STABILITY OF A DAMPED HYPERBOLIC TIMOSHENKO SYSTEM COUPLED WITH A HEAT EQUATION

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ABSTRACT

We consider a one-dimensional damped hyperbolic Timoshenko beam that is coupled with a heat equation. When its wave speeds are different, it is known that the Timoshenko beam that is coupled with a heat equation, under Cattaneo’s law, does not have exponential stability. With two internal dampings being introduced, in this paper, we show that the system under Cattaneo’s law is exponentially stable. We also show the exponential stability when the system is considered under Fourier’s law.

Key Words: Stability, Timoshenko beam, heat equation, Cattaneo’s law, Fourier’s law.

I. INTRODUCTION

In this paper, we consider the Timoshenko beams that are coupled with a heat equation and are modeled by Cattaneo’s law and Fourier’s law as: for $t \in (0, \infty) \times (0, L)$,

\[
\begin{align*}
\rho_1 \phi_x(t, x) - k(\phi_x(t, x) + \psi(t, x)), + \alpha \phi(t, x) &= 0 \\
\rho_2 \psi_x(t, x) - b \psi_x(t, x) + k(\phi_x(t, x) + \psi(t, x)) + \delta \theta_x(t, x) &= 0 \\
\rho_2 \theta_x(t, x) + q_1(t, x) + \partial_x(\psi(t, x)) &= 0 \\
\gamma \phi_x(t, x) + \beta q(t, x) + \theta_x(t, x) &= 0
\end{align*}
\]

(1)

and

\[
\begin{align*}
\rho_1 \phi_x(t, x) - k(\phi_x(t, x) + \psi(t, x)) + \alpha \phi(t, x) &= 0 \\
\rho_2 \psi_x(t, x) - b \psi_x(t, x) + k(\phi_x(t, x) + \psi(t, x)) + \delta \theta_x(t, x) &= 0 \\
\rho_2 \theta_x(t, x) - \frac{1}{\beta} \theta_x(t, x) + \partial_x(\psi(t, x)) &= 0
\end{align*}
\]

(2)

where the functions $\phi$ and $\psi$ describe the transverse displacement of a beam and the rotation angle of a filament, respectively. $\theta$ and $q$ denote the temperature and the heat flux, respectively; the physical parameters $\rho_1$, $k$, $\rho_2$, $b$, $\delta$, $\rho_3$, and $\beta$ are positive constants; and $\alpha$, $\gamma > 0$ denote the damping constants with respect to $\phi$, $\psi$. For more detailed physical background of the problems, refer to [1].

It is known that the positive parameter $\tau$ is the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature [2] and that $\tau = 0$ represents Fourier’s law, while $\tau > 0$ represents Cattaneo’s law.

The initial and boundary conditions for (1) and (2) are given as follows:

\[
\begin{align*}
\phi(0, x) &= \phi_0(x), \phi_x(0, x) = \phi_1(x) \\
\psi(0, x) &= \psi_0(x), \psi_x(0, x) = \psi_1(x) \\
\theta(0, x) &= \theta_0(x), q(0, x) = q_0(x)
\end{align*}
\]

(3)

\[
\begin{align*}
\phi_x(t, 0) &= \phi_x(t, L) = \psi(t, 0) = 0 \\
\psi(t, L) &= q(t, 0) = q(t, L) = 0
\end{align*}
\]

(4)

and

\[
\begin{align*}
\phi(0, x) &= \phi_0(x), \phi_x(0, x) = \phi_1(x) \\
\psi(0, x) &= \psi_0(x), \psi_x(0, x) = \psi_1(x) \\
\theta(0, x) &= \theta_0(x)
\end{align*}
\]

(5)

\[
\begin{align*}
\phi_x(t, 0) &= \phi_x(t, L) = \psi(t, 0) = 0 \\
\psi(t, L) &= \theta(t, 0) = \theta(t, L) = 0
\end{align*}
\]

(6)

Many results are already available in the literature. For the case of $\tau = 0$, if $\alpha = \gamma = 0$, that is, there are no internal dampings in the system, Soufyane [3] showed the exponential stability of the system under the condition that the system (1) has the same wave speeds, i.e.,

\[
\frac{\rho_1}{k} = \frac{\rho_2}{b}
\]

(7)

where $\delta \theta_x$ is replaced by a control function $b(x) \psi_x$, $b > 0$. Rivera and Sare [4] studied the stability of Timoshenko vibrating systems with past history. Sare and Racke [2] systematically studied the undamped system (1) with $\alpha = \gamma = 0$ and obtained the exponential stability of the system under Condition (7). When the wave speeds are different, the
exponential stability of (1) with one viscous damping, that is, \( \gamma = 0 \), is established [5] by the multiplier method. A weaker type of dissipation being presented only for \( \psi \) was considered [6] where \( \hat{\partial}_t \) is replaced by a memory term \( \int_0^t g(t-s)\psi_{ss}(s,x)dx \). Recently, Wang et al. [7] considered a Schrödinger equation that is coupled with a heat equation via the boundary coupling and showed the exponential stability and Gevrey regularity of the closed loop system. The stability for a system of serially connected vibrating strings with discontinuous displacement was considered in [8], and the spectral analysis for Euler-Bernoulli beam with input delay in the boundary control was presented in [9].

When \( \alpha = 0 \) in (2), with an approach adopted in Chen [10], Rivera and Racke [11] proved the exponential stability of the system if and only if \( \frac{\rho_1}{k} = \frac{\rho_2}{b} \) holds. Ammar-Khodja [6] proved the exponential stability of the common linear Timoshenko system

\[
\begin{align*}
\rho_1\psi_{tt} - k(\varphi_1 + \psi_1) &= 0 \\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_1 + \psi_1) + d\psi_1 &= 0
\end{align*}
\]

when \( \frac{\rho_1}{k} = \frac{\rho_2}{b} \) holds. When \( \alpha = \beta = 0 \), the first and second equations of (2) build a purely hyperbolic system [1], in which the energy is conserved and solutions, with respect to the energy, do not decay at all. The pair of the second and third equations of (2), those of moelasticity for \( k = 0 \) and those of linear thermoelasticity, for which the exponential stability for various boundary conditions has been proven [6].

In this paper, the frequency domain method is adopted to show the exponential stability for both Systems (1) and (2). For System (1), we assume that \( \rho_1, \rho_2 \neq 0 \) in this paper. It is noted that, since the heat equation is strongly coupled with the second equation of the Timoshenko beam, System (1) will be exponentially stable with \( \rho_1 \neq 0 \) and \( \rho_2 = 0 \), which can be deduced by the same argument in the paper (see also [5] for the argument with the multiplier methods).

The paper is organized as follows. In Section II, we present the well-posedness and exponential stability for System (1), and the well-posedness and exponential stability of System (2) are shown in Section III.

II. STABILITY OF SYSTEM (1)

In this section, we show the exponential stability of System (1) for the case \( \tau > 0 \). It is known that the total energy of the system (1) is given by:

\[
E(t) = \frac{1}{2} \int_0^L \left[ \rho_1 \varphi_1^2(t,x) + k(\varphi_1(t,x) + \psi_1(t,x))^2 + \rho_2 \psi_1^2(t,x) + b\psi_{xx}^2(t,x) + \tau \varphi_1^2(t,x) \right]dx.
\]

(8)

So, it is natural to consider the system (1) in the energy space. Let

\[
\begin{align*}
\mathcal{L}_w(0,L) &= \left\{ v \in L^2(0,L) \left| \int_0^L v(x)dx = 0 \right. \right\}, \\
\mathcal{H}_w(0,L) &= \left\{ v \in H^1(0,L) \left| \int_0^L v(x)dx = 0 \right. \right\}
\end{align*}
\]

and let

\[
\mathcal{H}_1 = H^1_0(0,L) \times \mathcal{L}_w(0,L) \times \mathcal{H}_w(0,L)
\]

be the Hilbert space with norm [2]

\[
\|v\|^2 = \int_0^L \left[ k[V_1(x)]^2 + \rho_1[V_2(x)]^2 + b[V_3(x)]^2 + \rho_2[V_4(x)]^2 + \rho_3[V_5(x)]^2 + \tau[V_6(x)]^2 \right]dx
\]

where \( V = (V_1, V_2, V_3, V_4, V_5, V_6) \in \mathcal{H}_1 \) and the subscript “x” denotes the differentiation of a function with respect to the space variable \( x \). Now, we define a system operator \( \mathcal{A}_1 \) as:

\[
\mathcal{A}_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{k}{\rho_1} & -\alpha & \frac{k}{\rho_1} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{b\omega_i^2}{\rho_2} & 0 & -\gamma & -\delta & 0 & 0 \\
0 & 0 & 0 & -\frac{\delta}{\rho_1} & 0 & -\frac{1}{\tau} \\
0 & 0 & 0 & 0 & -\frac{1}{\tau} & -\frac{\beta}{\tau}
\end{pmatrix}
\]

(10)

with the domain of \( \mathcal{A}_1 \):

\[
D(\mathcal{A}_1) = \left\{ V \in \mathcal{H}_1 : V_1, V_2, V_3 \in H^1_0(0,L), \quad V_4, V_5, V_6 \in H^1_w(0,L) \right\}
\]

(11)

Then, the system (1) can be rewritten as an evolution equation in \( \mathcal{H}_1 \):

\[
V_t(t) = \mathcal{A}_1 V(t), \quad V(t = 0) = V_0,
\]

(12)

where \( V(t) = (\varphi_1(t), \psi_1(t), \psi_2(t), \varphi_2(t), \psi_3(t)) \) and \( \dot{V}_0 = (\varphi_0, \psi_0, \psi_0, \varphi_0, \psi_0) \) is the initial condition. Here, the superscript “T” denotes the transpose of a vector or a matrix.

Now, we show the well-posedness of System (12).
Lemma II.1. Let $\mathcal{A}_1$ be defined by (10) and (11). Then, $\mathcal{A}_1$ is the infinitesimal generator of a $C_0$-semigroup of contractions $e^{\lambda t}$.

Proof. First, we show that $\mathcal{A}_1$ is dissipative. For each $V \in D(\mathcal{A}_1)$, where $V = (V_1, V_2, \ldots, V_6)^T$, we have

$$\mathcal{A}_1 V = \left( V_2 - \frac{k}{\rho_1} (V_{1,s} + V_{3,s}) - \alpha V_2, V_4, b, V_{3,s} - \frac{k}{\rho_2} (V_{1,s} + V_3), -\gamma V_4 - \frac{\delta}{\rho_2} V_{5,s} - \frac{1}{\rho_3} V_{4,s} - \frac{1}{\tau} V_{5,s} - \frac{\beta}{\tau} V_6 \right)^T.$$

So, we obtain:

$$\Re \langle \mathcal{A}_1 V, V \rangle_{\mathcal{H}_1} = -\alpha \rho_1 \int_0^T \| V_2 \|_1^2 \, dt - \gamma \rho_2 \int_0^T \| V_4 \|_1^2 \, dt - \frac{1}{\tau} \int_0^T \| V_{5,s} \|_1^2 \, dt \leq 0.$$

Hence, $\mathcal{A}_1$ is dissipative. Next, we show that $0 \in \rho(\mathcal{A}_1)$.

For any $F = (f_0, f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}_1$, consider the following equation,

$$\mathcal{A}_1 V = F$$

then we have

$$V_2(x) = f_0(x), \quad V_4(x) = f_1(x),$$

and $V_3(x), V_5(x), V_6(x)$ satisfy

$$k(V_{1,s}(x) + V_3(x)) - \alpha \rho_1 V_2(x) = \rho_1 f_2(x),$$

$$b V_{3,s}(x) - k(V_{1,s}(x) + V_3(x)) - \gamma \rho_2 V_4(x) - \delta V_{3,s}(x) = \rho_2 f_4(x),$$

$$-\delta V_{4,s}(x) - V_{6,s}(x) = \rho_1 f_3(x),$$

$$-V_{5,s}(x) - \beta V_6(x) = \tau f_5(x),$$

with the boundary conditions

$$\begin{align*}
V_{1,s}(0) &= V_{1,s}(L) = V_{1}(0) = V_{1}(L) = 0, \\
V_{4,s}(0) &= V_{4,s}(L) = V_{4}(0) = V_{4}(L) = 0.
\end{align*}$$

From (17) and $V_6(0) = V_4(0) = 0$, we have:

$$V_6(x) = -\delta f_3(x) - \rho_1 \int_0^x f_5(y) \, dy.$$
Theorem II.2. Let \( \mathcal{A}_t \) be defined by (10) and (11). Then, the semigroup \( e^{\mathcal{A}_t} \) generated by the operator \( \mathcal{A}_t \) is exponentially stable, i.e., there are two positive constants \( M, \mu \) such that
\[
\|S(t)\| \leq Me^{-\mu t}.
\]

By Theorem II.1, in order to show the exponential stability of System (12), we need to estimate the resolvent of \( \mathcal{A}_t \). By the resolvent equation
\[
V = (i\lambda - \mathcal{A}_t)^{-1}F, \quad \forall F \in \mathcal{H}_t, \lambda \in \mathbb{R}, \lambda \neq 0,
\]
where \( V = (V_1, \ldots, V_6) \in D(\mathcal{A}_t) \), we get
\[
i\lambda V_1 - V_2 = f_1, \tag{23}
\]
\[
i\lambda \rho V_2 - k(V_{a_1} + V_5) + \alpha \rho V_2 = \rho_3 f_2, \tag{24}
\]
\[
i\lambda V_3 - V_4 = f_3, \tag{25}
\]
\[
i\lambda \rho V_4 + k(V_{a_1} + V_5) - b V_{a_{3,xx}} + \gamma \rho V_4 + \delta V_{a_3,xx} = \rho_3 f_4, \tag{26}
\]
\[
i\lambda \rho V_5 + \delta V_{a_3,xx} + V_{a_3} = \rho_3 f_5, \tag{27}
\]
\[
i\lambda \tau V_6 + V_{a_3} + \beta V_{a_6} = \tau f_6. \tag{28}
\]

Now, we present the following four lemmas to show Theorem II.2.

Lemma II.2. Let \( \mathcal{A}_t \) be defined by (10) and (11). Then, the resolvent set \( \mathcal{R}(\mathcal{A}_t) \) of \( \mathcal{A}_t \) contains the imaginary axis.

Proof. We prove the result by contradiction. If the claim is not true, then there exist \( \omega \in \mathbb{R} \), a sequence \( (\beta_n) \subset \mathbb{R} \) with \( \beta_n \to \omega \), \( |\beta_n| < |\omega| \), and a sequence of functions
\[
\begin{align*}
V_n &= (V_{a_1}, V_{a_2}, V_{a_3}, V_{a_4}, V_{a_5}, V_{a_6})^T \in D(\mathcal{A}_t) \|V_n\|_{\mathcal{H}_t} \\
&= \int_0^t [k|V_{a_1}(x)|^2 + \rho_3|V_{a_2}(x)|^2 + b|V_{a_3,xx}(x)|^2 \tag{30} \\
&\quad + \rho_3|V_{a_4}(x)|^2 + \rho_3|V_{a_5}(x)|^2 + \tau|V_{a_6}(x)|^2]dx = 1
\end{align*}
\]
such that, as \( n \to \infty \),
\[
\|i\beta_n I - \mathcal{A}_t V_n\|_{\mathcal{H}_t} \to 0, \tag{31}
\]
which yields
\[
\begin{align*}
\langle i\beta_n I - \mathcal{A}_t V_n, V_n \rangle_{\mathcal{H}_t} &\to 0, \quad \text{in } L^2(0, L), \\
\langle (i\beta_n - \alpha) V_n, V_n \rangle_{\mathcal{H}_t} &\to 0, \quad \text{in } L^2(0, L), \\
\langle i\beta_n V_{a_3} - V_{a_4}, V_n \rangle_{\mathcal{H}_t} &\to 0, \quad \text{in } L^2(0, L), \\
\langle i\beta_n V_{a_3}, V_n \rangle_{\mathcal{H}_t} &\to 0, \quad \text{in } L^2(0, L). \tag{32}
\end{align*}
\]
By (31), we have
\[
\langle (i\beta_n I - \mathcal{A}_t) V_n, V_n \rangle_{\mathcal{H}_t} \to 0,
\]
which yields
\[
\begin{align*}
\Re \langle (i\beta_n I - \mathcal{A}_t) V_n, V_n \rangle_{\mathcal{H}_t} &= \alpha \rho \int_0^t |V_{a_2}|^2 dx \\
&\quad + \gamma \rho \int_0^t |V_{a_4}|^2 dx + \beta \int_0^t |V_{a_6}|^2 dx \to 0.
\end{align*}
\]
So, we have:
\[
V_{a_2} \to 0, \quad \text{in } L^2(0, L), \tag{33}
\]
\[
V_{a_4} \to 0, \quad \text{in } L^2(0, L), \tag{34}
\]
\[
V_{a_6} \to 0, \quad \text{in } L^2(0, L). \tag{35}
\]
Then, by (30), we have:
\[
k \|V_{a_1} + V_{a_3}\|_{\mathcal{H}_t}^2 + b \|V_{a_4}\|_{\mathcal{H}_t}^2 + \rho_3 \|V_{a_5}\|_{\mathcal{H}_t}^2 \to 1. \tag{36}
\]
On the other hand, by the fifth equation of (32), (34), (35) and the boundary conditions \( V_{a_3}(0) = V_{a_3}(L) = V_{a_5}(0) = V_{a_5}(L) = 0 \), we have:
\[
\begin{align*}
i\beta_n \rho_3 \|V_{a_5}\|_{\mathcal{H}_t}^2 &\to 0 \Rightarrow V_{a_5} \to 0. \tag{37}
\end{align*}
\]
This, together with (36), yields:
\[
k \|V_{a_1} + V_{a_3}\|_{\mathcal{H}_t}^2 + b \|V_{a_4}\|_{\mathcal{H}_t}^2 \to 1. \tag{38}
\]
Taking the inner product by \( V_{a_{3,xx}} \) with the third equation of (32) in \( L^2(0, L) \) and using the condition \( V_{a_3}(0) = V_{a_3}(L) = 0 \), then from (34) and (38), we have separately
\[
\begin{align*}
i\beta_n \|V_{a_5}\|_{\mathcal{H}_t}^2 \to 0 &\Rightarrow V_{a_5,xx} \to 0, \tag{39}
\end{align*}
\]
and
\[
k \|V_{a_1} + V_{a_3}\|_{\mathcal{H}_t}^2 \to 1. \tag{40}
\]
Similarly, taking the inner product by \( (V_{a_1} + V_{a_3}) \) with the fourth equation of (32) in \( L^2(0, L) \), we have

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By (37), (34) and
\[ V_{a,t}(0) + V_{a}(0) = V_{a,t}(L) + V_{a}(L) = 0, \]
we have separately
\[ \langle V_{a,t}, V_{a,t} + V_{a} \rangle_{L^2} \to 0 \]  \hspace{1cm} (42)
and
\[ \langle V_{a}, V_{a,t} + V_{a} \rangle_{L^2} \to 0. \]  \hspace{1cm} (43)

Hence, from (41), we have:
\[ k \|V_{a,t} + V_{a}\|_{L^2}^2 + \langle b V_{a,xx}, (V_{a,t} + V_{a}) \rangle_{L^2} \to 0. \]  \hspace{1cm} (44)

Moreover, taking the inner product by \( \frac{1}{k}(bV_{a,xx}) \) with the second equation of (32) in \( L^2(0,L) \) we have:
\[ (-i\beta_0 - \alpha) \frac{1}{k} \langle b V_{a,xx}, V_{a,xx} \rangle_{L^2} + \langle b V_{a,xx}, (V_{a,t} + V_{a}) \rangle_{L^2} \to 0. \]  \hspace{1cm} (45)

By (33), we have:
\[ \langle b V_{a,xx}, (V_{a,t} + V_{a}) \rangle_{L^2} \to 0. \]  \hspace{1cm} (46)

Hence, by (44) and (46), we have
\[ k \|V_{a,t} + V_{a}\|_{L^2}^2 \to 0, \]
which Contradicts (40). This completes the proof.

**Lemma II.3.** Suppose that \( \tau > 0 \) and the resolvent equation
\[ V = (i\lambda - A_0)^{-1}F \]
given by (23). Then for positive constants \( \alpha, \gamma, \beta, \rho_1, \rho_2 \), there exists a positive constant \( C \) independent of \( F \), such that
\[ \alpha \rho_1 \int_{0}^{t} |V_{a,t}|^2 dx + \gamma \rho_2 \int_{0}^{t} |V_{a}|^2 dx + \beta \int_{0}^{t} |V_{a}|^2 dx \leq C \|V_{a}\|_{W_1} \|F\|_{W_1}. \]

**Proof.** Multiplying (25) by \( \overline{V}_2 \), then, from \( V_{a,t}(0) = V_{a,t}(L) = V_{a}(0) = V_{a}(L) = 0 \), we have:
\[ i\alpha \rho_1 \int_{0}^{t} |V_{a,t}|^2 dx + \int_{0}^{t} (i\lambda + \alpha) \rho_1 \int_{0}^{t} |V_{a,t}|^2 dx - i\lambda k \int_{0}^{t} (V_{a,t} + \gamma V_{a})_{L^2} \]
\[ = \rho_1 \int_{0}^{t} f_2 \overline{V}_2 dx + \gamma \rho_2 \int_{0}^{t} |V_{a}|^2 dx + \beta \int_{0}^{t} |V_{a}|^2 dx. \]  \hspace{1cm} (47)

Multiplying (27) by \( \overline{V}_2 \), then, from \( V_{a}(0) = V_{a}(L) = 0 \), we have:
\[ (i\lambda + \gamma) \rho_2 \int_{0}^{t} |V_{a}|^2 dx + k \int_{0}^{t} (V_{a,t} + V_{a})_{L^2} \]
\[ + \beta \int_{0}^{t} |V_{a}|^2 dx - \int_{0}^{t} V_{a} \overline{V}_{a,xx} dx = \rho_1 \int_{0}^{t} f_3 \overline{V}_2 dx. \]  \hspace{1cm} (48)

From (26), we have \( \overline{V}_2 = -i\lambda \overline{V}_1 - \overline{f}_1 \), and by substituting \( \overline{V}_2 \)
into \( I_{21} \) and \( I_{22} \), we have:
\[ (i\lambda + \gamma) \rho_2 \int_{0}^{t} |V_{a}|^2 dx - \int_{0}^{t} V_{a} \overline{V}_{a,xx} dx \]
\[ - i\lambda b \int_{0}^{t} V_{a} \overline{V}_{a,xx} dx - \int_{0}^{t} V_{a} \overline{V}_{a,xx} dx = \rho_1 \int_{0}^{t} f_3 \overline{V}_2 dx + \gamma \rho_2 \int_{0}^{t} |V_{a}|^2 dx + k \int_{0}^{t} (V_{a,t} + V_{a})_{L^2} \]
\[ + k \int_{0}^{t} (V_{a,t} + V_{a})_{L^2} dx. \]  \hspace{1cm} (49)

Again, multiplying (28) by \( \overline{V}_2 \), we have:
\[ i\alpha \rho_1 \int_{0}^{t} |V_{a,t}|^2 dx + \int_{0}^{t} (i\lambda V_{a} + V_{a})_{L^2} \]
\[ = \rho_1 \int_{0}^{t} f_3 \overline{V}_2 dx + \int_{0}^{t} (V_{a,t} + V_{a})_{L^2} dx \]  \hspace{1cm} (50)

Multiplying (29) by \( \overline{V}_2 \), then, from \( V_{a}(0) = V_{a}(L) = 0 \), we have:
\[ i\lambda \tau \int_{0}^{t} |V_{a}|^2 dx - \int_{0}^{t} V_{a} \overline{V}_{a,xx} dx + \beta \int_{0}^{t} |V_{a}|^2 dx = \tau \int_{0}^{t} f_3 \overline{V}_2 dx. \]  \hspace{1cm} (51)

Again, from the Cauchy-Schwarz inequality, there exists a positive \( C \), such that
\[ k \|\overline{V}_2\|_{L^2}^2 + \tau \|V_{a}\|_{L^2}^2 + \rho_1 \|\overline{V}_2\|_{L^2}^2 + k \|V_{a,t} + V_{a}\|_{L^2} + \overline{f}_1 \]
\[ + b \|V_{a,xx}\|_{L^2} + \rho_1 \|\overline{V}_2\|_{L^2} \leq C \|V_{a}\|_{W_1} \|F\|_{W_1}. \]  \hspace{1cm} (52)

Adding (49), (50), and (51) together, then taking the real part and using (52), we finally get
\[ \alpha \rho_1 \int_{0}^{t} |V_{a,t}|^2 dx + \gamma \rho_2 \int_{0}^{t} |V_{a}|^2 dx + \beta \int_{0}^{t} |V_{a}|^2 dx \leq C \|V_{a}\|_{W_1} \|F\|_{W_1}. \]
This completes the proof.

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Lemma II.4. Suppose that \( \tau > 0 \) and the resolvent equation \( V = (i \lambda - \mathcal{A})^{-1} F \) is given by (23). Then, for positive constants \( \rho_3 \), there exists a positive constant \( C \) independent of \( F \), such that
\[
\rho_3 \int_0^t |V_{\lambda, t}^i| \, dt \leq C \|V\|_{\infty} \|F\|_{\infty}.
\]

Proof. Multiplying (28) by \( V_{\lambda} \), and using \( V_\lambda(0) = V_\lambda(L) = 0 \), we get:
\[
i \lambda \rho_3 \int_0^t |V_{\lambda, t}^i| \, dt + \delta \int_0^t V_{\lambda, t}^i \, dt = \rho_3 \int_0^t f_3 V_{\lambda, t}^i \, dt.
\]
By (29), we have \( V_\lambda = \frac{i \alpha - i \lambda \tau^2}{\lambda^2 \tau^2 + \beta^2} f_3 + \frac{i \alpha - \beta}{\lambda^2 \tau^2 + \beta^2} V_{\lambda, t}^i \). So, by substituting this equation into \( I_{23} \), we have:
\[
\delta \int_0^t V_{\lambda, t}^i \, dt + i \lambda \rho_3 \int_0^t |V_{\lambda, t}^i| \, dt + \frac{\beta - i \lambda \tau^2}{\lambda^2 \tau^2 + \beta^2} \int_0^t |V_{\lambda, t}^i| \, dt = \rho_3 \int_0^t f_3 V_{\lambda, t}^i \, dt.
\]
Adding (49) and (53) together and taking the real part yields:
\[
\frac{\beta}{\lambda^2 \tau^2 - \beta^2} \int_0^t |V_{\lambda, t}^i| \, dt \leq C \|V\|_{\infty} \|F\|_{\infty}.
\]
By Poincaré's inequality, for positive \( \rho_3 \), there exists a positive \( C \), such that
\[
\rho_3 \int_0^t |V_{\lambda, t}^i| \, dt \leq C \|V\|_{\infty} \|F\|_{\infty}.
\]
This completes the proof.

Lemma II.5. Suppose that \( \tau > 0 \) and the resolvent equation \( V = (i \lambda - \mathcal{A})^{-1} F \) is given by (23). Then, for positive constants \( b \), there exists a positive constant \( C \) independent of \( F \), such that
\[
k \int_0^t (V_{\lambda, t} + V_{\lambda}^i) \, dt + b \int_0^t |V_{\lambda, t}^i| \, dt \leq C \|V\|_{\infty} \|F\|_{\infty}.
\]

Proof. Multiplying (27) by \( V_{\lambda} \), we have:
\[
(i \lambda + \gamma) \rho_3 \int_0^t V_{\lambda, t} \, dt + k \int_0^t (V_{\lambda, t} + V_{\lambda}^i) \, dt = \rho_3 \int_0^t f_3 V_{\lambda, t} \, dt.
\]
By (26), we have \( V_{\lambda} = \frac{L}{\lambda} V_{\lambda}^i + \frac{i}{\lambda} f_3 \). Substitute this into \( I_{24}, I_{25} \), and using the fact \( f_3(0) = f_3(L) = 0 \), we have:
\[
k \int_0^t (V_{\lambda, t} + V_{\lambda}^i) \, dt + b \int_0^t |V_{\lambda, t}^i| \, dt \leq C \|V\|_{\infty} \|F\|_{\infty}.
\]
Adding (57)–(60) together, there exists a positive constant $C > 0$, such that
\[ \|V\|^2_{\mathcal{H}_2} \leq C \|F\|^2_{\mathcal{H}_1}. \]
This completes the proof.

### III. STABILITY OF SYSTEM (2)

In this section, we show the exponential stability of System (2) for the case $\tau = 0$. It is known that the total energy of the system (2) is given by
\[ E(t) = \frac{1}{2} \int_0^t \left[ \rho \phi^2(t, x) + k(\varphi(t, x) + \psi(t, x))^2 \right] \, dx \quad (61) \]
and it is natural to consider the system (2) in the energy space. Let
\[ \mathcal{H}_2 = H^2_0(0, L) \times L^2(0, L) \times H^2_0(0, L) \times L^2(0, L) \times L^2(0, L) \]
be the Hilbert space with norm
\[ \|V\| = \left\{ \int_0^t \left[ k[V_{i+1,3}(x) + V_3(x)]^2 + \rho_i[V_{i+1,2}(x)]^2 + \rho_i[V_3(x)]^2 \right] \, dx \right\}^{1/2} \]
where $V = (V_1, V_2, V_3, V_4, V_5) \in \mathcal{H}_2$. Now, we define a system operator $\mathcal{A}_2$ as
\[
\mathcal{A}_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
\frac{k}{\rho_i} \partial_x^2 & -\alpha & \frac{k}{\rho_i} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-\frac{k}{\rho_2} \partial_x & 0 & \frac{b}{\rho_2} \partial_x - \frac{k}{\rho_2} & 0 & -\frac{\delta}{\rho_2} \partial_x \\
0 & 0 & 0 & -\frac{\delta}{\rho_3} \partial_x & \frac{1}{\rho_3 \beta} \partial_x
\end{pmatrix}
\] (63)
with the domain of $\mathcal{A}_2$
\[ D(\mathcal{A}_2) = \left\{ V \in \mathcal{H}_2 \mid V_1, V_2, V_3 \in H^2_0(0, L), V_4 \in H^2_0(0, L), V_5 \right\}. \]
(64)

Then, the system (2) can be rewritten as an evolution equation in $\mathcal{H}_2$:
\[ V_i(t) = \mathcal{A}_2 V_i(t), \quad V_i(t_0) = V_i_0 \]
(65)
where $V(t) = (\varphi(t_0), \varphi(t_0), \psi(t_0), \psi(t_0), \theta(t_0))^T$ and $V_0 = (\varphi_0, \varphi_0, \psi_0, \psi_0, \theta_0)^T$ is the initial condition.

**Lemma III.1.** Let $\mathcal{A}_2$ be defined by (63) and (64). Then, $\mathcal{A}_2$ is the infinitesimal generator of a $C_0$-semigroup of contractions $e^{\mathcal{A}_2 t}$.

**Proof.** Since the proof is similar to Lemma II.1, we omit it here. This completes the proof.

Now, we get the exponential stability of System (65).

**Theorem III.1.** Let $\mathcal{A}_2$ be defined by (63) and (64). Then, the semigroup $e^{\mathcal{A}_2 t}$, generated by the operator $\mathcal{A}_2$, is exponentially stable, i.e., there are two positive constants $M, \mu$ such that
\[ \|S(t)\| \leq Me^{-\mu t}. \]
(66)

By Theorem II.1, in order to show the exponential stability of System (65), we need to estimate the resolvent of $\mathcal{A}_2$. By the resolvent equation
\[ V = (i\lambda - \mathcal{A}_2)^{-1} F, \quad \forall F \in \mathcal{H}_2, \lambda \in \mathbb{R}, \lambda \neq 0, \]
where $V = (V_1, \ldots, V_5)^T \in D(\mathcal{A}_2)$, we get
\[ i\lambda V_1 - V_5 = f_1, \]
(67)
\[ i\lambda \rho_2 V_2 - k(V_{i+1,3} + V_3) + \alpha \rho_2 V_2 = \rho_1 f_2, \]
(68)
\[ i\lambda V_3 - V_4 = f_3, \]
(69)
\[ i\lambda \rho_3 V_4 + k(V_{i+1,3} + V_3) - b V_{3,xx} + \delta V_{3,xx} = \rho_2 f_4, \]
(70)
\[ i\lambda \rho_3 V_5 + \delta V_{4,xx} - \frac{1}{\beta} V_{5,xx} = \rho_5 f_5. \]
(71)

**Lemma III.2.** Let $\mathcal{A}_2$ be defined by (63) and (64). Then, the resolvent set $\rho(\mathcal{A}_2)$ of $\mathcal{A}_2$ contains the imaginary axis.

**Proof.** We prove the result by a contradiction. If the claim is not true, then there exists $\omega \in \mathbb{R}$, a sequence $(\gamma_n) \subset \mathbb{R}$ with $\gamma_n \to \omega$, $|\gamma_n| < |\omega|$ and a sequence of functions
\[ V_n = (V_{n+1,3}, V_n, V_{n+1,3}, V_{n+1,3}, V_{n+1,3})^T \in D(\mathcal{A}_2) \]
\[ = \int_0^t \left[ k[V_{n+1,3}(x) + V_3(x)]^2 + \rho_i[V_{n+1,3}(x)]^2 + \rho_i[V_3(x)]^2 \right] \, dx \]
(72)
so that, as $n \to \infty$,
\[ \|(i\gamma_n I - \mathcal{A}_2)V_n\|_{\mathcal{H}_2} \to 0, \]
(73)
which yields:
\[
\begin{align*}
&\begin{cases}
i\gamma V_{a1} - V_{a2} \to 0 \quad \text{in} \quad H'(0, L), \\
i(\gamma + \alpha)\rho V_{a2} - k(V_{a1} + V_{a3})x \to 0 \quad \text{in} \quad L^2(0, L), \\
i\gamma V_{a3} - V_{a5} \to 0 \quad \text{in} \quad L^2(0, L), \\
i\gamma \rho_2 V_{a4} + k(V_{a1} + V_{a3}) - bV_{a3,xx} + \delta V_{a5,xx} \to 0 \quad \text{(74)}
\end{cases}
\end{align*}
\]

By (73), we have
\[
\langle (i\gamma I - \mathcal{A}_2) W_n, V_n \rangle \to 0,
\]
which yields:
\[
\text{Re} \langle (i\gamma I - \mathcal{A}_2) W_n, V_n \rangle = \alpha p \int_0^L |W_{a2}|^2 \, dx + \frac{1}{\beta} \int_0^L |V_{a5,xx}|^2 \, dx \to 0.
\]

So, we have
\begin{align*}
V_{a2} &\to 0, \quad V_{a5,xx} \to 0, \quad (75) \\
\text{then by (75) and the Poincaré's inequality, we have:} \\
V_{a5} &\to 0.
\end{align*}

Then, by (72), (75), and (76), we have:
\begin{align*}
k \|V_{a1} + V_{a3}\|^2 + b \|V_{a1}\|^2 + \rho_2 \|W_{a4}\|^2 &\to 1. \quad (77)
\end{align*}

On the other hand, by the fifth equation of (74), we have:
\[
\left\langle (i\gamma \rho_2 V_{a5} + \delta V_{a4,xx} - \frac{1}{\beta} V_{a5,xx} V_{a3,xx}) \right\rangle \to 0.
\]

Then, by (75), (76), and \(V_{a3}(0) = V_{a5}(L) = V_{a5,xx}(0) = V_{a5,xx}(L) = 0\), we have:
\[
\langle V_{a4,xx}, V_{a3,xx} \rangle \to 0. \quad (78)
\]

Taking the inner product by \(V_{a3,xx}\) with the third equation of (74) and \(V_{a3}(0) = V_{a5}(L) = V_{a5,xx}(0) = 0\), by (78), we have:
\[
-i\gamma \nu \|V_{a3,xx}\|^2 \to 0 \Rightarrow V_{a3,xx} \to 0. \quad (79)
\]

Then, by Poincaré’s inequality, we have:
\[
V_{a3} \to 0. \quad (80)
\]

Taking the inner product by \(V_{a4}\) with the third equation of (74), then, by (80), we have:
\[
\|V_{a4}\|^2 \to 0 \Rightarrow V_{a4} \to 0. \quad (81)
\]

Substituting (79) and (81) into (77), we have:
\[
k \|V_{a1} + V_{a3}\|^2 \to 1. \quad (82)
\]

Taking the inner product by \((V_{a1} + V_{a3})\) with the forth equation of (74), we have:
\[
i\gamma \rho_2 (V_{a4} + V_{a5}) + k \|V_{a1} + V_{a3}\|^2 - b(V_{a3,xx} + V_{a5,xx}) + \delta (V_{a5,xx} + V_{a3,xx}) \to 0,
\]
then by (75), (79), (81), and \(V_{a3}(0) = V_{a5}(L) = V_{a5,xx}(0) = V_{a5,xx}(L) = 0\), we have
\[
k \|V_{a1} + V_{a3}\|^2 \to 0. \quad (83)
\]

which contradicts (82). Thus, the (i) of Theorem II.1 is proved. This completes the proof.

**Lemma III.3.** Suppose that \(\nu = 0\) and the resolvent equation \(V = (i\lambda - \mathcal{A}_2)^{-1} F\) is given by (66). Then, for positive constants \(\alpha, \beta, \rho_1, \) there exists a positive constant \(C\) independent of \(F\), such that
\[
\alpha p \int_0^L |W_{a1}|^2 \, dx + \frac{1}{\beta} \int_0^L |W_{a5,xx}|^2 \, dx \leq C \|V\|_{H^2} \|F\|_{H^2}.
\]

**Proof.** Multiplying (68) by \(\overline{V}_1\), we have:
\[
(i\lambda + \alpha) \int_0^L |W_{a1}|^2 \, dx - k \int_0^L (V_{a1} + V_{a3}) \overline{V}_1 \, dx = \rho_1 \int_0^L f_{2x} \overline{V}_1 \, dx.
\]

By (67), we have \(\overline{V}_1 = -i\lambda \overline{V}_1 - \overline{F}_1\). Substitute \(\overline{V}_1\) into \(I_{30}\), and by \(V_{a1}(0) = V_{a1}(L) = V_{a3}(0) = V_{a3}(L) = 0\), we have:
\[
i\lambda \rho_1 \int_0^L |W_{a1}|^2 \, dx + i\lambda k \int_0^L (V_{a1} + V_{a3}) \overline{V}_1 \, dx + \alpha p \int_0^L |W_{a5,xx}|^2 \, dx = \rho_1 \int_0^L f_{2x} \overline{V}_1 \, dx.
\]

Multiplying (70) by \(\overline{V}_2\), we have
\[
\int_0^L \langle k(V_{a1} + V_{a3}) - bV_{a3,xx} \rangle \overline{V}_2 \, dx + i\lambda \rho_2 \int_0^L |W_{a4}|^2 \, dx + \delta \int_0^L V_{a5,xx} \overline{V}_2 \, dx = \rho_2 \int_0^L f_{4x} \overline{V}_2 \, dx
\]
and from (69), we have \(\overline{V}_2 = -i\lambda \overline{V}_2 - \overline{F}_2\). Substitute this into \(I_{31}\), and by \(V_{a3}(0) = V_{a3}(L) = 0\), we have:
\[
i\lambda \rho_2 \int_0^L |W_{a4}|^2 \, dx - i\lambda k \int_0^L (V_{a1} + V_{a3}) \overline{V}_2 \, dx - i\lambda b \int_0^L |V_{a5,xx}|^2 \, dx + \delta \int_0^L V_{a5,xx} \overline{V}_2 \, dx = \rho_2 \int_0^L f_{4x} \overline{V}_2 \, dx
\]
\[
+ k \int_0^L (V_{a1} + V_{a3}) \overline{V}_2 \, dx + h \int_0^L V_{a3,xx} \overline{V}_2 \, dx.
\]

Multiplying (71) by \(\overline{V}_3\), and by \(V_{a4}(0) = V_{a4}(L) = V_{a5,xx}(0) = V_{a5,xx}(L) = 0\), we have:
\[ i\lambda \rho \int_0^t |V_3|^2 dx - \delta \int_0^t V_4 \overline{V_3} dx + \frac{1}{\beta} \int_0^t |V_3|^2 dx = \rho \int_0^t f_3 \overline{V} dx. \]  

(86)

Adding (84), (85), and (86), then taking the real part, we obtain:

\[ \alpha \rho \int_0^t |V_1|^2 dx + \frac{1}{\beta} \int_0^t |V_3|^2 dx \leq C \|V\|_{\mathcal{W}_2} \|F\|_{\mathcal{W}_2}. \]

This completes the proof.

**Lemma III.4.** Suppose that \( \tau = 0 \) and the resolvent equation \( V = (i\lambda - \mathcal{A}_2)^{-1}F \) is given by (66). Then, for positive constants \( \rho_2 \), there exists a positive constant \( C \) independent of \( F \), such that

\[ k \int_0^t |V_{1,3} + V_3|^2 dx + \rho_2 \int_0^t |V_4|^2 dx \leq C \|V\|_{\mathcal{W}_2} \|F\|_{\mathcal{W}_2}. \]

\[ \|V\|_{\mathcal{W}_2} \leq \|F\|_{\mathcal{W}_2} \]

**Proof.** Multiplying (70) by \( V_3 \), we have:

\[ i\lambda \rho \int_0^t V_4 \overline{V_3} dx + k \int_0^t (V_{1,3} + V_3) \overline{V_3} dx + \rho_2 \int_0^t |V_4|^2 dx \]

By (69), we have \( V_3 = -\frac{i}{\lambda} V_4 - \frac{i}{\lambda} f_3 \). Substitute \( V_3 \) into (72), (73), and (74), and by \( V_3(0) = V_3(L) = V_4(0) = V_4(L) = 0 \), we have:

\[ k \int_0^t (V_{1,3} + V_3) \overline{V_3} dx + b \int_0^t V_3 \overline{V_3} dx + i\delta \int_0^t V_3 \overline{V_3} dx \]

Multiplying (68) by \( V_1 \), then, by \( V_{1,3}(0) = V_{1,3}(L) = V_3(0) = V_3(L) = 0 \), we have:

\[ (i\lambda + \alpha) \int_0^t V_2 \overline{V_3} dx + k \int_0^t (V_{1,3} + V_3) \overline{V_3} dx + \rho_1 \int_0^t f_2 \overline{V} dx. \]

\[ \frac{i\alpha}{\lambda} \int_0^t |V_3|^2 dx + k \int_0^t (V_{1,3} + V_3) \overline{V_3} dx + \rho_1 \int_0^t f_2 \overline{V} dx. \]

Again, from Cauchy-Schwarz inequality, we have:

\[ \int_0^t V_{1,3} \overline{V_3} dx \leq C \|V_{1,3}\| \|V_3\| \leq C (\|V_{1,3}\|^2 + \|V_3\|^2) \]

(89)

Adding (87) and (88), then taking the real part and using (89) with it, we have:

\[ k \int_0^t |V_{1,3} + V_3|^2 dx + \rho_2 \int_0^t |V_4|^2 dx \leq C \|V\|_{\mathcal{W}_2} \|F\|_{\mathcal{W}_2}. \]

This completes the proof.

**Lemma III.5.** Suppose that \( \tau = 0 \) and the resolvent equation \( V = (i\lambda - \mathcal{A}_2)^{-1}F \) is given by (66). Then, for positive constants \( b \), there exists a positive constant \( C \) independent of \( F \), such that

\[ b \int_0^t |V_{3,3}|^2 dx \leq C \|V\|_{\mathcal{W}_2} \|F\|_{\mathcal{W}_2}. \]

**Proof.** Multiplying (70) by \( V_3 \), we have:

\[ i\lambda \rho \int_0^t V_4 \overline{V_3} dx + k \int_0^t (V_{1,3} + V_3) \overline{V_3} dx - b \int_0^t V_{3,3} \overline{V_3} dx + b \int_0^t |V_3|^2 dx \]

By (69), we have \( V_3 = -\frac{i}{\lambda} V_4 - \frac{i}{\lambda} f_3 \). Substitute \( V_3 \) into (72), (73), and (74), and by \( V_3(0) = V_3(L) = V_4(0) = V_4(L) = 0 \), we have:

\[ -b \int_0^t V_{3,3} \overline{V_3} dx + i\delta \int_0^t V_3 \overline{V_3} dx \]

\[ + \rho_2 \int_0^t f_3 \overline{V_3} dx + \rho_1 \int_0^t f_2 \overline{V} dx. \]

\[ -b \int_0^t V_{3,3} \overline{V_3} dx + i\delta \int_0^t V_3 \overline{V_3} dx \]

\[ + \rho_2 \int_0^t f_3 \overline{V_3} dx + i\lambda \rho \int_0^t f_3 \overline{V_3} dx. \]

Again, from Cauchy-Schwarz inequality, we have:

\[ \int_0^t V_{3,3} \overline{V_3} dx \leq C \|V_{3,3}\| \|V_3\| \leq C (\|V_{3,3}\|^2 + \|V_3\|^2) \]

(90)

Then, by adding (88) and (90) together, taking the real part, and using Poincaré’s inequality, by (91) we have:

\[ b \int_0^t |V_{3,3}|^2 dx \leq C \|V\|_{\mathcal{W}_2} \|F\|_{\mathcal{W}_2}. \]

This completes the proof.

Now, we prove Theorem III.1.

**Proof of Theorem III.1.** We use Theorem II.1 to prove the exponential stability. First, from Lemma III.2, the resolvent set \( \rho(\mathcal{A}_2) \) of \( \mathcal{A}_2 \) contains the imaginary axis, which satisfies Condition (i) of Theorem II.1. By the resolvent equation \( V = (i\lambda - \mathcal{A}_2)^{-1}F \) given by (66), we have \( V \) and \( F \), where \( V = (V_1, V_2, V_3, V_4, V_5)^T \) and \( F = (f_1, f_2, f_3, f_4, f_5)^T \) satisfy (67)–(71). Then, from Lemma III.3, Lemma III.4 and Lemma
III.5, we see that there exists $C > 0$, independent of $\lambda$ (and $F$, $V$), such that

$$\rho_1 \| V \|_{H_2}^2 + \rho_2 \| V \|_{H_2}^2 \leq C \| F \|_{H_2}^2,$$  \hspace{1cm} (92)

$$k \| V \|_{H_2}^2 + \rho_2 \| V \|_{H_2}^2 \leq C \| F \|_{H_2}^2,$$  \hspace{1cm} (93)

$$b \| V \|_{H_2}^2 \leq C \| F \|_{H_2}^2.$$  \hspace{1cm} (94)

Adding (92)–(94) together, then there exists a positive $C$, such that

$$\| V \|_{H_2}^2 \leq C \| F \|_{H_2}^2.$$  \hspace{1cm} (95)

This completes the proof.

IV. CONCLUSIONS

A one-dimensional damped hyperbolic Timoshenko beam that is coupled with a heat equation is considered. With internal dampings being introduced and the help of the frequency domain method, Timoshenko beam under Cattaneo’s and Fourier’s law is showed to be exponentially stable, respectively.

REFERENCES


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