mathbb{R}, \end{cases} \tag{1.2}$$

where γ is a positive constant. Although the suggested torque control is very common in literature, the motivation of proposing such a shear force control is twofold: (i) it has been noticed in practice (see [16]) that there are cases in which it is easier to measure shear force than others. In fact, one way of measuring shear force, inter alia, is to approximate it by the bending strain as it is found by experiments that such treatment is quite acceptable [16]; (ii) the experimental results conducted in [16] have demonstrated that shear force feedback control is effective not only in damping out vibrations but also in maintaining a satisfactory performance in the motion of flexible systems, which is not the case when conventional vibration control methods are applied. The reader is referred to the articles [15, 16] for further details about the advantages of using a shear force control, its physical interpretation, and its implementation.

Now, what can be said about the mathematical aspect of proposing such a feedback law? To answer this question, the reader can check that, contrary to previous articles related to such a system, we end up with a nondissipative closed-loop system since no apparent dissipation property emerges from the usual computations related to the system energy. This case obviously presents, on one hand, a more challenging mathematical problem as the lack of dissipativity prevents applicability of classical semigroup theorems which often resolve the well-posedness problem. Moreover, the issue of stability of solutions of the system cannot be addressed by using standard methods such as the energy and multiplier methods (see [5, 9, 14, 17, 19]) or the frequency-domain method based on Huang's result [12] as in [6]. (See also [4] for a dissipative simple beam system.) On the other hand, it turns out that one may encounter similar situations of nondissipative systems (see, for instance, [7, 8, 11]), where boundary controls lead to an unboundedness on the boundary that is not controlled by the system energy. Thus, the model under consideration is not only of practical interest but also, mathematically speaking, considerably more complicated.

The novelty of the current work, compared to the previous ones, lies in the fact that we do take into consideration the presence of a new type of boundary control, namely, a shear force control, in the feedback law. Based on the discussion above, a mathematical difficulty arises with regard to the semigroup generation as well as the stability of the system. Despite this situation, it will be shown that for any desired angular velocity ϖ smaller than a well-defined value, the beam vibrations can be forced to decay exponentially to zero while the disk will rotate with the desired angular velocity ϖ. The key idea of the proof is to adopt the Riesz basis approach, as in [11], to establish the exponential stability of a linear subsystem and then deduce the desired result for the nonlinearly coupled hybrid system (1.1)–(1.2). Note that for the linear system, the critical maximum angular velocity is shown to be less than the square root of the first eigenvalue of the related self-adjoint positive operator. This outcome extends many previous results in a number of directions. First, we can recover the situation in [11], where their system has no coupling. Indeed, employing the techniques of [11] with a number of changes born out of necessity, we prove the semigroup generation and provide a complete characterization of the uniform exponential stability of a subsystem based on the angular velocity ϖ. Of course, these results could be obtained by invoking perturbation theory of operators but at the expense of stronger conditions on the angular velocity ϖ than the one we shall present. Second, we are able to establish (as in [2]) the Gevrey regularity of a subsystem, whereas in [11] it has been proved that a simpler system possesses a semigroup which is only differentiable. Last but not least, we show that the major findings of [6, 14, 17, 19], namely, the exponential stability of the disk-beam system (1.1), also hold with the feedback law (1.2), notwithstanding that in this case the dissipativity of the system is lost.

Now, let us briefly present an overview of this paper. In section 2, we formulate the closed-loop system in an evolution equation in an appropriate state space. Next, we prove in section 3 the exponential stability of an uncoupled subsystem. Section 4 is devoted to the proof of the main result of this work, namely, the exponential stability of the global closed-loop system. In section 5, we show the relevance of the theoretical results through several numerical simulations. Finally, our conclusions are given in section 6.

2. Preliminaries. For sake of simplicity and without loss of generality, assume that $EI = \rho = \ell = 1$ (in fact a simple change of variables will lead to unit physical parameters). Then, let

$$H_c^n = \Big\{ f \in H^n(0,1); f(0) = f_x(0) = 0 \Big\} \text{ for } n = 2, 3, \dots \tag{2.1}$$

and consider the state space $\mathcal{X}$ defined by

$$\mathcal{X} = H_c^2 \times L^2(0,1) \times \mathbb{R} = \mathcal{H} \times \mathbb{R},$$

equipped with the following inner product:

$$\langle (y, z, \omega), (\tilde{y}, \tilde{z}, \tilde{\omega}) \rangle = \int_0^1 \left(y_{xx}\tilde{y}_{xx} - \varpi^2 y\tilde{y} + z\tilde{z} \right) dx + \omega\tilde{\omega}. \tag{2.2}$$

One can readily check that $\mathcal{X}$ and a fortiori $\mathcal{H} = H_c^2 \times L^2(0,1)$ are Hilbert spaces provided that $|\varpi| < \sqrt{\mu_1}$, where μ_1 is the first eigenvalue of the self-adjoint positive operator $\mathbf{B}$ given by

$$\begin{cases} (\mathbf{B}\phi)(x) = \phi''''(x), \\ \mathcal{D}(\mathbf{B}) = \{\phi \in L^2(0,1) | B\phi \in L^2(0,1), \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0\}. \end{cases} \tag{2.3}$$

Specifically, $\sqrt{\mu_1} \simeq 3.516$ satisfies $1 + \cos(\sqrt[4]{\mu_1})\cosh(\sqrt[4]{\mu_1}) = 0$ (see Lemma 6.1). In addition, for $|\varpi| < \sqrt{\mu_1}$, we have necessarily

$$1 + \cos(\sqrt{|\varpi|})\cosh(\sqrt{|\varpi|}) > 0. \tag{2.4}$$

Subsequently, let $z = y_t$, $\Phi = (y, z, \omega)$ and define on $\mathcal{X}$ the operators $\mathcal{A}$ and $\mathcal{P}$ by

$$\begin{aligned} &\mathcal{D}(\mathcal{A}) = \Big\{ (y, z, \omega) \in H_c^4 \times H_c^2 \times \mathbb{R};\ y_{xx}(1) = 0,\ y_{xxx}(1) = \alpha z_x(1) \Big\}, \\ &\mathcal{A}\Phi = \Big(z, -y_{xxxx} + \varpi^2 y, 0 \Big), \quad \text{for any } \Phi = (y, z, \omega) \in \mathcal{D}(\mathcal{A}) \end{aligned} \tag{2.5}$$

and $\mathcal{P}$ is a nonlinear operator in $\mathcal{X}$ defined by

$$\mathcal{P}\Psi = \Big(0, (\omega^2 - \varpi^2)y, \frac{-\gamma(\omega - \varpi) - 2\,\omega < y, z >_{L^2(0,1)}}{I_d + \|y\|^2_{L^2(0,1)}}\Big), \ \ \forall\, \Psi \in \mathcal{X}. \tag{2.6}$$

Thereafter, the closed-loop system (1.1)–(1.2) can be brought to the standard form of a differential equation in $\mathcal{X}$,

$$\Phi_t(t) = (\mathcal{A} + \mathcal{P})\,\Phi(t), \tag{2.7}$$

which, in turn, can be written as

$$\Phi_t(t) = \left[\begin{pmatrix} A^{\varpi} & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{P}\right]\Phi(t). \tag{2.8}$$

Here A^{ϖ} is an unbounded linear operator defined by

$$\mathcal{D}(A^{\varpi}) = \Big\{\phi = (y, z) \in H_c^4 \times H_c^2;\ y_{xx}(1) = 0,\, y_{xxx}(1) = \alpha z_x(1)\Big\}, \tag{2.9}$$

and for $\phi \in \mathcal{D}(A^{\varpi})$,

$$A^{\varpi}\phi = \Big(z, -y_{xxxx} + \varpi^2 y\Big). \tag{2.10}$$

We shall also consider, in the space $\mathcal{H} = H_c^2 \times L^2(0,1)$, the uncoupled subsystem

$$\begin{cases} \phi_t(t) = A^{\varpi}\phi(t), \\ \phi(0) = \phi_0, \end{cases} \tag{2.11}$$

and show that $|\varpi| = \sqrt{\mu_1}$ is the critical angular velocity with regard to the stability of (2.11), that is, that for any $|\varpi| \geq \sqrt{\mu_1}$, system (2.11) would be unstable (see Lemma 3.6, Theorem 3.7, and Remark 3.8).

3. Well-posedness and stability of the linear subsystem (2.11).

This section is intended to provide a detailed analysis of the linear system (2.11).

3.1. Well-posedness of the subsystem (2.11).

We have the following result.

LEMMA 3.1. *Assume that* $|\varpi| < \sqrt{\mu_1}$. *Let* A^{ϖ} *be defined by* (2.9)–(2.10). *Then,* $(A^{\varpi})^{-1}$ *exists and is compact on* $\mathcal{H}$. *Hence,* $\sigma(A^{\varpi})$, *the spectrum of* A^{ϖ}, *consists of isolated eigenvalues of finite algebraic multiplicity only.*

Proof. Given $\Phi = (f, g) \in \mathcal{H}$, solve $A^{\varpi}\Psi = \Phi$ for $\Psi = (u, v) \in \mathcal{D}(A^{\varpi})$ to get $v = f$ and u must be a solution of

$$-u''''(x) + \varpi^2 u(x) = g(x),\ \ u(0) = u'(0) = u''(1) = 0,\ u'''(1) = \alpha v'(1) = \alpha f'(1).$$

Let

$$F(x) = u(x) - \frac{\alpha}{6} f'(1)\left[x^3 - 3x^2\right].$$

Then F satisfies

$$\begin{cases} F''''(x) - \varpi^2 F(x) = -g(x) + \dfrac{\alpha}{6} f'(1)\varpi^2\left[x^3 - 3x^2\right], \\ F(0) = F'(0) = F''(1) = F'''(1) = 0. \end{cases} \tag{3.1}$$

Thus, (3.1) has a unique solution if and only if the homogeneous equation of (3.1) only admits a zero solution. Consequently, we first consider the homogeneous equation of (3.1)

$$\begin{cases} F_1''''(x) - \varpi^2 F_1(x) = 0, \\ F_1(0) = F_1'(0) = F_1''(1) = F_1'''(1) = 0. \end{cases} \tag{3.2}$$

Since the fundamental solution of (3.2) has the form

$$F_1(x) = c_1 e^{\sqrt{|\varpi|}x} + c_2 e^{-\sqrt{|\varpi|}x} + c_3 e^{i\sqrt{|\varpi|}x} + c_4 e^{-i\sqrt{|\varpi|}x},$$

one can substitute the above expression of F_1 into the boundary conditions of (3.2) to get

$$\begin{cases} c_1 + c_2 + c_3 + c_4 = 0, \\ c_1 - c_2 + ic_3 - ic_4 = 0, \\ c_1 e^{\sqrt{|\varpi|}} + c_2 e^{-\sqrt{|\varpi|}} - c_3 e^{i\sqrt{|\varpi|}} - c_4 e^{-i\sqrt{|\varpi|}} = 0, \\ c_1 e^{\sqrt{|\varpi|}} - c_2 e^{-\sqrt{|\varpi|}} - c_3 i e^{i\sqrt{|\varpi|}} + c_4 i e^{-i\sqrt{|\varpi|}} = 0. \end{cases} \tag{3.3}$$

Note that the determinant of the coefficient matrix of (3.3) is

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ e^{\sqrt{|\varpi|}} & e^{-\sqrt{|\varpi|}} & -e^{i\sqrt{|\varpi|}} & -e^{-i\sqrt{|\varpi|}} \\ e^{\sqrt{|\varpi|}} & -e^{-\sqrt{|\varpi|}} & -ie^{i\sqrt{|\varpi|}} & ie^{-i\sqrt{|\varpi|}} \end{vmatrix} = 8i\left[1 + \cosh\sqrt{|\varpi|}\cos\sqrt{|\varpi|}\right] \neq 0,$$

where we have used the fact $|\varpi| < \sqrt{\mu_1}$ and (2.4). This enables us to obtain the unique solution $F(x)$ to (3.1) and consequently there is a unique $u(x) = F(x) + \frac{\alpha}{6}f'(1)\left[x^3 - 3x^2\right]$. Therefore, $(A^{\varpi})^{-1}$ exists and is compact on $\mathcal{H}$ by the Sobolev embedding theorem, and $\sigma(A^{\varpi})$, the spectrum of A^{ϖ}, consists of isolated eigenvalues of finite algebraic multiplicity only. The proof is complete. ∎

3.2. Eigenvalue problems. It is an easy task to check that the equation $A^{\varpi}(f, g) = \lambda(f, g)$ is equivalent to writing that $g = \lambda f$ and f satisfies the eigenvalue problem

$$\begin{cases} \lambda^2 f(x) + f''''(x) - \varpi^2 f(x) = 0, \quad 0 < x < 1, \\ f(0) = f'(0) = f''(1) = 0, \\ f'''(1) = \alpha\lambda f'(1). \end{cases} \tag{3.4}$$

LEMMA 3.2. *If $\alpha|\varpi| = 2$, then $\lambda = -|\varpi|$ is an eigenvalue of A^{ϖ} and $(x^3 - 3x^2, -|\varpi|(x^3 - 3x^2))$ is an associated eigenfunction of A^{ϖ}.*

Proof. When $\lambda = -|\varpi|$ and $\alpha|\varpi| = 2$, the system (3.4) changes

$$\begin{cases} f''''(x) = 0, & 0 < x < 1, \\ f(0) = f'(0) = f''(1) = 0, \ f'''(1) = -2f'(1), \end{cases}$$

and hence $f = x^3 - 3x^2$ is a nontrivial solution. Therefore, $\lambda = -|\varpi|$ is an eigenvalue of A^{ϖ} and $(x^3 - 3x^2, -|\varpi|(x^3 - 3x^2))$ is an associated eigenfunction of A^{ϖ}. ∎

Let $\lambda \neq |\varpi|$. Then, the fundamental solution of (3.4) is

$$f(x) = c_1 e^{a(\lambda)\sqrt{i}x} + c_2 e^{-a(\lambda)\sqrt{i}x} + c_3 e^{a(\lambda)i\sqrt{i}x} + c_4 e^{-a(\lambda)i\sqrt{i}x}, \tag{3.5}$$

where

$$a(\lambda) = \sqrt[4]{\lambda^2 - \varpi^2}. \tag{3.6}$$

Inserting (3.5) in the boundary conditions of (3.4), we have

$$\begin{cases} c_1 + c_2 + c_3 + c_4 = 0, \\ c_1 - c_2 + ic_3 - ic_4 = 0, \\ c_1 e^{a(\lambda)\sqrt{i}} + c_2 e^{-a(\lambda)\sqrt{i}} - c_3 e^{a(\lambda)i\sqrt{i}} - c_4 e^{-a(\lambda)i\sqrt{i}} = 0, \\ c_1 \left[ia^2(\lambda) - \alpha\lambda\right] e^{a(\lambda)\sqrt{i}} - c_2 \left[ia^2(\lambda) - \alpha\lambda\right] e^{-a(\lambda)\sqrt{i}} \\ \quad + c_3 \left[a^2(\lambda) - i\alpha\lambda\right] e^{a(\lambda)i\sqrt{i}} - c_4 \left[a^2(\lambda) - i\alpha\lambda\right] e^{-a(\lambda)i\sqrt{i}} = 0. \end{cases} \tag{3.7}$$

Clearly, λ is an eigenvalue of (3.4) if and only if λ is a zero of the determinant $G(\lambda)$ of the coefficient matrix of (3.7), where $G(\lambda)$ is given by

$$G(\lambda) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ e^{a(\lambda)\sqrt{i}} & e^{-a(\lambda)\sqrt{i}} & -e^{a(\lambda)i\sqrt{i}} & -e^{-a(\lambda)i\sqrt{i}} \\ G_1(\lambda) & G_2(\lambda) & G_3(\lambda) & G_4(\lambda) \end{vmatrix}. \tag{3.8}$$

Here

$$\begin{cases} G_1(\lambda) = \left[ia^2(\lambda) - \alpha\lambda\right] e^{a(\lambda)\sqrt{i}}, \quad G_2(\lambda) = -\left[ia^2(\lambda) - \alpha\lambda\right] e^{-a(\lambda)\sqrt{i}}, \\ G_3(\lambda) = \left[a^2(\lambda) - i\alpha\lambda\right] e^{a(\lambda)i\sqrt{i}}, \quad G_4(\lambda) = -\left[a^2(\lambda) - i\alpha\lambda\right] e^{-a(\lambda)i\sqrt{i}}. \end{cases}$$

A straightforward computation leads to the following result. We omit the details here.

LEMMA 3.3. *Let $G(\lambda)$ be given by* (3.8). *Then, $G(\lambda) = -2G_*(\lambda)$, where*

$$\begin{aligned} G_*(\lambda) = \ & 4a^2(\lambda) + \left[a^2(\lambda) - \alpha\lambda\right]\left[e^{\sqrt{2}a(\lambda)i} + e^{-\sqrt{2}a(\lambda)i}\right] \\ & + \left[a^2(\lambda) + \alpha\lambda\right]\left[e^{\sqrt{2}a(\lambda)} + e^{-\sqrt{2}a(\lambda)}\right]. \end{aligned} \tag{3.9}$$

Thus, for $\lambda \neq -|\varpi|$, $\lambda \in \sigma(A^\varpi)$ if and only if $G_*(\lambda) = 0$.

Now, we can state another result.

LEMMA 3.4. *Let $\lambda \in \sigma(A^\varpi)$. Then, we have necessarily* $\mathrm{Re}\lambda \neq 0$.

Proof. In the event that $\alpha|\varpi| = 2$ and according to Lemma 3.2, $\lambda = -|\varpi|$ is an eigenvalue of A^ϖ and the desired result is obtained. On the other hand, for each $\lambda = is$, $s \in \mathbb{R}$, we have

$$a(\lambda) = \sqrt[4]{-s^2 - \varpi^2} = \sqrt{i}\sqrt[4]{s^2 + \varpi^2} = \sqrt{i}a_1(s), \text{ where } a_1(s) = \sqrt[4]{s^2 + \varpi^2} > 0$$

and by (3.9),

$$\begin{aligned}
-iG_*(\lambda) &= 4a_1^2(s) + \left[a_1^2(s) - \alpha s\right]\left[e^{\sqrt{2}a_1(s)i\sqrt{i}} + e^{-\sqrt{2}a_1(s)i\sqrt{i}}\right]\\
&\quad + \left[a_1^2(s) + \alpha s\right]\left[e^{\sqrt{2}a_1(s)\sqrt{i}} + e^{-\sqrt{2}a_1(s)\sqrt{i}}\right]\\
&= 4a_1^2(s) + 2a_1^2(s)\Big[e^{-a_1(s)}\cos(a_1(s)) + e^{a_1(s)}\cos(a_1(s))\Big]\\
&\quad + 2iks\Big[e^{a_1(s)}\sin(a_1(s)) - e^{-a_1(s)}\sin(a_1(s))\Big]\\
&= 4a_1^2(s) + 4a_1^2(s)\cosh(a_1(s))\cos(a_1(s)) + 4iks\sinh(a_1(s))\sin(a_1(s)) \neq 0.
\end{aligned}$$

This implies that $\mathrm{Re}\lambda \neq 0$, for $\lambda \in \sigma(A^{\varpi})$. ∎

LEMMA 3.5. *Assume that $\alpha > 0$, and $|\varpi| < \sqrt{\mu_1}$. Let $\lambda = c + di \in \sigma(A^{\varpi})$, $c, d \in \mathbb{R}$. Then, we have $c < 0$ whenever $d = 0$.*

Proof. Let $d = 0$. Therefore

$$a(\lambda) = \sqrt[4]{c^2 - \varpi^2} = \begin{cases} \sqrt[4]{c^2 - \varpi^2} = a_2(c), & |c| > |\varpi|,\\ \sqrt[4]{-(\varpi^2 - c^2)} = \sqrt{i}\sqrt[4]{\varpi^2 - c^2} = \sqrt{i}a_3(c), & 0 < |c| < |\varpi|, \end{cases}$$

where

$$a_2(c) = \sqrt[4]{c^2 - \varpi^2},\ a_3(c) = \sqrt[4]{\varpi^2 - c^2} > 0.$$

We shall consider three cases:

1. If $c > |\varpi|$ and $d = 0$, then we get

$$\begin{aligned}
G_*(\lambda) = \;& 4a_2^2(c) + \left[a_2^2(c) - \alpha c\right]\left[e^{\sqrt{2}a_2(c)i} + e^{-\sqrt{2}a_2(c)i}\right]\\
& + \left[a_2^2(c) + \alpha c\right]\left[e^{\sqrt{2}a_2(c)} + e^{-\sqrt{2}a_2(c)}\right].
\end{aligned}$$

Since

$$\begin{aligned}
& 4a_2^2(c) + \left[a_2^2(c) + \alpha c\right]\left[e^{\sqrt{2}a_2(c)} + e^{-\sqrt{2}a_2(c)}\right]\\
\geq\; & 4a_2^2(c) + 2\left[a_2^2(c) + \alpha c\right] > 2\left|a_2^2(c) - \alpha c\right|\\
\geq\; & \left|\left|a_2^2(c) - \alpha c\right|\left[e^{\sqrt{2}a_2(c)i} + e^{-\sqrt{2}a_2(c)i}\right]\right|,
\end{aligned}$$

we get $G_*(\lambda) \neq 0$ and whence a contradiction.

2. When $0 \leq c < |\varpi|$ and $d = 0$, we have

$$\begin{aligned}
G_*(\lambda) &= 4ia_3^2(c) + \left[ia_3^2(c) - \alpha c\right]\left[e^{\sqrt{2}\sqrt{i}a_3(c)i} + e^{-\sqrt{2}\sqrt{i}a_3(c)i}\right]\\
&\quad + \left[ia_3^2(c) + \alpha c\right]\left[e^{\sqrt{2}\sqrt{i}a_3(c)} + e^{-\sqrt{2}\sqrt{i}a_3(c)}\right]\\
&= 4ia_3^2(c) + ia_3^2(c)\left[2e^{-a_3(c)}\cos(a_3(c)) + 2e^{a_3(c)}\cos(a_3(c))\right]\\
&\quad + \alpha c\left[-2ie^{-a_3(c)}\sin(a_3(c)) + 2ie^{a_3(c)}\sin(a_3(c))\right]\\
&= 4i\left[a_3^2(c) + a_3^2(c)\cosh(a_3(c))\cos(a_3(c)) + \alpha c\sinh(a_3(c))\sin(a_3(c))\right]. \qquad (3.10)
\end{aligned}$$

In light of the condition $|\varpi| < \sqrt{\mu_1}$, it can be deduced that

$$0 < a_3(c) = \sqrt[4]{\varpi^2 - c^2} \leq \sqrt{|\varpi|} < \sqrt[4]{\mu_1} \simeq \sqrt{3.516} < 2. \tag{3.11}$$

Hence we have $\sin(a_3(c)) > 0$ and in view of (2.4), it follows that $1 + \cosh(a_3(c))\cos(a_3(c)) > 0$. Therefore, $G_*(\lambda) \neq 0$ which contradicts the fact that $\lambda \in \sigma(A^\varpi)$.

3. Assume $c = |\varpi|$, whereupon, $\lambda^2 = \varpi^2$. Then the eigenvalue problem (3.4) changes

$$\begin{cases} f''''(x) = 0, & 0 < x < 1, \\ f(0) = f'(0) = f''(1) = 0, \\ f'''(1) = \alpha|\varpi| f'(1). \end{cases} \tag{3.12}$$

An elementary computation shows that this system has only the trivial solution. Herewith, $c = |\varpi| \notin \sigma(A^\varpi)$.

Whichever, we conclude from the three cases that $c < 0$. The proof is achieved. ∎

LEMMA 3.6. *Suppose that $\alpha > 0$ and $|\varpi| \geq \sqrt{\mu_1}$. Then, there exists at least one nonnegative real eigenvalue of A^ϖ.*

Proof. Without loss of generality, we assume that $|\varpi| \in [\sqrt{\mu_1}, \sqrt{\mu_2})$, where μ_2 is the second eigenvalue of the self-adjoint positive operator $\mathbf{B}$ given by (2.3). Let $\lambda = c \in \mathbb{R}$. Invoking (3.10) and setting

$$h(c) = a_3^2(c) + a_3^2(c)\cosh(a_3(c))\cos(a_3(c)) + \alpha c \sinh(a_3(c))\sin(a_3(c)),$$

we obtain h is continuous in $[0, |\varpi|]$ and

$$h(0) = 1 + \cosh\sqrt{|\varpi|}\cos\sqrt{|\varpi|} \leq 0.$$

Moreover, for $c_* = \sqrt{|\varpi|^2 - \frac{1}{2}\mu_1}$, we have $c_* \in (0, |\varpi|)$ and $\varpi^2 - c_*^2 = \frac{1}{2}\mu_1 < \mu_1$. Thereby (3.11) yields

$$h(c_*) = a_3^2(c_*) + a_3^2(c_*)\cosh(a_3(c_*))\cos(a_3(c_*)) + \alpha c \sinh(a_3(c_*))\sin(a_3(c_*)) > 0.$$

Therefore, there exists $c_0 \in [0, c_*)$ such that $h(c_0) = 0$ and whence $c_0 \geq 0$ is an eigenvalue of A^ϖ. ∎

THEOREM 3.7. *Assume that $\alpha > 0$, and $|\varpi| < \sqrt{\mu_1}$. If $\lambda \in \sigma(A^\varpi)$, then* $\mathrm{Re}(\lambda) < 0$.

Proof. Let $\lambda = a + bi \in \sigma(A^\varpi)$. When $b \neq 0$, it follows from Lemma 6.3 of the appendix that $\mathrm{Re}(\lambda) < 0$. In turn, if $b = 0$, then we have $\mathrm{Re}(\lambda) < 0$ according to Lemma 3.5. The desired result is then proved. ∎

Remark 3.8. Let $\alpha > 0$ and $\lambda \in \sigma(A^\varpi)$. In the light of Lemma 3.6 and Theorem 3.7, we can claim that the assumption $|\varpi| < \sqrt{\mu_1}$ is indeed a sufficient and necessary condition for getting the important property $\mathrm{Re}(\lambda) < 0$.

3.3. Riesz basis property. This subsection will be concerned with the study of the Riesz basis property for the linear system (2.11).

THEOREM 3.9. *Assume that $\alpha > 0$, $\alpha \neq 1$, and $|\varpi| < \sqrt{\mu_1}$. Let A^{ϖ} be defined by (2.9)–(2.10). Then $\sigma(A^{\varpi}) = \{\lambda_n, \overline{\lambda_n}, n \in \mathbb{N}\}$ and as $n \to \infty$, λ_n has the following asymptotic expansion:*

$$\lambda_n = \begin{cases} n\pi \ln\left(\frac{\alpha-1}{\alpha+1}\right) + \left[n^2\pi^2 - \frac{1}{4}\left\{\ln\left(\frac{\alpha-1}{\alpha+1}\right)\right\}^2\right] i + \mathcal{O}(n^{-3}), & \alpha > 1, \\ \left(n-\frac{1}{2}\right)\pi \ln\left(\frac{1-\alpha}{\alpha+1}\right) + \left[\left(n-\frac{1}{2}\right)^2\pi^2 - \frac{1}{4}\left\{\ln\left(\frac{1-\alpha}{\alpha+1}\right)\right\}^2\right] i + \mathcal{O}(n^{-3}), & 0 < \alpha < 1. \end{cases} \tag{3.13}$$

Hence, we have

$$\mathrm{Re}(\lambda_n) \to -\infty, \quad \text{as } n \to \infty. \tag{3.14}$$

Proof. In view of the property $\mathrm{Re}(\lambda) < 0$, which has been claimed and proved in Theorem 3.7, and since all eigenvalues of A^{ϖ} are symmetric about the real axis from the eigenvalue problem (3.4), we only need to consider those $\lambda \in \sigma(A^{\varpi})$ satisfying $\frac{\pi}{2} \leq \arg(\lambda) \leq \pi$.

For the further study of (3.4), it is advantageous to set $\lambda = \rho^2$. Thus

$$\frac{\pi}{2} \leq \arg(\lambda) \leq \pi \Leftrightarrow \rho \in \mathcal{S} = \left\{\rho \in \mathbb{C} \,\middle|\, \frac{\pi}{4} \leq \arg(\rho) \leq \frac{\pi}{2}\right\}. \tag{3.15}$$

When $\rho \in \mathcal{S}$ and $|\rho| \to \infty$, by (3.6), we have

$$\begin{cases} a(\lambda) = \sqrt[4]{\lambda^2 - \varpi^2} = \sqrt[4]{\rho^4 - \varpi^2} = \rho\left(1 - \frac{1}{4}\frac{\varpi^2}{\rho^4} + \mathcal{O}(\rho^{-8})\right), \\ \left|e^{\sqrt{2}\rho i}\right| = \mathcal{O}(e^{-c\rho}), \\ e^{\sqrt{2}a(\lambda)i} = e^{\sqrt{2}i\rho}\left(1 - i\frac{\sqrt{2}}{4}\frac{\varpi^2}{\rho^3} + \mathcal{O}(\rho^{-6})\right) = \mathcal{O}(e^{-c\rho}), \end{cases} \tag{3.16}$$

where $c > 0$ is a positive constant. Then, invoking Lemma 3.3, we claim that the equation $G(\lambda) = 0$, together with $\rho \in \mathcal{S}$, yields

$$\rho^{-2}e^{\sqrt{2}\rho i}G_*(\lambda) = 1 - \alpha + [1+\alpha]\, e^{\sqrt{2}(1+i)\rho} + \mathcal{O}(\rho^{-4}) = 0. \tag{3.17}$$

The solution of $1 - \alpha + [1+\alpha]\, e^{\sqrt{2}(1+i)\rho} = 0$ has the form

$$\tilde{\rho}_n = \begin{cases} \frac{1}{\sqrt{2}i}\left[2n\pi i + \ln\left|\frac{\alpha-1}{\alpha+1}\right|\right], & \alpha > 1, \\ \frac{1}{\sqrt{2}i}\left[(2n+1)\pi i + \ln\left|\frac{\alpha-1}{\alpha+1}\right|\right], & \alpha < 1, \end{cases} \quad n \in \mathbb{N},$$

by applying the Rouché theorem. Thus, the asymptotic solutions of (3.17), as $n \to \infty$, are given by

$$\rho_n = \begin{cases} \frac{1}{\sqrt{2}(1+i)}\left[2n\pi i + \ln\left(\frac{\alpha-1}{\alpha+1}\right)\right] + \mathcal{O}(n^{-4}), & \alpha > 1, \\ \frac{1}{\sqrt{2}(1+i)}\left[(2n-1)\pi i + \ln\left(\frac{1-\alpha}{\alpha+1}\right)\right] + \mathcal{O}(n^{-4}), & \alpha < 1. \end{cases} \tag{3.18}$$

This implies that, as $n \to \infty$,

$$\lambda_n = \begin{cases} n\pi \ln\left(\frac{\alpha-1}{\alpha+1}\right) + \left[n^2\pi^2 - \frac{1}{4}\left\{\ln\left(\frac{\alpha-1}{\alpha+1}\right)\right\}^2\right] i + \mathcal{O}(n^{-3}), & \alpha > 1, \\ (n-\frac{1}{2})\pi \ln\left(\frac{1-\alpha}{\alpha+1}\right) + \left[\left(n-\frac{1}{2}\right)^2\pi^2 - \frac{1}{4}\left\{\ln\left(\frac{1-\alpha}{\alpha+1}\right)\right\}^2\right] i + \mathcal{O}(n^{-3}), & \alpha < 1, \end{cases}$$

which is the desired result (3.13). ∎

We turn now to the establishment of the asymptotic expansion of the eigenfunctions.

THEOREM 3.10. *Assume that* $\alpha > 0$, $\alpha \neq 1$, *and* $|\varpi| < \sqrt{\mu_1}$. *Let* $\sigma(A^\varpi) = \{\lambda_n, \overline{\lambda_n}, n \in \mathbb{N}\}$ *be the eigenvalues of* A^ϖ *and let* λ_n *be given by* (3.13). *Then the corresponding eigenfunctions* $\{(f_n, \lambda_n f_n), (\overline{f_n}, \overline{\lambda}_n \overline{f}_n), n \in \mathbb{N}\}$ *have the following asymptotic expansion: as* $n \to \infty$,

$$\begin{cases} \lambda_n f_n(x) = (1+i)e^{\rho_n\sqrt{i}(1+x)} + (1-i)e^{\rho_n\sqrt{i}(1-x)} \\ \qquad\qquad - 2e^{\rho_n\sqrt{i}}e^{\rho_n i\sqrt{i}x} + (1-i)e^{\rho_n i\sqrt{i}(1-x)} + \mathcal{O}(n^{-3}), \\ f_n''(x) = -(1-i)e^{\rho_n\sqrt{i}(1+x)} + (1+i)e^{\rho_n\sqrt{i}(1-x)} \\ \qquad\qquad + 2ie^{\rho_n\sqrt{i}}e^{\rho_n i\sqrt{i}x} - (1+i)e^{\rho_n i\sqrt{i}(1-x)} + \mathcal{O}(n^{-3}), \end{cases} \tag{3.19}$$

where ρ_n *is given by* (3.18). *Furthermore,* $(f_n, \lambda_n f_n)$ *are approximately normalized in the sense that there exist positive constants* c_1 *and* c_2 *independent on* n *such that*

$$c_1 \leq \|f_n''\|_{L^2(0,1)},\ \|\lambda_n f_n\|_{L^2(0,1)} \leq c_2. \tag{3.20}$$

Proof. Recollecting (3.4)–(3.5) and (3.8), it can be seen that the eigenfunction f corresponding to the eigenvalue $\lambda = \rho^2$, with $\rho \in \mathcal{S}$, is given by

$$f(x) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ e^{a(\lambda)\sqrt{i}} & e^{-a(\lambda)\sqrt{i}} & -e^{a(\lambda)i\sqrt{i}} & -e^{-a(\lambda)i\sqrt{i}} \\ e^{a(\lambda)\sqrt{i}x} & e^{-a(\lambda)\sqrt{i}x} & e^{a(\lambda)i\sqrt{i}x} & e^{-a(\lambda)i\sqrt{i}x} \end{vmatrix}.$$

A tedious but direct computation gives

$$\begin{aligned} e^{a(\lambda)[i\sqrt{i}+\sqrt{i}]}f(x) = &- \left[-2ie^{a(\lambda)i\sqrt{i}} + (1-i)e^{a(\lambda)[2i\sqrt{i}+\sqrt{i}]} - (1+i)e^{a(\lambda)\sqrt{i}}\right] e^{a(\lambda)\sqrt{i}x} \\ &+ \left[-2ie^{a(\lambda)[i\sqrt{i}+\sqrt{i}]} - (1+i)e^{a(\lambda)[2i\sqrt{i}+\sqrt{i}]} + (1-i)\right] e^{a(\lambda)\sqrt{i}(1-x)} \\ &- \left[(1-i)e^{a(\lambda)[i\sqrt{i}+2\sqrt{i}]} + (1+i)e^{ia(\lambda)\sqrt{i}} + 2e^{a(\lambda)\sqrt{i}}\right] e^{a(\lambda)i\sqrt{i}x} \\ &+ \left[(1+i)e^{2a(\lambda)\sqrt{i}} + (1-i) + 2e^{a(\lambda)[i\sqrt{i}+\sqrt{i}]}\right] e^{a(\lambda)i\sqrt{i}(1-x)} \end{aligned}$$

and

$$\begin{aligned} e^{a(\lambda)[i\sqrt{i}+\sqrt{i}]}f''(x) = &-a^2(\lambda)\left[2e^{a(\lambda)i\sqrt{i}} + (1+i)e^{a(\lambda)[2i\sqrt{i}+\sqrt{i}]} + (1-i)e^{a(\lambda)\sqrt{i}}\right] e^{a(\lambda)\sqrt{i}x} \\ &+a^2(\lambda)\left[2e^{a(\lambda)[i\sqrt{i}+\sqrt{i}]} + (1-i)e^{a(\lambda)[2i\sqrt{i}+\sqrt{i}]} + (1+i)\right] e^{a(\lambda)\sqrt{i}(1-x)} \\ &-a^2(\lambda)\left[-(1+i)e^{a(\lambda)[i\sqrt{i}+2\sqrt{i}]} + (1-i)e^{ia(\lambda)\sqrt{i}} - 2ie^{a(\lambda)\sqrt{i}}\right] e^{a(\lambda)i\sqrt{i}x} \\ &+a^2(\lambda)\left[(1-i)e^{2a(\lambda)\sqrt{i}} - (1+i) - 2ie^{a(\lambda)[i\sqrt{i}+\sqrt{i}]}\right] e^{a(\lambda)i\sqrt{i}(1-x)}. \end{aligned}$$

Furthermore, amalgamating (3.15) and (3.16) with $\rho \in \mathcal{S}$, we have, as $\rho \to \infty$,

$$\begin{cases} e^{a(\lambda)\sqrt{i}} = e^{\sqrt{i}\rho}\left(1+\mathcal{O}(\rho^{-3})\right), \\ e^{a(\lambda)i\sqrt{i}} = e^{i\sqrt{i}\rho}\left(1+\mathcal{O}(\rho^{-3})\right) = \mathcal{O}(e^{-c\rho}), \\ e^{a(\lambda)[2i\sqrt{i}+\sqrt{i}]} = e^{[2i\sqrt{i}+\sqrt{i}]\rho}\left(1+\mathcal{O}(\rho^{-3})\right) = \mathcal{O}(e^{-c\rho}), \\ e^{a(\lambda)[i\sqrt{i}+\sqrt{i}]} = e^{[i\sqrt{i}+\sqrt{i}]\rho}\left(1+\mathcal{O}(\rho^{-3})\right) = \mathcal{O}(e^{-c\rho}), \end{cases}$$

where c is a positive constant. Hence, for $\rho \in \mathcal{S}$ and $\rho \to \infty$, there holds

$$\begin{aligned} e^{a(\lambda)[i\sqrt{i}+\sqrt{i}]}f(x) &= (1+i)e^{a(\lambda)\sqrt{i}}e^{a(\lambda)\sqrt{i}x} + (1-i)e^{a(\lambda)\sqrt{i}(1-x)} \\ &\quad - 2e^{a(\lambda)\sqrt{i}}e^{a(\lambda)i\sqrt{i}x} + (1-i)e^{a(\lambda)i\sqrt{i}(1-x)} + \mathcal{O}(e^{-c\rho}) \\ &= (1+i)e^{\rho\sqrt{i}(1+x)} + (1-i)e^{\rho\sqrt{i}(1-x)} \\ &\quad - 2e^{\rho\sqrt{i}}e^{\rho i\sqrt{i}x} + (1-i)e^{\rho i\sqrt{i}(1-x)} + \mathcal{O}(\rho^{-3}), \end{aligned}$$

and

$$\begin{aligned} \rho^{-2}e^{a(\lambda)[i\sqrt{i}+\sqrt{i}]}f''(x) &= -(1-i)e^{\rho\sqrt{i}(1+x)} + (1+i)e^{\rho\sqrt{i}(1-x)} \\ &\quad + 2ie^{\rho\sqrt{i}}e^{\rho i\sqrt{i}x} - (1+i)e^{\rho i\sqrt{i}(1-x)} + \mathcal{O}(\rho^{-3}). \end{aligned}$$

Thereafter, the asymptotic expansions in (3.19) can be deduced from the above two formulas by setting $f_n(x) = \rho_n^{-2}e^{a(\lambda_n)[i\sqrt{i}+\sqrt{i}]}f(x)$. Finally, to vindicate the theorem, it remains to show (3.20). To proceed, we notice from (3.18) that

$$\rho_n\sqrt{i} = \begin{cases} n\pi i + \dfrac{1}{2}\ln\left(\dfrac{\alpha-1}{\alpha+1}\right) + \mathcal{O}(n^{-4}), & \alpha > 1, \\ \left(n-\dfrac{1}{2}\right)\pi i + \dfrac{1}{2}\ln\left(\dfrac{1-\alpha}{\alpha+1}\right) + \mathcal{O}(n^{-4}), & \alpha < 1, \end{cases}$$

and

$$\rho_n i\sqrt{i} = \begin{cases} -n\pi + i\dfrac{1}{2}\ln\left(\dfrac{\alpha-1}{\alpha+1}\right) + \mathcal{O}(n^{-4}), & \alpha > 1, \\ -\left(n-\dfrac{1}{2}\right)\pi + i\dfrac{1}{2}\ln\left(\dfrac{1-\alpha}{\alpha+1}\right) + \mathcal{O}(n^{-4}), & \alpha < 1. \end{cases}$$

Thus

$$\begin{cases} \|e^{\rho_n\sqrt{i}(1+x)}\|^2_{L^2(0,1)} = \left[\ln\left(\dfrac{|\alpha-1|}{1+\alpha}\right)\right]^{-1}\left[\left(\dfrac{|\alpha-1|}{1+\alpha}\right)^2 - \dfrac{|\alpha-1|}{1+\alpha}\right], \\ \|e^{\rho_n\sqrt{i}(1-x)}\|^2_{L^2(0,1)} = \left[\ln\left(\dfrac{|\alpha-1|}{1+\alpha}\right)\right]^{-1}\left[\dfrac{|\alpha-1|}{1+\alpha} - 1\right], \\ \|e^{\rho_n i\sqrt{i}x}\|^2_{L^2(0,1)} = \mathcal{O}(n^{-1}), \quad \|e^{\rho_n i\sqrt{i}(1-x)}\|^2_{L^2(0,1)} = \mathcal{O}(n^{-1}). \end{cases}$$

These, together with (3.19), yield (3.20). ∎

The task ahead is to show the Riesz basis property for the linear system (2.11) governed by the operator (2.9)–(2.10).

THEOREM 3.11. *Assume that* $\alpha > 0, \alpha \neq 1$, *and* $|\varpi| < \sqrt{\mu_1}$. *Let* A^{ϖ} *be the operator defined by* (2.9)–(2.10). *Then there is a set of generalized eigenfunctions* *which form a Riesz basis in $\mathcal{H}$ and each eigenvalue of A^{ϖ} with sufficiently large moduli is algebraically simple.*

Proof. In order to show the Riesz basis, we introduce another operator on $\mathcal{H}$:

$$\begin{cases} A^0\phi = \Big(z, -y_{xxxx}\Big), \\ \mathcal{D}(A^0) = \Big\{\phi = (y,z) \in H_c^4 \times H_c^2;\ y_{xx}(1) = 0,\ y_{xxx}(1) = \alpha z_x(1)\Big\}. \end{cases} \tag{3.21}$$

Clearly, A^0 is the system operator, with compact resolvent, for the linear subsystem (2.11) with $\varpi = 0$. This operator has been intensively studied in [11]. Indeed, it has been shown in [11, Theorem 4.1] that the generalized eigenfunctions $\{(f_{n0}, \lambda_{n0}f_{n0}), (\overline{f_{n0}}, \overline{\lambda}_{n0}\overline{f}_{n0}), n \in \mathbb{N}\}$ of A^0, where λ_{n0} denotes the eigenvalues of A^0, form a Riesz basis in $\mathcal{H}$ and in [11, Proposition 3.1] that each eigenvalue λ_{n0} of A^0 with sufficiently large moduli is algebraically simple. Furthermore, similar arguments and computations for A^{ϖ} will lead us to claim that λ_{n0} and $(f_{n0}, \lambda_{n0}f_{n0})$ have the same asymptotic expressions (3.13) and (3.19), respectively. Consequently, there is an $N > 0$ large enough such that

$$\sum_{n\geq N} \|(f_n, \lambda_n f_n) - (f_{n0}, \lambda_{n0}f_{n0})\|_{\mathcal{H}}^2 = \sum_{n\geq N} \mathcal{O}(n^{-6}) < \infty,$$

and the same is true for their conjugates. By using the extended Bari's theorem [10, Theorem 6.3], we claim that the generalized eigenfunctions $\{(f_n, \lambda_n f_n), (\overline{f}_n, \overline{\lambda}_n \overline{f}_n), n \in \mathbb{N}\}$ of A^{ϖ} form a Riesz basis in $\mathcal{H}$ and each eigenvalue of A^{ϖ} with sufficiently large moduli is algebraically simple. The proof is complete. ∎

3.4. Exponential stability and Gevrey regularity. It is the intention of this subsection to show the exponential stability and Gevrey regularity for linear system (2.11). First, we establish the exponential stability result for (2.11).

THEOREM 3.12. *Assume that $\alpha > 0, \alpha \neq 1$, and $|\varpi| < \sqrt{\mu_1}$. Let A^{ϖ} be the operator defined by* (2.9)–(2.10). *Then A^{ϖ} generates a C_0-semigroup $e^{A^{\varpi}t}$ in $\mathcal{H}$. Moreover, $e^{A^{\varpi}t}$ satisfies the spectrum-determined growth condition: $S(A^{\varpi}) = \omega(A^{\varpi})$, where $S(A^{\varpi}) = \sup_{\lambda\in\sigma(A^{\varpi})} \mathrm{Re}(\lambda)$ is the spectral bound and $\omega(A^{\varpi})$ is the growth order of the semigroup $e^{A^{\varpi}t}$. Furthermore, $e^{A^{\varpi}t}$ is exponentially stable in $\mathcal{H}$.*

Proof. The C_0-semigroup and spectrum-determined growth condition directly follow from the Riesz basis property of A^{ϖ} claimed by Theorem 3.11, whereupon we only need to verify the exponential stability, which can be proved by the eigenvalues distribution of A^{ϖ}. In fact, let $\lambda \in \sigma(A^{\varpi})$. Then, invoking Theorem 3.7, we have $\mathrm{Re}(\lambda) < 0$. Moreover, recall that $\mathrm{Re}(\lambda) \to -\infty$ as $|\lambda| \to \infty$ by means of (3.14). Hence, the exponential stability is achieved as the spectrum-determined growth condition holds. The proof is complete. ∎

We turn now to the proof of the regularity of the C_0-semigroup $e^{A^{\varpi}t}$ generated by A^{ϖ}, namely, $e^{A^{\varpi}t}$ is of Gevrey class δ with any $\delta > 2$, which is the semigroup class between the differentiable semigroup class and the analytical ones.

DEFINITION 3.13 (see [2, 22]). *A C_0-semigroup $S(t)$ in a Hilbert space H is of a Gevrey class $\delta > 1$ for $t > t_0$ if $S(t)$ is infinitely differentiable for $t > t_0$ and for every compact subset $K \subset (t_0, \infty)$ and each $\theta > 0$, there is a constant $C = C(K, \theta)$ such that*

$$\|S^{(n)}(t)\| \leq C\theta^n (n!)^{\delta},\ \ \forall t \in K,\ n = 0, 1, 2, \ldots.$$

Clearly, if $\delta = 1$, then $S(t)$ is analytic.

In order to get the Gevrey regularity of the system (2.11), we need the following theorem established in [20, Theorem 13].

THEOREM 3.14. *Let A be an infinitesimal generator of a C_0-semigroup e^{At} and let A be a Riesz-spectral operator in a Hilbert space, that is, the generalized eigenfunctions of A form a Riesz basis in the Hilbert space. Then the following assertions are equivalent:*

1. *e^{At} is of Gevrey class $\delta \geq 1$ for $t > 0$.*
2. *There is $b > 0$, $a \in \mathbb{R}$ such that*

$$\sigma(A) \subset \left\{ \lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq a - b|\mathrm{Im}\lambda|^{1/\delta} \right\}. \tag{3.22}$$

Now we are in a position to establish the Gevrey regularity for the system (2.11).

THEOREM 3.15. *Assume that $\alpha > 0, \alpha \neq 1$, and $|\varpi| < \sqrt{\mu_1}$. Let A^{ϖ} be the operator defined by* (2.9)–(2.10)*. Then the semigroup $e^{A^{\varpi} t}$, generated by A^{ϖ}, is of a Gevrey class $\delta > 2$ with $t_0 = 0$.*

Proof. From (3.13) and (3.14), it is found that (3.22) is satisfied if we take $\delta = 2$, $b = |\ln r|$, and $a > 0$ large enough. Therefore, the semigroup $e^{A^{\varpi} t}$, generated by A^{ϖ}, is of a Gevrey class $\delta > 2$ with $t_0 = 0$. ∎

Remark 3.16. Note that when $\varpi = 0$, it is shown in [11] that the corresponding system operator A^0 generates a differentiable C_0-semigroup and the regularity for $e^{A^0 t}$ is extended to be a Gevrey class $\delta > 2$ with $t_0 = 0$ in [2].

4. The closed-loop system (1.1)–(1.2). The aim of this section is to prove the following result.

THEOREM 4.1. *Assume that each desired angular velocity ϖ satisfies $|\varpi| < \sqrt{\mu_1}$ and the feedback gain $\alpha \neq 1$. Then, for each initial data $\Phi_0 \in \mathcal{D}(\mathcal{A})$, the corresponding solution $\Phi(t)$ of the closed-loop system* (2.7) *exponentially tends to the equilibrium point $(0_{\mathcal{H}}, \varpi)$ in $\mathcal{X}$ as $t \to \infty$.*

Proof. First, recall that the closed-loop system (1.1)–(1.2) or equivalently (2.7) can be written in a form involving the operator A^{ϖ} (see the system (2.8)). Subsequently, invoking the facts that A^{ϖ} generates a C_0-semigroup $e^{A^{\varpi} t}$ in $\mathcal{H}$ (see Theorem 3.12) and $\mathcal{P}$ is continuously differentiable [25], we deduce that for any $\Phi_0 = (\phi_0, \omega_0) \in \mathcal{H} \times \mathbb{R}$, there is a unique local mild solution $\Phi = (\phi, \omega) \in C([0, T]; \mathcal{H} \times \mathbb{R})$ of the system (2.7), for $T > 0$, and Φ is given by the variation of constant formula [21]. Moreover, a regularity result [21] leads us to conclude that each local solution of (2.8), with initial data in $\mathcal{D}(A^{\varpi}) \times \mathbb{R}$, is a strong one. Then, using the same approach as in [24], one can show that each strong solution exists globally and is bounded. Thus, $\int_0^{+\infty} (\omega(t) - \varpi)^2 dt$ converges and the solution $(\phi(t), \omega(t))$ is bounded in $\mathcal{X}$. This implies, thanks to Barbalat's lemma [13], that $\lim_{t \to +\infty} \omega(t) = \varpi$. Herewith, for any $\nu > 0$, there exists T_0 sufficiently large such that for each $t \geq T_0$, we have

$$|\omega^2(t) - \varpi^2| < \nu. \tag{4.1}$$

Thereafter, inspired by the works of [25] and [14], one can establish the stability of the system (1.1)–(1.2). In fact, it suffices to consider the compact operator $P(u, v) = (0, u)$ for any $(u, v) \in \mathcal{H}$. Then, the solution $\Phi(t)$ of the global system (2.8) stemming from $\Phi_0 = (\phi_0, \omega_0) \in \mathcal{D}(\mathcal{A})$ can be written as $\Phi(t) = \big(\phi(t), \omega(t)\big)$, where $\phi(t) = (y(\cdot, t), y_t(\cdot, t))$ is the unique solution of the subsystem (2.11) perturbed by the operator $(\omega^2 - \varpi^2)P$. Indeed, given a positive real number T, the solution of (2.8) is given by

$$\phi(t) = e^{(t-T)A^{\varpi}}\phi(T) + \int_T^t e^{(t-s)A^{\varpi}}(\omega^2(s) - \varpi^2)P\phi(s)\,ds \ \ \forall t \geq T, \tag{4.2}$$

and $\omega(t)$ is solution of the ordinary differential equation

$$\dot{\omega}(t) = \frac{-\gamma(\omega - \varpi) - 2\,\omega(t) < y, y_t >_{L^2(0,1)}}{I_d + \|y\|^2_{L^2(0,1)}}. \tag{4.3}$$

Using Theorem 3.12 and (4.1)–(4.2), we get for t sufficiently large ($t \geq T_0$)

$$\|e^{\varsigma t}\phi(t)\|_{\mathcal{H}} \leq \kappa\|e^{\varsigma T}\phi(T)\|_{\mathcal{H}} + \nu\kappa\int_T^t \|e^{\varsigma s}\phi(s)\|_{\mathcal{H}}\,ds \tag{4.4}$$

for some positive constants κ and ς. Applying Gronwall's inequality to (4.4), we get the exponential stability of $\phi(t)$ in $\mathcal{H}$ provided that $\nu < \frac{\varsigma}{\kappa}$. Finally, one can show the nonuniform exponential convergence of ω toward ϖ in $\mathbb{R}$ by using exactly the same arguments as put forth in [25]. ▯

5. Numerical simulations. This section will be concerned with the validation, through some numerical examples, of the theoretical results previously stated and proved in this work. In particular, primary emphasis is placed on the illustration of the exponential stability result. Also, we shall give prominence to the importance of the critical value $\sqrt{\mu_1} \simeq 3.516$ which has been used as a sufficient condition for all our results.

For simplicity, we confine ourselves to the case $\omega(t) = \varpi > 0$ in the closed-loop system (2.7) whose energy-norm is (see (2.2))

$$E_0(t) = -\frac{1}{2}\varpi^2\int_0^1 y^2dx + \frac{1}{2}\int_0^1 (y_t^2 + y_{xx}^2)dx.$$

Throughout this section, we will assume the initial displacement $y_0(x) = \frac{x^2}{2}$ and the initial velocity $y_1(x) = \frac{x^2}{2}$. Subsequently, using MATLAB we shall consider three situations depending on the values of the feedback gain α. To be more precise, we will investigate the behavior of the system when $\alpha = 0.7$ or $\alpha = 1.5$, α has either small or large values, and finally $\alpha = 1$.

First situation: $\boldsymbol{\alpha = 0.7}$ or $\boldsymbol{\alpha = 1.5}$. Examining Figures 1–2, it can be seen that the system energy decays in a short time as long as $\varpi < \sqrt{\mu_1} \simeq 3.5160$. This agrees with Theorem 4.1 with regard to the stability of the system. Furthermore, the spectrum of the system lies in the left-half plane (see Figures 3–4). This validates Theorem 3.7 as the real parts of the eigenvalues are negative.

In turn, if $\varpi = 3.6 > \sqrt{\mu_1}$, we notice that the spectrum gets at least one eigenvalue with nonnegative real part (see Figure 5). This clearly ties in with the statement of Lemma 3.6.

In a similar vein to Figures 1–5, the outcomes of Theorem 3.7 and Lemma 3.6 are also depicted in Figures 6–8 when the feedback gain $\alpha = 1.5$. The reader can observe that we have exactly the same conclusions whether $\varpi < \sqrt{\mu_1}$ or $\varpi > \sqrt{\mu_1}$.

Second situation: $\boldsymbol{\alpha}$ has small or large values. Motivated by the interest in the behavior of the spectrum when α takes either small or large values, two cases will be treated.

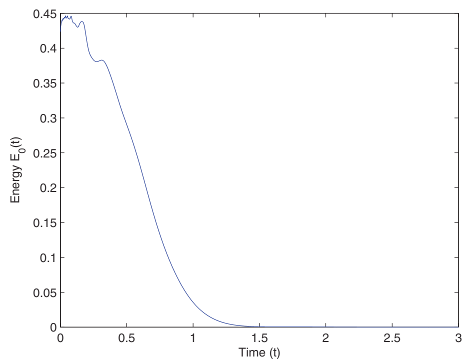

FIG. 1. *Energy decay with* $\varpi = 2$ *and* $\alpha = 0.7$.

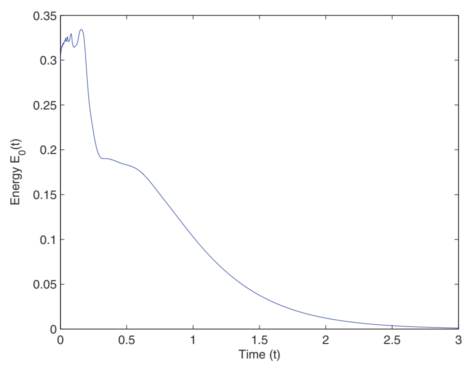

FIG. 2. *Energy decay with* $\varpi = 3$ *and* $\alpha = 0.7$.

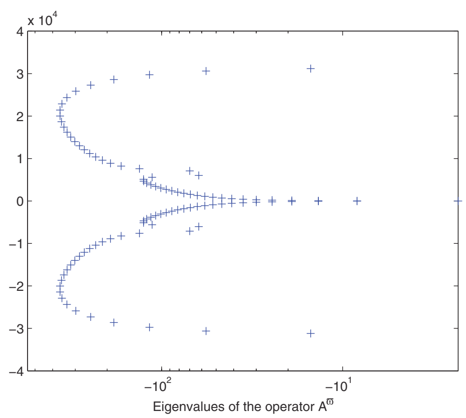

FIG. 3. *Spectrum with* $\varpi = 2$ *and* $\alpha = 0.7$.

In the first case, we assume that $\varpi = 2$ (spectrum with negative real part). Here, it is apparent from previous figures such as Figure 3 and Figures 9–10 that the spectrum moves to the right as $\alpha < 1$ gets small values. This illustrates the outcome of Theorem 3.9, namely, the second situation in (3.13). This can be explained by the fact that (3.13) shows that the real part of the eigenvalues is governed, as a function of α, by the coefficients $\ln(\frac{1-\alpha}{\alpha+1})$, which is a decreasing negative function.

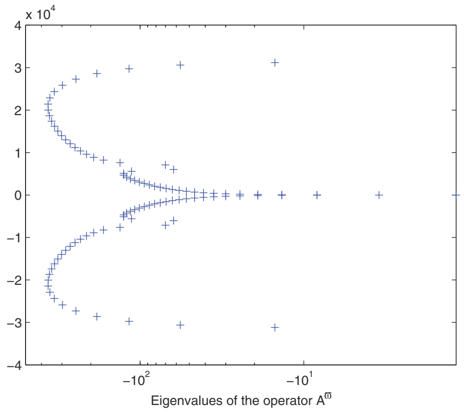

FIG. 4. *Spectrum with* $\varpi = 3$ *and* $\alpha = 0.7$.

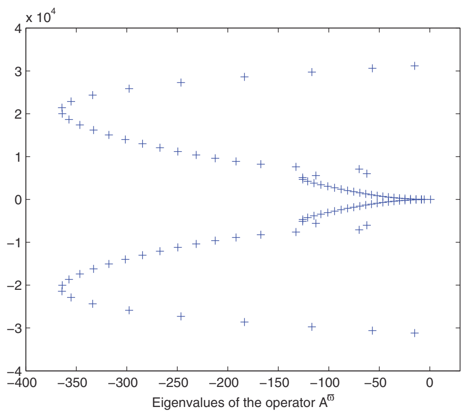

FIG. 5. *Spectrum with* $\varpi = 3.6$ *and* $\alpha = 0.7$*: eigenvalue with real part equals* 1.

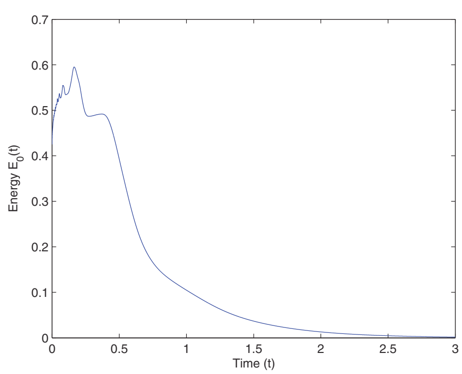

FIG. 6. *Energy decay with* $\varpi = 2$ *and* $\alpha = 1.5$.

The finding of Theorem 3.9, namely, the first situation in (3.13), is illustrated in Figures 11–12 together with Figure 7. This is due to the fact that $\ln(\frac{\alpha-1}{\alpha+1})$ is an increasing negative function and hence the real part of the eigenvalues increases for larger values of $\alpha > 1$.

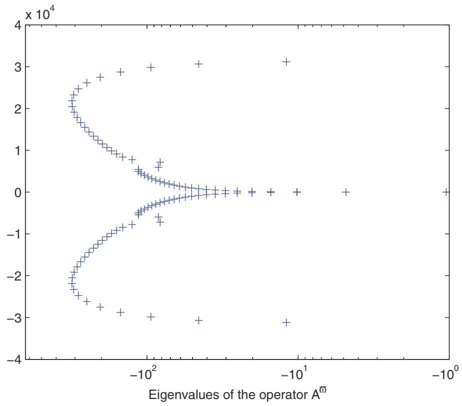

FIG. 7. *Spectrum with* $\varpi = 2$ *and* $\alpha = 1.5$.

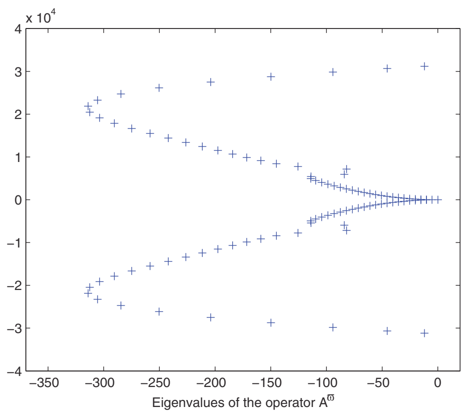

FIG. 8. *Spectrum with* $\varpi = 3.6$ *and* $\alpha = 1.5$*: eigenvalue with positive real part.*

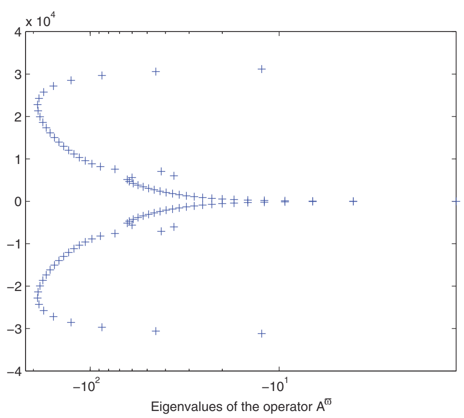

FIG. 9. *Spectrum with* $\varpi = 2$ *and* $\alpha = 0.4$.

To end this part, let us deal with the second case $\varpi = 4$ (at least one eigenvalue with positive real part). Thereafter, the spectrum moves to the right whether $\alpha > 1$ gets larger values (see Figures 13–14) or smaller values $\alpha < 1$ (see Figures 15–16).

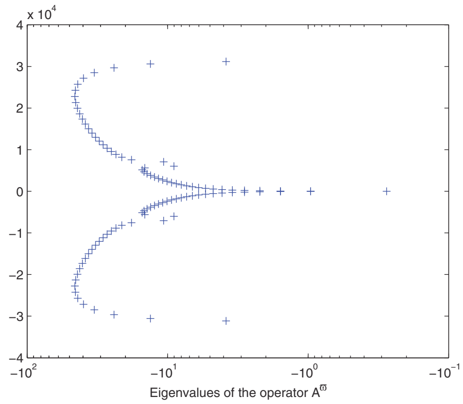

FIG. 10. *Spectrum with* $\varpi = 2$ *and* $\alpha = 0.1$.

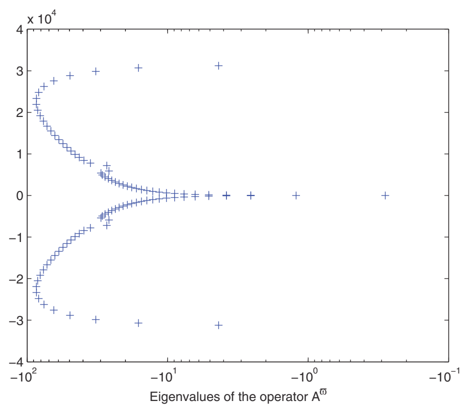

FIG. 11. *Spectrum with* $\varpi = 2$ *and* $\alpha = 5$.

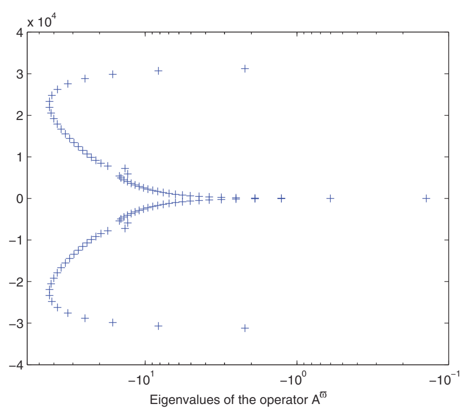

FIG. 12. *Spectrum with* $\varpi = 2$ *and* $\alpha = 10$.

Third situation: $\alpha = 1$. Besides the situations considered heretofore, there is an interesting instance in which the spectrum has a definite distinction in its shape (see Figures 17–18). Notwithstanding, Figures 17–18 depict the same conclusions as before with regard to the eigenvalues real part. To be more precise, when $\alpha = 1$,

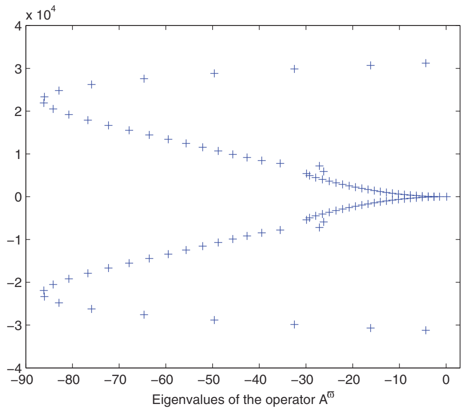

FIG. 13. *Spectrum with* $\varpi = 4$ *and* $\alpha = 5$.

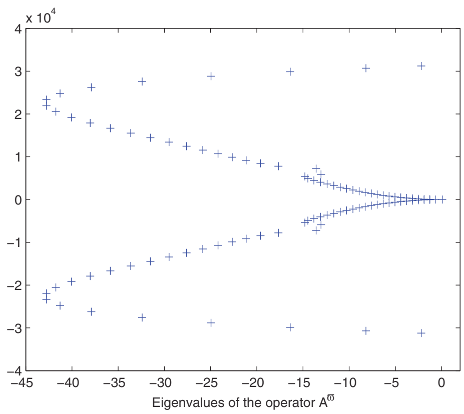

FIG. 14. *Spectrum with* $\varpi = 4$ *and* $\alpha = 10$.

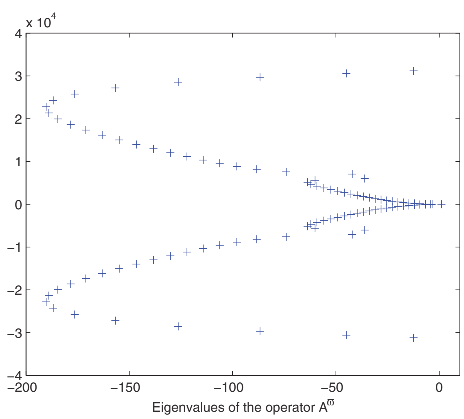

FIG. 15. *Spectrum with* $\varpi = 4$ *and* $\alpha = 0.4$.

Figure 17 illustrates the outcome of Theorem 3.7 as the real part is negative, while Figure 18 confirms Lemma 3.6 since the condition $\varpi < \sqrt{\mu_1}$ is violated and hence there is at least one eigenvalue with positive real part.

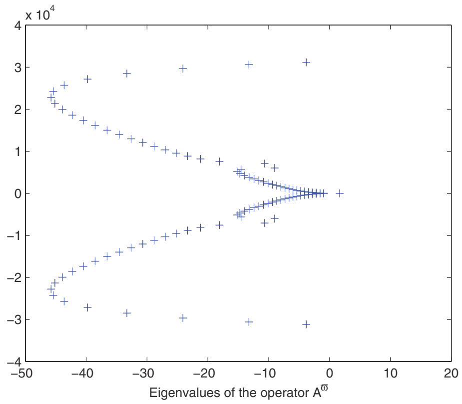

FIG. 16. *Spectrum with* $\varpi = 4$ *and* $\alpha = 0.1$.

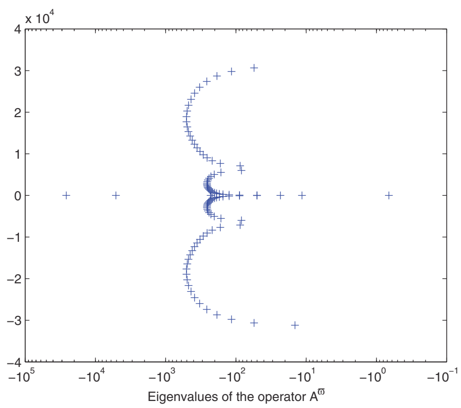

FIG. 17. *Spectrum with* $\varpi = 3$ *and* $\alpha = 1$.

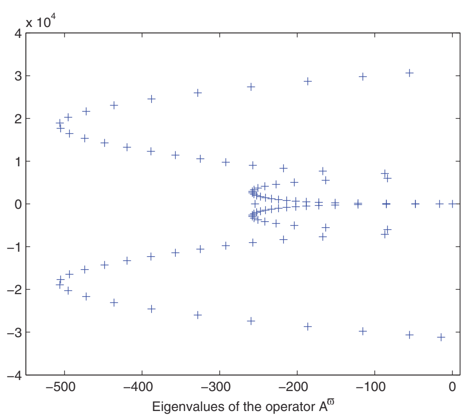

FIG. 18. *Spectrum with* $\varpi = 3.6$ *and* $\alpha = 1$.

Finally, it is worth mentioning that in order to highlight the first eigenvalue of the spectrum, especially in the event that its real part is nonnegative, we had to zoom some figures (as in Figures 5, 8, 13–16, and 18). Of course, whether we scale or not, the general shape of the spectrum remains the same in each specific situation.

6. Conclusion. This paper addressed the stabilization problem of a flexible beam attached to the center of a rotating disk. We proved that for any given small constant angular velocity less than the square root of the first eigenvalue of the related self-adjoint positive operator, the beam vibrations are suppressed and the disk rotates with a constant angular velocity provided that both torque control and boundary shear force control are exerted, respectively, on the disk and at the free end of the beam. Several numerical simulations are also conducted to ascertain the theoretical results.

Appendix. In this appendix, we present and show some lemmas which are needed for the proof of Theorem 3.7. We shall proceed as in [15] but with a number of changes born out of necessity.

First of all, integrating the differential equation of (3.4) over $(1, x)$ twice yields

$$f''(x) - \alpha\lambda f'(1)(x-1) + \left(\lambda^2 - \varpi^2\right)\int_1^x (x-\tau)f(\tau)d\tau = 0.$$

Replacing x by $1-x$ yields

$$f''(1-x) + \alpha\lambda f'(1)x + \left(\lambda^2 - \varpi^2\right)\int_1^{1-x} (1-x-\tau)f(\tau)d\tau = 0.$$

Let

$$\phi(x) := \int_1^{1-x} (1-x-\tau)f(\tau)d\tau.$$

Consequently, $\phi''(x) = f(1-x)$ and $\phi \neq 0$. A direct computation shows that ϕ satisfies

$$\begin{cases} \lambda^2\phi(x) - \alpha x\lambda\phi'''(0) + \phi''''(x) - \varpi^2\phi(x) = 0, & 0 < x < 1, \\ \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0, \end{cases} \tag{6.1}$$

and hence $\lambda \in \sigma(A^{\varpi})$ if and only if there exist $\phi \neq 0 \in H^2(0,1)$ and $\lambda \in \mathbb{C}$ satisfying (6.1). Moreover, (6.1) can be rewritten as follows:

$$(\lambda^2 - \varpi^2)\phi(x) - \alpha x\lambda\phi'''(0) + \mathbf{B}\phi(x) = 0, \tag{6.2}$$

where $\mathbf{B}$ is a linear operator in $L^2(0,1)$ defined by (2.3). The main properties of $\mathbf{B}$ are settled in the following lemma [15].

LEMMA 6.1. *Let* $\mathbf{B}$ *be given by* (2.3) *and let* $\{(\mu_n, \varphi_n(x))\}_{n=1}^{\infty}$ *be the eigenpairs of* $\mathbf{B}$. *Then* $\mathbf{B}$ *is a self-adjoint and positive operator. Moreover, for* $n \geq 1$, *the following properties hold:*

1. $\mu_n = \beta_n^4$, *with* β_n *satisfying* $1 + \cos(\beta_n)\cosh(\beta_n) = 0$ *and* $\beta_n = \mathcal{O}(n) > 0$.
2. *The set* $\{\varphi_n(x)\}_{n=1}^{\infty}$ *forms an orthogonal basis on* $L^2(0,1)$ *and* $\varphi_n(x)$ *has the following asymptotic expression:*

$$\varphi_n(x) = -\frac{1+\gamma_n}{2}\exp(\beta_n x) - \frac{1-\gamma_n}{2}\exp(-\beta_n x) + \gamma_n\sin(\beta_n x) + \cos(\beta_n x),$$

where

$$\gamma_n = -\frac{\exp(\beta_n) - \sin(\beta_n) + \cos(\beta_n)}{\exp(\beta_n) + \sin(\beta_n) + \cos(\beta_n)} \to -1 \qquad \text{as } n \to \infty$$

and $\gamma_n < 0$ *for every* $n \geq 1$.

3. *Each* $y \in L^2(0,1)$ *can be expressed as follows:*

$$y = \sum_{n=1}^{\infty} b_n \varphi_n(x), \tag{6.3}$$

where

$$\begin{cases} b_n = -\dfrac{2}{\beta_n^2}\|\varphi_n\|^{-2}, \quad \beta_n^2 = -\dfrac{1}{2}\varphi_n''(0), \\ \beta_n^3 = -\dfrac{1}{2\gamma_n}\varphi_n'''(0), \quad \|\varphi_n\| = \mathcal{O}(1). \end{cases} \tag{6.4}$$

Our next result is the following.

LEMMA 6.2. *Let* $\lambda \in \sigma(A^{\varpi})$ *with* $\mathrm{Im}(\lambda) \neq 0$ *and let* $\alpha > 0$. *Then,* λ *satisfies the following equation involving the entire function:*

$$F(\lambda) = 1 + \frac{1}{2}\alpha\lambda + 4\alpha \sum_{n=1}^{\infty} \frac{\lambda(\lambda^2 - \varpi^2)}{\lambda^2 - \varpi^2 + \mu_n}\gamma_n \beta_n^{-3}\|\phi_n\|^{-2} = 0. \tag{6.5}$$

Proof. According to Lemma 3.4, we have $\mathrm{Re}(\lambda) \neq 0$, for all $\lambda \in \sigma(A^{\varpi})$, which necessitates that for $\lambda \in \sigma(A^{\varpi})$ with $\mathrm{Im}(\lambda) \neq 0$, we have $\lambda^2 - \varpi^2 \in \rho(-\mathbf{B})$. Thereafter, solving (6.2) yields

$$\phi(x) = \alpha\lambda\phi'''(0)((\lambda^2 - \varpi^2)I + \mathbf{B})^{-1}x. \tag{6.6}$$

Moreover, integrating both sides of (6.2) with respect to x from 0 to 1, we obtain

$$\begin{aligned} 0 &= (\lambda^2 - \varpi^2)\int_0^1 \phi(x)dx - \frac{1}{2}\alpha\lambda\phi'''(0) + \int_0^1 \mathbf{B}\phi(x)dx \\ &= (\lambda^2 - \varpi^2)\int_0^1 \phi(x)dx - \frac{1}{2}\alpha\lambda\phi'''(0) - \phi'''(0). \end{aligned}$$

Thus, we get

$$(\lambda^2 - \varpi^2)\int_0^1 \phi(x)dx - \left[1 + \frac{1}{2}\alpha\lambda\right]\phi'''(0) = 0. \tag{6.7}$$

Substituting (6.6) into (6.7), we obtain

$$\left\{\alpha\lambda(\lambda^2 - \varpi^2)\int_0^1 (\lambda^2 - \varpi^2 + \mathbf{B})^{-1}x dx - \left[1 + \frac{1}{2}\alpha\lambda\right]\right\}\phi'''(0) = 0. \tag{6.8}$$

Since $\phi'''(0) = 0$ implies that $\phi(x) = 0$, we can claim that $\lambda \in \sigma(A^{\varpi})$ with $\mathrm{Im}(\lambda) \neq 0$ if and only if λ satisfies

$$F(\lambda) = 1 + \frac{1}{2}\alpha\lambda - \alpha\lambda(\lambda^2 - \varpi^2)\int_0^1 (\lambda^2 - \varpi^2 + \mathbf{B})^{-1}x dx = 0. \tag{6.9}$$

Furthermore, thanks to (6.3) and (6.9), we have

$$\begin{aligned}
F(\lambda) &= 1 + \frac{1}{2}\alpha\lambda - \alpha\lambda(\lambda^2 - \varpi^2)\int_0^1 (\lambda^2 - \varpi^2 + \mathbf{B})^{-1}\left[\sum_{n=1}^{\infty} b_n\phi_n(x)\right]dx \\
&= 1 + \frac{1}{2}\alpha\lambda - \alpha\lambda(\lambda^2 - \varpi^2)\sum_{n=1}^{\infty} b_n \int_0^1 (\lambda^2 - \varpi^2 + \mathbf{B})^{-1}\phi_n(x)dx \\
&= 1 + \frac{1}{2}\alpha\lambda - \alpha\lambda(\lambda^2 - \varpi^2)\sum_{n=1}^{\infty} \frac{b_n}{\lambda^2 - \varpi^2 + \mu_n}\int_0^1 \phi_n(x)dx \\
&= 1 + \frac{1}{2}\alpha\lambda - \alpha\lambda(\lambda^2 - \varpi^2)\sum_{n=1}^{\infty} \frac{b_n}{\lambda^2 - \varpi^2 + \mu_n}\frac{1}{\mu_n}\int_0^1 \phi_n''''(x)dx \\
&= 1 + \frac{1}{2}\alpha\lambda + \alpha\lambda(\lambda^2 - \varpi^2)\sum_{n=1}^{\infty} \frac{b_n}{\lambda^2 - \varpi^2 + \mu_n}\frac{1}{\mu_n}\phi_n'''(0) \\
&= 1 + \frac{1}{2}\alpha\lambda - 2\alpha\lambda(\lambda^2 - \varpi^2)\sum_{n=1}^{\infty} \frac{b_n}{\lambda^2 - \varpi^2 + \mu_n}\frac{\gamma_n}{\mu_n}\beta_n^3 \\
&= 1 + \frac{1}{2}\alpha\lambda + 4\alpha\sum_{n=1}^{\infty} \frac{\lambda(\lambda^2 - \varpi^2)}{\lambda^2 - \varpi^2 + \mu_n}\frac{\gamma_n}{\mu_n}\beta_n\|\phi_n\|^{-2} \\
&= 1 + \frac{1}{2}\alpha\lambda + 4\alpha\sum_{n=1}^{\infty} \frac{\lambda(\lambda^2 - \varpi^2)}{\lambda^2 - \varpi^2 + \mu_n}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2} = 0,
\end{aligned}$$

which is the desired result (6.5). ∎

LEMMA 6.3. *Let λ, with $\mathrm{Im}(\lambda) \neq 0$, be a zero of $F(\lambda)$, and let $\alpha > 0$. Then $\mathrm{Re}(\lambda) < 0$.*

Proof. Let $\lambda = a + bi$, $a, b \neq 0 \in \mathbb{R}$ be a zero of $F(\lambda)$. Noting that

$$\lambda^2 = (a + bi)^2 = a^2 - b^2 + 2abi$$

and

$$\begin{aligned}
\frac{\lambda(\lambda^2 - \varpi^2)}{\lambda^2 - \varpi^2 + \mu_n} &= \frac{\lambda(\lambda^2 - \varpi^2)(\overline{\lambda^2} - \varpi^2 + \mu_n)}{|\lambda^2 - \varpi^2 + \mu_n|^2} \\
&= \frac{\left[|\lambda|^4 - (\lambda^2 + \overline{\lambda^2})\varpi^2 - \varpi^4 - \varpi^2\mu_n + \lambda^2\mu_n\right]\lambda}{|\lambda^2 - \varpi^2 + \mu_n|^2} \\
&= \frac{\left[|\lambda|^4 - 2(a^2 - b^2)\varpi^2 - \varpi^4 - \varpi^2\mu_n + (a^2 - b^2 + 2abi)\mu_n\right]\lambda}{|\lambda^2 - \varpi^2 + \mu_n|^2} \\
&= \frac{\left[|\lambda|^4 - 2(a^2 - b^2)\varpi^2 - \varpi^4 - \varpi^2\mu_n + (a^2 - b^2)\mu_n\right]a - 2ab^2\mu_n}{|\lambda^2 - \varpi^2 + \mu_n|^2} \\
&\quad + i\frac{\left[|\lambda|^4 - 2(a^2 - b^2)\varpi^2 - \varpi^4 - \varpi^2\mu_n + (a^2 - b^2)\mu_n\right]b + 2a^2b\mu_n}{|\lambda^2 - \varpi^2 + \mu_n|^2}
\end{aligned}$$

$$= \frac{\left[|\lambda|^4 - 2(a^2-b^2)\varpi^2 - \varpi^4 - \varpi^2\mu_n + (a^2-3b^2)\mu_n\right]a}{|\lambda^2-\varpi^2+\mu_n|^2} + i\,\frac{\left[|\lambda|^4 - 2(a^2-b^2)\varpi^2 - \varpi^4 - \varpi^2\mu_n + (3a^2-b^2)\mu_n\right]b}{|\lambda^2-\varpi^2+\mu_n|^2},$$

we deduce from (6.5) that

(6.10)
$$1+\frac{1}{2}\alpha a+4\alpha a\sum_{n=1}^{\infty}\frac{|\lambda|^4 - 2(a^2-b^2)\varpi^2 - \varpi^4 - \varpi^2\mu_n + (a^2-3b^2)\mu_n}{|\lambda^2-\varpi^2+\mu_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}=0$$

and

(6.11)
$$\frac{1}{2}\alpha b+4\alpha b\sum_{n=1}^{\infty}\frac{|\lambda|^4 - 2(a^2-b^2)\varpi^2 - \varpi^4 - \varpi^2\mu_n + (3a^2-b^2)\mu_n}{|\lambda^2-\varpi^2+\mu_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}=0.$$

From (6.10), we have $a \neq 0$. Observing that $b \neq 0$, it follows from (6.11) that

$$\frac{1}{2}\alpha+4\alpha\sum_{n=1}^{\infty}\frac{|\lambda|^4 - 2(a^2-b^2)\varpi^2 - \varpi^4 - \varpi^2\mu_n + (3a^2-b^2)\mu_n}{|\lambda^2-\varpi^2+\mu_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}=0$$

or

$$\frac{1}{2}\alpha+4\alpha\sum_{n=1}^{\infty}\frac{|\lambda|^4 - 2(a^2-b^2)\varpi^2 - \varpi^4 - \varpi^2\mu_n}{|\lambda^2-\varpi^2+\mu_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2} = -4\alpha\sum_{n=1}^{\infty}\frac{\mu_n(3a^2-b^2)}{|\lambda^2-\varpi^2+\mu_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}.$$

On the other hand, (6.10) yields

$$\begin{aligned}0 &= \frac{1}{a}+\frac{1}{2}\alpha+4\alpha\sum_{n=1}^{\infty}\frac{|\lambda|^4 - 2(a^2-b^2)\varpi^2 - \varpi^4 - \varpi^2\mu_n + (a^2-3b^2)\mu_n}{|\lambda^2-\varpi^2+\mu_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}\\
&= \frac{1}{a}+4\alpha\sum_{n=1}^{\infty}\frac{-\mu_n(3a^2-b^2)+\mu_n(a^2-3b^2)}{|\lambda^2-\varpi^2+\mu_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}\\
&= \frac{1}{a}+4\alpha\sum_{n=1}^{\infty}\frac{-2\mu_n(a^2+b^2)}{|\lambda^2+\mu_n|^2}\gamma_n\beta_n^{-3}\|\phi_n\|^{-2}\\
&= \frac{1}{a}+4\alpha\sum_{n=1}^{\infty}\frac{-2\beta_n|\lambda|^2}{|\lambda^2+\mu_n|^2}\gamma_n\|\phi_n\|^{-2},\end{aligned}$$

and hence

$$\frac{1}{a}=4\alpha\sum_{n=1}^{\infty}\frac{2\beta_n|\lambda|^2}{|\lambda^2+\mu_n|^2}\gamma_n\|\phi_n\|^{-2}.$$

Lastly, recalling that $a=\mathrm{Re}(\lambda)$, $\beta_n=\mathcal{O}(n)>0$, and $\gamma_n<0$ (see the first and second assertions of Lemma 6.1), the conclusion immediately follows. ∎

Acknowledgments. The authors are grateful to the editor and anonymous referees for their valuable comments and useful suggestions.

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