Asian Journal of Control, Vol. 16, No. 2, pp. 546–555, March 2014
Published online 29 May 2013 in Wiley Online Library (wileyonlinelibrary.com) DOI: 10.1002/asjc.739

STABILITY OF A DAMPED HYPERBOLIC TIMOSHENKO SYSTEM COUPLED WITH A HEAT EQUATION

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ABSTRACT

We consider a one-dimensional damped hyperbolic Timoshenko beam that is coupled with a heat equation. When its wave speeds are different, it is known that the Timoshenko beam that is coupled with a heat equation, under Cattaneo's law, does not have exponential stability. With two internal dampings being introduced, in this paper, we show that the system under Cattaneo's law is exponentially stable. We also show the exponential stability when the system is considered under Fourier's law.

Key Words: Stability, Timoshenko beam, heat equation, Cattaneo's law, Fourier's law.

I. INTRODUCTION

In this paper, we consider the Timoshenko beams that are coupled with a heat equation and are modeled by Cattaneo's law and Fourier's law as: for $(t,x) \in (0,\infty)\times(0,L)$,

$$\begin{cases} \rho_1\varphi_{tt}(t,x) - k(\varphi_x(t,x)+\psi(t,x))_x + \alpha\rho_1\varphi_t(t,x) = 0 \\ \rho_2\psi_{tt}(t,x) - b\psi_{xx}(t,x) + k(\varphi_x(t,x)+\psi(t,x)) \\ \quad + \delta\theta_x(t,x) + \gamma\rho_2\psi_t(t,x) = 0 \\ \rho_3\theta_t(t,x) + q_x(t,x) + \delta\psi_{tx}(t,x) = 0 \\ \tau q_t(t,x) + \beta q(t,x) + \theta_x(t,x) = 0 \end{cases} \tag{1}$$

and

$$\begin{cases} \rho_1\varphi_{tt}(t,x) - k(\varphi_x(t,x)+\psi(t,x))_x + \alpha\rho_1\varphi_t(t,x) = 0 \\ \rho_2\psi_{tt}(t,x) - b\psi_{xx}(t,x) + k(\varphi_x(t,x)+\psi(t,x)) + \delta\theta_x(t,x) = 0 \\ \rho_3\theta_t(t,x) - \dfrac{1}{\beta}\theta_{xx}(t,x) + \delta\psi_{tx}(t,x) = 0 \end{cases} \tag{2}$$

where the functions φ and ψ describe the transverse displacement of a beam and the rotation angle of a filament, respectively. θ and q denote the temperature and the heat flux, respectively; the physical parameters ρ_1, k, ρ_2, b, δ, ρ_3, and β are positive constants; and α, $\gamma > 0$ denote the damping constants with respect to φ_t, ψ_t. For more detailed physical background of the problems, refer to [1].

It is known that the positive parameter τ is the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature [2] and that $\tau = 0$ represents Fourier's law, while $\tau > 0$ represents Cattaneo's law.

Manuscript received April 16, 2012; revised July 22, 2012; accepted January 16, 2013.

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This work was supported by National Natural Science Foundation of China.

The initial and boundary conditions for (1) and (2) are given as follows:

$$\begin{cases} \varphi(0,x) = \varphi_0(x), \varphi_t(0,x) = \varphi_1(x) \\ \psi(0,x) = \psi_0(x), \psi_t(0,x) = \psi_1(x) \\ \theta(0,x) = \theta_0(x), q(0,x) = q_0(x) \end{cases} \tag{3}$$

$$\begin{cases} \varphi_x(t,0) = \varphi_x(t,L) = \psi(t,0) = 0 \\ \psi(t,L) = q(t,0) = q(t,L) = 0 \end{cases} \tag{4}$$

and

$$\begin{cases} \varphi(0,x) = \varphi_0(x), \varphi_t(0,x) = \varphi_1(x) \\ \psi(0,x) = \psi_0(x), \psi_t(0,x) = \psi_1(x) \\ \theta(0,x) = \theta_0(x) \end{cases} \tag{5}$$

$$\begin{cases} \varphi_x(t,0) = \varphi_x(t,L) = \psi(t,0) = 0 \\ \psi(t,L) = \theta(t,0) = \theta(t,L) = 0 \end{cases} \tag{6}$$

Many results are already available in the literature. For the case of $\tau = 0$, if $\alpha = \gamma = 0$, that is, there are no internal dampings in the system, Soufyane [3] showed the exponential stability of the system under the condition that the system (1) has the same wave speeds, *i.e.*,

$$\frac{\rho_1}{k} = \frac{\rho_2}{b} \tag{7}$$

where $\delta\theta_x$ is replaced by a control function $b(x)\psi_x$, $b > 0$. Rivera and Sare [4] studied the stability of Timoshenko vibrating systems with past history. Sare and Racke [2] systematically studied the undamped system (1) with $\alpha = \gamma = 0$ and obtained the exponential stability of the system under Condition (7). When the wave speeds are different, the exponential stability of (1) with one viscous damping, that is, $\gamma = 0$, is established [5] by the multiplier method. A weaker type of dissipation being presented only for ψ was considered [6] where $\delta\theta_x$ is replaced by a memory term $\int_0^L g(t-s)\psi_{xx}(s,x)ds$. Recently, Wang *et al*. [7] considered a Schrödinger equation that is coupled with a heat equation via the boundary coupling and showed the exponential stability and Gevrey regularity of the closed loop system. The stability for a system of serially connected vibrating strings with discontinuous displacement was considered in [8], and the spectral analysis for Euler-Bernoulli beam with input delay in the boundary control was presented in [9].

When $\alpha = 0$ in (2), with an approach adopted in Chen [10], Rivera and Racke [11] proved the exponential stability of the system if and only if $\frac{\rho_1}{k} = \frac{\rho_2}{b}$ holds. Ammar-Khdja [6] proved the exponential stability of the common linear Timoshenko system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + d\psi_t = 0 \end{cases}$$

when $\frac{\rho_1}{k} = \frac{\rho_2}{b}$ holds. When $\alpha = \delta = 0$, the first and second equations of (2) build a purely hyperbolic system [1], in which the energy is conserved and solutions, with respect to the energy, do not decay at all. The pair of the second and third equations of (2), those of moelasticity for $k = 0$ and those of linear thermoelasticity, for which the exponential stability for various boundary conditions has been proven [6].

In this paper, the frequency domain method is adopted to show the exponential stability for both Systems (1) and (2). For System (1), we assume that $\rho_1, \rho_2 \neq 0$ in this paper. It is noted that, since the heat equation is strongly coupled with the second equation of the Timoshenko beam, System (1) will be exponentially stable with $\rho_1 \neq 0$ and $\rho_2 = 0$, which can be deduced by the same argument in the paper (see also [5] for the argument with the multiplier methods).

The paper is organized as follows. In Section II, we present the well-posedness and exponential stability for System (1), and the well-posedness and exponential stability of System (2) are shown in Section III.

II. STABILITY OF SYSTEM (1)

In this section, we show the exponential stability of System (1) for the case $\tau > 0$. It is known that the total energy of the system (1) is given by:

$$E(t) = \frac{1}{2}\int_0^L [\rho_1\varphi_t^2(t,x) + k(\varphi_x(t,x) + \psi(t,x))^2 + \rho_2\psi_t^2(t,x) + b\psi_x^2(t,x) + \rho_3\theta^2(t,x) + \tau q^2(t,x)]dx. \tag{8}$$

So, it is natural to consider the system (1) in the energy space. Let

$$L_*^2(0,L) = \left\{ v \in L^2(0,L) \middle| \int_0^L v(x)dx = 0 \right\},$$
$$H_*^1(0,L) = \left\{ v \in H^1(0,L) \middle| \int_0^L v(x)dx = 0 \right\},$$

and let

$$\mathcal{H}_1 = H_*^1(0,L) \times L_*^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times L_*^2(0,L) \times L^2(0,L) \tag{9}$$

be the Hilbert space with norm [2]

$$\|V\|^2 = \int_0^L [k|V_{1,x}(x) + V_3(x)|^2 + \rho_1|V_2(x)|^2 + b|V_{3,x}(x)|^2 + \rho_2|V_4(x)|^2 + \rho_3|V_5(x)|^2 + \tau|V_6(x)|^2]dx$$

where $V = (V_1, V_2, V_3, V_4, V_5, V_6) \in \mathcal{H}_1$ and the subscript "x" denotes the differentiation of a function with respect to the space variable x. Now, we define a system operator $\mathcal{A}_1$ as:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{k}{\rho_1}\partial_x^2 & -\alpha & \frac{k}{\rho_1}\partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{-k}{\rho_2}\partial_x & 0 & \frac{b\partial_x^2 - k}{\rho_2} & -\gamma & \frac{-\delta}{\rho_2}\partial_x & 0 \\ 0 & 0 & 0 & \frac{-\delta}{\rho_3}\partial_x & 0 & \frac{-1}{\rho_3}\partial_x \\ 0 & 0 & 0 & 0 & \frac{-1}{\tau}\partial_x & -\frac{\beta}{\tau} \end{pmatrix} \tag{10}$$

with the domain of $\mathcal{A}_1$:

$$D(\mathcal{A}_1) = \left\{ V \in \mathcal{H}_1 \middle| \begin{array}{l} V_1, V_3 \in H^2(0,L) \\ V_2, V_5 \in H_*^1(0,L) \\ V_{1,x}, V_4, V_6 \in H_0^1(0,L) \end{array} \right\}. \tag{11}$$

Then, the system (1) can be rewritten as an evolution equation in $\mathcal{H}_1$:

$$V_t(t) = \mathcal{A}_1 V(t), \quad V(t=0) = V_0, \tag{12}$$

where $V(t) = (\varphi(t,\cdot),\ \varphi_t(t,\cdot),\ \psi(t,\cdot),\ \psi_t(t,\cdot),\ \theta(t,\cdot),\ q(t,\cdot))^T$ and where $V_0 = (\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1,\ \theta_0,\ q_0)^T$ is the initial condition. Here, the superscript "T" denotes the transpose of a vector or a matrix.

Now, we show the well-posedness of System (12).

Lemma II.1. Let $\mathcal{A}_1$ be defined by (10) and (11). Then, $\mathcal{A}_1$ is the infinitesimal generator of a C_0-semigroup of contractions $e^{\mathcal{A}_1 t}$.

Proof. First, we show that $\mathcal{A}_1$ is dissipative. For each $V \in D(\mathcal{A}_1)$, where $V = (V_1, V_2, \ldots, V_6)^T$, we have

$$\mathcal{A}_1 V = \left(V_2, \frac{k}{\rho_1}(V_{1,xx} + V_{3,x}) - \alpha V_2, V_4, \frac{b}{\rho_2} V_{3,xx} - \frac{k}{\rho_2}(V_{1,x} + V_3) - \gamma V_4 - \frac{\delta}{\rho_2} V_{5,x}, -\frac{\delta}{\rho_3} V_{4,x} - \frac{1}{\rho_3} V_{6,x}, -\frac{1}{\tau} V_{5,x} - \frac{\beta}{\tau} V_6 \right)^T$$

So, we obtain:

$$\mathrm{Re}\langle \mathcal{A}_1 V, V\rangle_{\mathcal{H}_1} = -\alpha\rho_1 \int_0^L |V_2|^2\,dx - \gamma\rho_2 \int_0^L |V_4|^2\,dx - \beta \int_0^L |V_6|^2\,dx \le 0.$$

Hence, $\mathcal{A}_1$ is dissipative. Next, we show that $0 \in \rho(\mathcal{A}_1)$.

For any $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}_1$, consider the following equation,

$$\mathcal{A}_1 V = F \tag{13}$$

then we have

$$V_2(x) = f_1(x), \quad V_4(x) = f_3(x), \tag{14}$$

and $V_1(x)$, $V_3(x)$, $V_5(x)$, $V_6(x)$ satisfy

$$k(V_{1,x}(x) + V_3(x))_x - \alpha\rho_1 V_2(x) = \rho_1 f_2(x), \tag{15}$$

$$bV_{3,xx}(x) - k(V_{1,x}(x) + V_3(x)) - \gamma\rho_2 V_4(x) - \delta V_{5,x}(x) = \rho_2 f_4(x), \tag{16}$$

$$-\delta V_{4,x}(x) - V_{6,x}(x) = \rho_3 f_5(x), \tag{17}$$

$$-V_{5,x}(x) - \beta V_6(x) = \tau f_6(x), \tag{18}$$

with the boundary conditions

$$\begin{cases} V_{1,x}(0) = V_{1,x}(L) = V_3(0) = V_3(L) = 0, \\ V_4(0) = V_4(L) = V_6(0) = V_6(L) = 0. \end{cases}$$

From (17) and $V_6(0) = V_4(0) = 0$, we have:

$$V_6(x) = -\delta f_3(x) - \rho_3 \int_0^x f_5(y)\,dy. \tag{19}$$

This, together with (18) and $\int_0^L V_5(x)dx = 0$, yields:

$$V_5(x) = -\int_0^x (\beta V_6(y) + \tau f_6(y))dy + \frac{1}{L}\int_0^L \int_0^x (\beta V_6(y) + \tau f_6(y))dydx. \tag{20}$$

From (15) and $V_{1,x}(0) = V_3(0) = 0$, we have:

$$V_{1,x}(x) + V_3(x) = \int_0^x \left[\frac{\alpha\rho_1}{k} f_1(y) + \frac{\rho_1}{k} f_2(y) \right] dy. \tag{21}$$

Also, from (16), (21), and $V_3(0) = V_3(L) = 0$, we have:

$$V_3(x) = \frac{\rho_1}{b}\int_0^x \int_0^\xi \int_0^y [\alpha f_1(\tau) + f_2(\tau)]\,d\tau dy d\xi + \frac{\rho_2}{b}\int_0^x \int_0^\xi [\gamma f_3(y) + f_4(y)]\,dyd\xi - \frac{\delta}{b} x V_5(0) + \frac{\delta}{b}\int_0^x V_5(\xi)d\xi + xV_{3,x}(0), \tag{22}$$

where

$$V_{3,x}(0) = -\frac{\rho_1}{Lb}\int_0^L \int_0^\xi \int_0^y [\alpha f_1(\tau) + f_2(\tau)]\,d\tau dy d\xi - \frac{\rho_2}{Lb}\int_0^L \int_0^\xi [\gamma f_3(y) + f_4(y)]\,dyd\xi - \frac{\delta}{Lb}\int_0^L V_5(\xi)d\xi + \frac{\delta}{b} V_5(0),$$

and

$$V_5(0) = \frac{1}{L}\int_0^L \int_0^x (\beta V_6(\xi) + \tau f_6(\xi))d\xi dx.$$

Substitute (22) into (21) to get V_1. Hence, we have $V = (V_1, V_2, V_3, V_4, V_5, V_6)^T \in D(\mathcal{A}_1)$ and $0 \in \rho(\mathcal{A}_1)$. Therefore, by the Lumer-Philips theorem [13], p.14, $\mathcal{A}_1$ generates a C_0-semigroup of contractions $e^{\mathcal{A}_1 t}$ in $\mathcal{H}_1$. This completes the proof.

Now, we are going to show the stability of System (12). Here, we will use the following theorem from semigroup theory [12].

Theorem II.1. A semigroup of contractions $\{e^{t\mathcal{A}}\}_{t\ge 0}$ in a Hilbert space with norm $\|.\|$ is exponentially stable if and only if

(i) the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains the imaginary axis.

(ii) $\limsup_{\lambda\to\infty} \|(i\lambda - \mathcal{A})^{-1}\| < \infty$.

Theorem II.2. Let $\mathcal{A}_1$ be defined by (10) and (11). Then, the semigroup $e^{\mathcal{A}_1 t}$, generated by the operator $\mathcal{A}_1$, is exponentially stable, *i.e.*, there are two positive constants M, μ such that

$$\|S(t)\| \le Me^{-\mu t}.$$

By Theorem II.1, in order to show the exponential stability of System (12), we need to estimate the resolvent of $\mathcal{A}_1$. By the resolvent equation

$$V = (i\lambda - \mathcal{A}_1)^{-1}F, \quad \forall F \in \mathcal{H}_1, \lambda \in \mathbb{R}, \lambda \neq 0, \tag{23}$$

where $V = (V_1, \dots, V_6)^T \in D(\mathcal{A}_1)$, we get

$$i\lambda V_1 - V_2 = f_1, \tag{24}$$

$$i\lambda\rho_1 V_2 - k(V_{1,x} + V_3)_x + \alpha\rho_1 V_2 = \rho_1 f_2, \tag{25}$$

$$i\lambda V_3 - V_4 = f_3, \tag{26}$$

$$i\lambda\rho_2 V_4 + k(V_{1,x} + V_3) - bV_{3,xx} + \gamma\rho_2 V_4 + \delta V_{5,x} = \rho_2 f_4, \tag{27}$$

$$i\lambda\rho_3 V_5 + \delta V_{4,x} + V_{6,x} = \rho_3 f_5, \tag{28}$$

$$i\lambda\tau V_6 + V_{5,x} + \beta V_6 = \tau f_6. \tag{29}$$

Now, we present the following four lemmas to show Theorem II.2.

Lemma II.2. Let $\mathcal{A}_1$ be defined by (10) and (11). Then, the resolvent set $\rho(\mathcal{A}_1)$ of $\mathcal{A}_1$ contains the imaginary axis.

Proof. We prove the result by a contradiction. If the claim is not true, then there exists $\omega \in \mathbb{R}$, a sequence $(\beta_n)_n \subset \mathbb{R}$ with $\beta_n \to \omega$, $|\beta_n| < |\omega|$, and a sequence of functions

$$\begin{cases} V_n = (V_{n1}, V_{n2}, V_{n3}, V_{n4}, V_{n5}, V_{n6})^T \in D(\mathcal{A}_1) \|V_n\|_{\mathcal{H}_1} \\ \quad = \int_0^L [k|V_{n1,x}(x) + V_{n3}(x)|^2 + \rho_1 |V_{n2}(x)|^2 + b|V_{n3,x}(x)|^2 \\ \quad + \rho_2 |V_{n4}(x)|^2 + \rho_3 |V_{n5}(x)|^2 + \tau |V_{n6}(x)|^2] dx = 1 \end{cases} \tag{30}$$

such that, as $n \to \infty$,

$$\|(i\beta_n I - \mathcal{A}_1)V_n\|_{\mathcal{H}_1} \to 0, \tag{31}$$

which yields

$$\begin{cases} i\beta_n V_{n1} - V_{n2} \to 0 \quad \text{in} \quad H^1(0, L), \\ (i\beta_n + \alpha)\rho_1 V_{n2} - k(V_{n1,x} + V_{n3})_x \to 0 \\ \text{in} \quad L^2(0, L), i\beta_n V_{n3} - V_{n4} \to 0 \quad \text{in} \quad L^2(0, L), \\ i\beta_n \rho_2 V_{n4} + k(V_{n1,x} + V_{n3}) - bV_{n3,xx} + \delta V_{n5,x} + \gamma\rho_2 V_{n4} \to 0 \\ \text{in} \quad L^2(0, L), i\beta_n \rho_3 V_{n5} + \delta V_{n4,x} + V_{n6,x} \to 0 \\ \text{in} \quad L^2_*(0, L), i\beta_n \tau V_{n6} + V_{n5,x} + \beta V_{n6} \to 0 \quad \text{in} \quad L^2(0, L). \end{cases} \tag{32}$$

By (31), we have

$$\langle (i\beta_n I - \mathcal{A}_1)V_n, V_n \rangle_{\mathcal{H}_1} \to 0,$$

which yields

$$\begin{aligned} \mathrm{Re}\langle (i\beta_n I - \mathcal{A}_1)V_n, V_n \rangle_{\mathcal{H}_1} &= \alpha\rho_1 \int_0^L |V_{n2}|^2 \, dx \\ &+ \gamma\rho_2 \int_0^L |V_{n4}|^2 \, dx + \beta \int_0^L |V_{n6}|^2 \, dx \to 0. \end{aligned}$$

So, we have:

$$V_{n2} \to 0, \quad \text{in} \quad L^2(0, L), \tag{33}$$

$$V_{n4} \to 0, \quad \text{in} \quad L^2(0, L), \tag{34}$$

$$V_{n6} \to 0, \quad \text{in} \quad L^2(0, L). \tag{35}$$

Then, by (30), we have:

$$k\|V_{n1,x} + V_{n3}\|_{L^2}^2 + b\|V_{n3,x}\|_{L^2}^2 + \rho_3 \|V_{n5}\|_{L^2}^2 \to 1. \tag{36}$$

On the other hand, by the fifth equation of (32), (34), (35) and the boundary conditions $V_{n4}(0) = V_{n4}(L) = V_{n6}(0) = V_{n6}(L) = 0$, we have:

$$i\beta_n \rho_3 \|V_{n5}\|_{L^2}^2 \to 0 \Rightarrow V_{n5} \to 0. \tag{37}$$

This, together with (36), yields:

$$k\|V_{n1,x} + V_{n3}\|_{L^2}^2 + b\|V_{n3,x}\|_{L^2}^2 \to 1. \tag{38}$$

Taking the inner product by $V_{n3,xx}$ with the third equation of (32) in $L^2(0,L)$ and using the condition $V_{n3}(0) = V_{n3}(L) = 0$, then from (34) and (38), we have separately

$$i\beta_n \|V_{n3,x}\|_{L^2}^2 \to 0 \Rightarrow V_{n3,x} \to 0, \tag{39}$$

and

$$k\|V_{n1,x} + V_{n3}\|_{L^2}^2 \to 1. \tag{40}$$

Similarly, taking the inner product by $(V_{n1,x} + V_{n3})$ with the fourth equation of (32) in $L^2(0,L)$, we have

$$\langle (i\beta_n\rho_2+\gamma\rho_2)V_{n4}-bV_{n3,xx},V_{n1,x}+V_{n3}\rangle_{L^2} + \langle \delta V_{n5,x},V_{n1,x}+V_{n3}\rangle_{L^2}+k\|V_{n1,x}+V_{n3}\|_{L^2}^2\to 0. \tag{41}$$

By (37), (34) and

$$V_{n1,x}(0)+V_{n3}(0)=V_{n1,x}(L)+V_{n3}(L)=0,$$

we have separately

$$\langle V_{n5,x},V_{n1,x}+V_{n3}\rangle_{L^2}\to 0 \tag{42}$$

and

$$\langle V_{n4},V_{n1,x}+V_{n3}\rangle_{L^2}\to 0. \tag{43}$$

Hence, from (41), we have:

$$k\|V_{n1,x}+V_{n3}\|_{L^2}^2+\langle bV_{n3,x},(V_{n1,x}+V_{n3})_x\rangle_{L^2}\to 0. \tag{44}$$

Moreover, taking the inner product by $\frac{1}{k}(bV_{n3,x})$ with the second equation of (32) in $L^2(0,L)$ we have:

$$(-i\beta_n-\alpha)\frac{\rho_1}{k}\langle bV_{n3,x},V_{n2}\rangle_{L^2}+\langle bV_{n3,x},(V_{n1,x}+V_{n3})_x\rangle_{L^2}\to 0. \tag{45}$$

By (33), we have:

$$\langle bV_{n3,x},(V_{n1,x}+V_{n3})_x\rangle_{L^2}\to 0. \tag{46}$$

Hence, by (44) and (46), we have

$$k\|V_{n1,x}+V_{n3}\|_{L^2}^2\to 0,$$

which Contradicts (40). This completes the proof.

Lemma II.3. Suppose that $\tau>0$ and the resolvent equation $V=(i\lambda-\mathcal{A}_1)^{-1}F$ is given by (23). Then for positive constants α, γ, β, ρ_1, ρ_2, there exists a positive constant C independent of F, such that

$$\alpha\rho_1\int_0^L|V_2|^2\,dx+\gamma\rho_2\int_0^L|V_4|^2\,dx+\beta\int_0^L|V_6|^2\,dx\le C\|V\|_{\mathcal{H}_1}\|F\|_{\mathcal{H}_1}.$$

Proof. Multiplying (25) by $\overline{V_2}$, then, from $V_{1,x}(0)=V_{1,x}(L)=V_3(0)=V_3(L)=0$, we have:

$$i\lambda\rho_1\int_0^L|V_2|^2\,dx+k\int_0^L(V_{1,x}+V_3)\underbrace{\overline{V_{2,x}}}_{I_{20}}\,dx + \alpha\rho_1\int_0^L|V_2|^2\,dx=\rho_1\int_0^L f_2\overline{V_2}dx.$$

Noting from (24) that we have $\overline{V_2}=-i\lambda\overline{V_1}-\overline{f_1}$, and by substituting $\overline{V_2}$ into I_{20}, we have:

$$(i\lambda+\alpha)\rho_1\int_0^L|V_2|^2\,dx-i\lambda k\int_0^L(V_{1,x}+V_3)\overline{V_{1,x}}dx =\rho_1\int_0^L f_2\overline{V_2}dx+k\int_0^L(V_{1,x}+V_3)\overline{f_{1,x}}dx. \tag{47}$$

Multiplying (27) by $\overline{V_4}$, then, from $V_4(0)=V_4(L)=0$, we have:

$$(i\lambda+\gamma)\rho_2\int_0^L|V_4|^2\,dx+k\int_0^L(V_{1,x}+V_3)\underbrace{\overline{V_4}}_{I_{21}}\,dx + b\int_0^L V_{3,x}\underbrace{\overline{V_{4,x}}}_{I_{22}}\,dx-\delta\int_0^L V_5\overline{V_{4,x}}dx=\rho_2\int_0^L f_4\overline{V_4}dx.$$

From (26), we have $\overline{V_4}=-i\lambda\overline{V_3}-\overline{f_3}$, and by substituting $\overline{V_4}$ into I_{21} and I_{22}, we have:

$$(i\lambda+\gamma)\rho_2\int_0^L|V_4|^2\,dx-\delta\int_0^L V_5\overline{V_{4,x}}dx - i\lambda b\int_0^L|V_{3,x}|^2\,dx-i\lambda k\int_0^L(V_{1,x}+V_3)\overline{V_3}dx =\rho_2\int_0^L f_4\overline{V_4}dx+b\int_0^L V_{3,x}\overline{f_{3,x}}dx+k\int_0^L(V_{1,x}+V_3)\overline{f_3}dx. \tag{48}$$

Adding (47) and (48) together, we have:

$$(i\lambda+\alpha)\rho_1\int_0^L|V_2|^2\,dx-i\lambda k\int_0^L|V_{1,x}+V_3|^2\,dx + (i\lambda+\gamma)\rho_2\int_0^L|V_4|^2\,dx-i\lambda b\int_0^L|V_{3,x}|^2\,dx-\delta\int_0^L V_5\overline{V_{4,x}}dx =\rho_1\int_0^L f_2\overline{V_2}dx+\rho_2\int_0^L f_4\overline{V_4}dx+b\int_0^L V_{3,x}\overline{f_{3,x}}dx + k\int_0^L(V_{1,x}+V_3)(\overline{f_{1,x}}+\overline{f_3})dx. \tag{49}$$

Again, multiplying (28) by $\overline{V_5}$, we have:

$$i\lambda\rho_3\int_0^L|V_5|^2\,dx+\int_0^L(\delta V_{4,x}+V_{6,x})\overline{V_5}dx=\rho_3\int_0^L f_5\overline{V_5}dx. \tag{50}$$

Multiplying (29) by $\overline{V_6}$, then, from $V_6(0)=V_6(L)=0$, we have:

$$i\lambda\tau\int_0^L|V_6|^2\,dx-\int_0^L V_5\overline{V_{6,x}}dx+\beta\int_0^L|V_6|^2\,dx=\tau\int_0^L f_6\overline{V_6}dx. \tag{51}$$

Again, from the Cauchy-Schwarz inequality, there exists a positive C, such that

$$k\|\overline{V_2}\|\|f_2\|+\tau\|\overline{V_6}\|\|f_6\|+\rho_2\|\overline{V_4}\|\|f_4\|+k\|V_{1,x}+V_3\|\|\overline{f_{1,x}}+\overline{f_3}\| + b\|V_{3,x}\|\|\overline{f_{3,x}}\|+\rho_3\|\overline{V_5}\|\|f_5\|\le C\|V\|\|F\|. \tag{52}$$

Adding (49), (50), and (51) together, then taking the real part and using (52), we finally get

$$\alpha\rho_1\int_0^L|V_2|^2\,dx+\gamma\rho_2\int_0^L|V_4|^2\,dx+\beta\int_0^L|V_6|^2\,dx\le C\|V\|_{\mathcal{H}_1}\|F\|_{\mathcal{H}_1}.$$

This completes the proof.

Lemma II.4. Suppose that $\tau > 0$ and the resolvent equation $V = (i\lambda - \mathcal{A})^{-1}F$ is given by (23). Then, for positive constants ρ_3, there exists a positive constant C independent of F, such that

$$\rho_3 \int_0^L |V_5|^2 \, dx \le C \|V\|_{\mathcal{H}_1} \|F\|_{\mathcal{H}_1}.$$

Proof. Multiplying (28) by $\overline{V_5}$, and using $V_6(0) = V_6(L) = 0$, we get:

$$i\lambda\rho_3 \int_0^L |V_5|^2 \, dx + \delta \int_0^L V_{4,x} \overline{V_5} dx - \int_0^L \underbrace{V_6 \overline{V_{5,x}}}_{I_{23}} dx = \rho_3 \int_0^L f_5 \overline{V_5} dx.$$

By (29), we have $V_6 = \frac{\tau\beta - i\lambda\tau^2}{\lambda^2\tau^2 + \beta^2} f_6 + \frac{i\lambda\tau - \beta}{\lambda^2\tau^2 + \beta^2} V_{5,x}$. So, by substituting this equation into I_{23}, we have:

$$\begin{aligned} &\delta \int_0^L V_{4,x} \overline{V_5} dx + i\lambda\rho_3 \int_0^L |V_5|^2 \, dx + \frac{\beta - i\lambda\tau}{\lambda^2\tau^2 + \beta^2} \int_0^L |V_{5,x}|^2 \, dx \\ &= \rho_3 \int_0^L f_5 \overline{V_5} dx + \frac{\tau\beta - i\lambda\tau^2}{\lambda^2\tau^2 + \beta^2} \int_0^L f_6 \overline{V_{5,x}} dx. \end{aligned} \tag{53}$$

Adding (49) and (53) together and taking the real part yields:

$$\frac{\beta}{\lambda^2\tau^2 + \beta^2} \int_0^L |V_{5,x}|^2 \, dx \le C \|V\|_{\mathcal{H}_1} \|F\|_{\mathcal{H}_1}.$$

By Poincaré's inequality, for positive ρ_3, there exists a positive C, such that

$$\rho_3 \int_0^L |V_5|^2 \, dx \le C \|V\|_{\mathcal{H}_1} \|F\|_{\mathcal{H}_1}.$$

This completes the proof.

Lemma II.5. Suppose that $\tau > 0$ and the resolvent equation $V = (i\lambda - \mathcal{A}_1)^{-1}F$ is given by (23). Then, for positive constants b, there exists a positive constant C independent of F, such that

$$k \int_0^L |V_{1,x} + V_3|^2 \, dx + b \int_0^L |V_{3,x}|^2 \, dx \le C \|V\|_{\mathcal{H}_1} \|F\|_{\mathcal{H}_1}.$$

Proof. Multiplying (27) by $\overline{V_3}$, we have:

$$\begin{aligned} &(i\lambda + \gamma)\rho_2 \int_0^L \underbrace{V_4 \overline{V_3}}_{I_{24}} dx + k \int_0^L (V_{1,x} + V_3) \overline{V_3} dx \\ &+ \delta \int_0^L \underbrace{V_{5,x} \overline{V_3}}_{I_{25}} dx + b \int_0^L |V_{3,x}|^2 \, dx = \rho_2 \int_0^L f_4 \overline{V_3} dx. \end{aligned}$$

By (26), we have $\overline{V_3} = \frac{i}{\lambda} \overline{V_4} + \frac{i}{\lambda} \overline{f_3}$. Substitute this into I_{24}, I_{25}, and using the fact $f_3(0) = f_3(L) = 0$, we have:

$$\begin{aligned} &k \int_0^L (V_{1,x} + V_3) \overline{V_3} dx + b \int_0^L |V_{3,x}|^2 \, dx + \frac{i\gamma\rho_2}{\lambda} \int_0^L |V_4|^2 \, dx \\ &+ \frac{i\delta}{\lambda} \int_0^L V_{5,x} \overline{V_4} dx = \rho_2 \int_0^L f_4 \overline{V_3} dx + \rho_2 \int_0^L |V_4|^2 \, dx \\ &+ (\rho_2 - \frac{i\gamma\rho_2}{\lambda}) \int_0^L V_4 \overline{f_3} dx + \frac{i\delta}{\lambda} \int_0^L V_5 \overline{f_{3,x}} dx. \end{aligned} \tag{54}$$

Again, from the Cauchy-Schwarz inequality, we have:

$$\begin{aligned} \int_0^L V_{5,x} \overline{V_4} dx &\le C \|V_{5,x}\| \|V_4\| \le C(\|V_{5,x}\|^2 + \|V_4\|^2) \\ &\le C \|V\|_{\mathcal{H}_1} \|F\|_{\mathcal{H}_1}. \end{aligned} \tag{55}$$

Multiplying (25) by $\overline{V_1}$, we have:

$$(i\lambda + \alpha)\rho_1 \int_0^L \underbrace{V_2 \overline{V_1}}_{I26} dx + k \int_0^L (V_{1,x} + V_3) \overline{V_{1,x}} dx = \rho_1 \int_0^L f_2 \overline{V_1} dx.$$

By (24), we have $\overline{V_1} = \frac{i}{\lambda} \overline{V_2} + \frac{i}{\lambda} \overline{f_1}$, and by substituting this into I_{26}, we have:

$$\begin{aligned} &k \int_0^L (V_{1,x} + V_3) \overline{V_{1,x}} dx + \frac{i\alpha\rho_1}{\lambda} \int_0^L |V_2|^2 \, dx \\ &= \rho_1 \int_0^L f_2 \overline{V_1} dx + \rho_1 \int_0^L |V_2|^2 \, dx + \left(1 - \frac{i\alpha}{\lambda}\right) \rho_1 \int_0^L V_2 \overline{f_1} dx. \end{aligned} \tag{56}$$

Adding (54) and (56) together, then taking the real part and using (55), we obtain:

$$k \int_0^L |V_{1,x} + V_3|^2 \, dx + b \int_0^L |V_{3,x}|^2 \, dx \le C \|V\|_{\mathcal{H}_1} \|F\|_{\mathcal{H}_1}.$$

This completes the proof.

Now, we prove Theorem II.2.

Proof of Theorem II.2. We use Theorem II.1 to prove the exponential stability. First, from Lemma II.2, the resolvent set $\rho(\mathcal{A}_1)$ of $\mathcal{A}_1$ contains the imaginary axis, which satisfies the condition (i) of Theorem II.1. By the resolvent equation $V = (i\lambda - \mathcal{A}_1)^{-1}F$ given by (23), we have V and F, where $V = (V_1, V_2, V_3, V_4, V_5, V_6)^T$ and $f = (f_1, f_2, f_3, f_4, f_5, f_6)^T$ satisfy (24)–(29). Then, from Lemma II.3, II.4 and II.5, we see that there exists $C > 0$, independent of λ (and F, V), such that

$$\rho_1 \|V_2\|^2 + \rho_2 \|V_4\|^2 \le C \|V\|_{\mathcal{H}_1} \|F\|_{\mathcal{H}_1}, \tag{57}$$

$$\tau \|V_6\|^2 \le C \|V\|_{\mathcal{H}_1} \|F\|_{\mathcal{H}_1}, \tag{58}$$

$$\rho_3 \|V_5\|^2 \le C \|V\|_{\mathcal{H}_1} \|F\|_{\mathcal{H}_1}, \tag{59}$$

$$k \|V_{1,x} + V_3\|^2 + b \|V_{3,x}\|^2 \le C \|V\|_{\mathcal{H}_1} \|F\|_{\mathcal{H}_1}. \tag{60}$$

Adding (57)–(60) together, there exists a positive constant $C > 0$, such that

$$\|V\|_{\mathcal{H}_1}^2 \le C\|F\|_{\mathcal{H}_1}^2.$$

This completes the proof.

III. STABILITY OF SYSTEM (2)

In this section, we show the exponential stability of System (2) for the case $\tau = 0$. It is known that the total energy of the system (2) is given by

$$E(t) = \frac{1}{2}\int_0^L [\rho_1\varphi_t^2(t,x) + k(\varphi_x(t,x) + \psi(t,x))^2 + \rho_2\psi_t^2(t,x) + b\psi_x^2(t,x) + \rho_3\theta^2(t,x)]dx \tag{61}$$

and it is natural to consider the system (2) in the energy space. Let

$$\mathcal{H}_2 = H_*^1(0,L)\times L_*^2(0,L)\times H_0^1(0,L)\times L^2(0,L)\times L^2(0,L) \tag{62}$$

be the Hilbert space with norm

$$\|V\|^2 = \int_0^L [k|V_{1,x}(x) + V_3(x)|^2 + \rho_1|V_2(x)|^2 + b|V_{3,x}(x)|^2 + \rho_2|V_4(x)|^2 + \rho_3|V_5(x)|^2]dx$$

where $V = (V_1, V_2, V_3, V_4, V_5) \in \mathcal{H}_2$. Now, we define a system operator $\mathcal{A}_2$ as

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{k}{\rho_1}\partial_x^2 & -\alpha & \frac{k}{\rho_1}\partial_x & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{k}{\rho_2}\partial_x & 0 & \frac{b}{\rho_2}\partial_x^2 - \frac{k}{\rho_2} & 0 & -\frac{\delta}{\rho_2}\partial_x \\ 0 & 0 & 0 & -\frac{\delta}{\rho_3}\partial_x & \frac{1}{\rho_3\beta}\partial_{xx} \end{pmatrix} \tag{63}$$

with the domain of $\mathcal{A}_2$

$$D(\mathcal{A}_2) = \left\{ V \in \mathcal{H}_2 \middle| \begin{array}{l} V_1, V_3, V_5 \in H^2(0,L) \\ V_{1,x}, V_4, V_5 \in H_0^1(0,L) \\ V_2 \in H_*^1(0,L) \end{array} \right\}. \tag{64}$$

Then, the system (2) can be rewritten as an evolution equation in $\mathcal{H}_2$:

$$V_t(t) = \mathcal{A}_2 V(t), \quad V(t=0) = V_0, \tag{65}$$

where $V(t) = (\varphi(t,\cdot),\ \varphi_t(t,\cdot),\ \psi(t,\cdot),\ \psi_t(t,\cdot),\ \theta(t,\cdot))^T$ and $V_0 = (\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1,\ \theta_0)^T$ is the initial condition.

Lemma III.1. Let $\mathcal{A}_2$ be defined by (63) and (64). Then, $\mathcal{A}_2$ is the infinitesimal generator of a C_0-semigroup of contractions $e^{\mathcal{A}_2 t}$.

Proof. Since the proof is similar to Lemma II.1, we omit it here. This completes the proof.

Now, we get the exponential stability of System (65).

Theorem III.1. Let $\mathcal{A}_2$ be defined by (63) and (64). Then, the semigroup $e^{\mathcal{A}_2 t}$, generated by the operator $\mathcal{A}_2$, is exponentially stable, *i.e.*, there are two positive constants M, μ such that

$$\|S(t)\| \le Me^{-\mu t}.$$

By Theorem II.1, in order to show the exponential stability of System (65), we need to estimate the resolvent of $\mathcal{A}_2$. By the resolvent equation

$$V = (i\lambda - \mathcal{A}_2)^{-1}F, \quad \forall F \in \mathcal{H}_2, \lambda \in \mathbb{R}, \lambda \ne 0, \tag{66}$$

where $V = (V_1, \dots, V_5)^T \in D(\mathcal{A}_2)$, we get

$$i\lambda V_1 - V_2 = f_1, \tag{67}$$

$$i\lambda\rho_1 V_2 - k(V_{1,x} + V_3)_x + \alpha\rho_1 V_2 = \rho_1 f_2, \tag{68}$$

$$i\lambda V_3 - V_4 = f_3, \tag{69}$$

$$i\lambda\rho_2 V_4 + k(V_{1,x} + V_3) - bV_{3,xx} + \delta V_{5,x} = \rho_2 f_4, \tag{70}$$

$$i\lambda\rho_3 V_5 + \delta V_{4,x} - \frac{1}{\beta}V_{5,xx} = \rho_3 f_5. \tag{71}$$

Lemma III.2. Let $\mathcal{A}_2$ be defined by (63) and (64). Then, the resolvent set $\rho(\mathcal{A}_2)$ of $\mathcal{A}_2$ contains the imaginary axis.

Proof. We prove the result by a contradiction. If the claim is not true, then there exists $\omega \in \mathbb{R}$, a sequence $(\gamma_n)_n \subset \mathbb{R}$ with $\gamma_n \to \omega$, $|\gamma_n| < |\omega|$ and a sequence of functions

$$\begin{cases} V_n = (V_{n1}, V_{n2}, V_{n3}, V_{n4}, V_{n5})^T \in D(\mathcal{A}_2)\|V_n\|_{\mathcal{H}_2} \\ \quad = \int_0^L [k|V_{n1,x}(x) + V_{n3}(x)|^2 + \rho_1|V_{n2}(x)|^2 + b|V_{n3,x}(x)|^2 \\ \quad + \rho_2|V_{n4}(x)|^2 + \rho_3|V_{n5}(x)|^2]dx = 1 \end{cases} \tag{72}$$

so that, as $n \to \infty$,

$$\|(i\gamma_n I - \mathcal{A}_2)V_n\|_{\mathcal{H}_2} \to 0, \tag{73}$$

which yields:

$$\begin{cases} i\gamma_n V_{n1} - V_{n2} \to 0 \quad \text{in} \quad H^1(0,L), \\ i(\gamma_n+\alpha)\rho_1 V_{n2} - k(V_{n1,x}+V_{n3})_x \to 0 \quad \text{in} \quad L^2(0,L), \\ i\gamma_n V_{n3} - V_{n4} \to 0 \quad \text{in} \quad L^2(0,L), \\ i\gamma_n \rho_2 V_{n4} + k(V_{n1,x}+V_{n3}) - bV_{n3,xx} + \delta V_{n5,x} \to 0 \\ \text{in} \quad L^2(0,L), \\ i\gamma_n\rho_3 V_{n5} + \delta V_{n4,x} - \dfrac{1}{\beta}V_{n5,xx} \to 0 \quad \text{in} \quad L^2_*(0,L). \end{cases} \tag{74}$$

By (73), we have

$$\langle (i\gamma_n I - \mathcal{A}_2)V_n, V_n\rangle \to 0,$$

which yields:

$$\mathrm{Re}\langle (i\gamma_n I - \mathcal{A}_2)V_n, V_n\rangle_{\mathcal{H}_2} = \alpha\rho_1\int_0^L |V_{n2}|^2\,dx + \frac{1}{\beta}\int_0^L |V_{n5,x}|^2\,dx \to 0.$$

So, we have

$$V_{n2}\to 0, \quad V_{n5,x}\to 0, \tag{75}$$

then by (75) and the Poincaré's inequality, we have:

$$V_{n5}\to 0. \tag{76}$$

Then, by (72), (75), and (76), we have:

$$k\|V_{n1,x}+V_{n3}\|^2 + b\|V_{n3,x}\|^2 + \rho_2\|V_{n4}\|^2 \to 1. \tag{77}$$

On the other hand, by the fifth equation of (74), we have:

$$\left\langle (i\gamma_n\rho_3 V_{n5} + \delta V_{n4,x} - \frac{1}{\beta}V_{n5,xx}, V_{n3,x}\right\rangle \to 0.$$

Then, by (75), (76), and $V_{n3}(0) = V_{n3}(L) = V_{n5,x}(0) = V_{n5,x}(L) = 0$, we have:

$$\langle V_{n4,x}, V_{n3,x}\rangle \to 0. \tag{78}$$

Taking the inner product by $V_{n3,xx}$ with the third equation of (74) and $V_{n3}(0) = V_{n3}(L) = V_{n4}(0) = V_{n4}(L) = 0$, by (78), we have:

$$-i\gamma_n\|V_{n3,x}\|^2 \to 0 \Rightarrow V_{n3,x}\to 0. \tag{79}$$

Then, by Poincaré's inequality, we have:

$$V_{n3}\to 0. \tag{80}$$

Taking the inner product by V_n4 with the third equation of (74), then, by (80), we have:

$$\|V_{n4}\|^2 \to 0 \Rightarrow V_{n4}\to 0. \tag{81}$$

Substituting (79) and (81) into (77), we have:

$$k\|V_{n1,x}+V_{n3}\|^2 \to 1. \tag{82}$$

Taking the inner product by $(V_{n1,x} + V_{n3})$ with the forth equation of (74), we have

$$i\gamma_n\rho_2\langle V_{n4}, V_{n1,x}+V_{n3}\rangle + k\|V_{n1,x}+V_{n3}\|^2 - b\langle V_{n3,xx}, V_{n1,x}+V_{n3}\rangle + \delta\langle V_{n5,x}, V_{n1,x}+V_{n3}\rangle \to 0,$$

then by (75), (79), (81), and $V_{n1,x}(0) = V_{n1,x}(L) = V_{n3}(0) = V_{n3}(L) = 0$, we have

$$k\|V_{n1,x}+V_{n3}\|^2 \to 0. \tag{83}$$

which contradicts (82). Thus, the (*i*) of Theorem II.1 is proved. This completes the proof.

Lemma III.3. Suppose that $\tau = 0$ and the resolvent equation $V = (i\lambda - \mathcal{A}_2)^{-1}F$ is given by (66). Then, for positive constants α, β, ρ_1, there exists a positive constant C independent of F, such that

$$\alpha\rho_1\int_0^L |V_2|^2\,dx + \frac{1}{\beta}\int_0^L |V_{5,x}|^2\,dx \le C\|V\|_{\mathcal{H}_2}\|F\|_{\mathcal{H}_2}.$$

Proof. Multiplying (68) by $\overline{V_2}$, we have:

$$(i\lambda+\alpha)\rho_1\int_0^L |V_2|^2\,dx - k\int_0^L (V_{1,x}+V_3)_x\underbrace{\overline{V_2}}_{I_{30}}\,dx = \rho_1\int_0^L f_2\overline{V_2}dx.$$

By (67), we have $\overline{V_2} = -i\lambda\overline{V_1} - \overline{f_1}$. Substitute $\overline{V_2}$ into I_{30}, and by $V_{1,x}(0) = V_{1,x}(L) = V_3(0) = V_3(L) = 0$, we have:

$$\begin{aligned} &i\lambda\rho_1\int_0^L |V_2|^2\,dx - i\lambda k\int_0^L (V_{1,x}+V_3)\overline{V_{1,x}}dx \\ &+ \alpha\rho_1\int_0^L |V_2|^2\,dx = \rho_1\int_0^L f_2\overline{V_2}dx + k\int_0^L (V_{1,x}+V_3)\overline{f_{1,x}}dx. \end{aligned} \tag{84}$$

Multiplying (70) by $\overline{V_4}$, we have

$$\int_0^L [k(V_{1,x}+V_3) - bV_{3,xx}]\underbrace{\overline{V_4}}_{I_{31}}\,dx + i\lambda\rho_2\int_0^L |V_4|^2\,dx + \delta\int_0^L V_{5,x}\overline{V_4}dx = \rho_2\int_0^L f_4\overline{V_4}dx$$

and from (69), we have $\overline{V_4} = -i\lambda\overline{V_3} - \overline{f_3}$. Substitute this into I_{31}, and by $V_3(0) = V_3(L) = 0$, we have:

$$\begin{aligned} &i\lambda\rho_2\int_0^L |V_4|^2\,dx - i\lambda k\int_0^L (V_{1,x}+V_3)\overline{V_3}dx - i\lambda b\int_0^L |V_{3,x}|^2\,dx \\ &+ \delta\int_0^L V_{5,x}\overline{V_4}dx = \rho_2\int_0^L f_4\overline{V_4}dx \\ &+ k\int_0^L (V_{1,x}+V_3)\overline{f_3}dx + b\int_0^L V_{3,x}\overline{f_{3,x}}dx. \end{aligned} \tag{85}$$

Multiplying (71) by $\overline{V_5}$, and by $V_4(0) = V_4(L) = V_{5,x}(0) = V_{5,x}(L) = 0$, we have:

$$i\lambda\rho_3\int_0^L |V_5|^2\,dx - \delta\int_0^L V_4\overline{V_{5,x}}dx + \frac{1}{\beta}\int_0^L |V_{5,x}|^2\,dx = \rho_3\int_0^L f_5\overline{V_5}dx. \tag{86}$$

Adding (84), (85), and (86), then taking the real part, we obtain:

$$\alpha\rho_1\int_0^L |V_2|^2\,dx + \frac{1}{\beta}\int_0^L |V_{5,x}|^2\,dx \le C\|V\|_{\mathcal{H}_2}\|F\|_{\mathcal{H}_2}.$$

This completes the proof.

Lemma III.4. Suppose that $\tau = 0$ and the resolvent equation $V = (i\lambda - \mathcal{A}_2)^{-1}F$ is given by (66). Then, for positive constants ρ_2, there exists a positive constant C independent of F, such that

$$k\int_0^L |V_{1,x} + V_3|^2\,dx + \rho_2\int_0^L |V_4|^2\,dx \le C\|V\|_{\mathcal{H}_2}\|F\|_{\mathcal{H}_2}.$$

Proof. Multiplying (70) by $\overline{V_3}$, we have:

$$i\lambda\rho_2\int_0^L V_4\underbrace{\overline{V_3}}_{I_{32}}\,dx + k\int_0^L (V_{1,x}+V_3)\overline{V_3}dx - b\int_0^L \underbrace{V_{3,xx}\overline{V_3}}_{I_{33}}\,dx + \delta\int_0^L V_{5,x}\underbrace{\overline{V_3}}_{I_{34}}\,dx = \rho_2\int_0^L f_4\overline{V_3}dx,$$

By (69), we have $V_3 = -\frac{i}{\lambda}V_4 - \frac{i}{\lambda}f_3$. Substitute V_3 into I_{32}, I_{33}, and I_{34}, and by $V_3(0) = V_3(L) = V_4(0) = V_4(L) = 0$, we have:

$$\begin{aligned} k\int_0^L (V_{1,x}+V_3)\overline{V_3}dx + \frac{b}{\lambda^2}\int_0^L |V_{4,x}|^2\,dx + \frac{i\delta}{\lambda}\int_0^L V_{5,x}\overline{V_4}dx \\ = \rho_2\int_0^L f_4\overline{V_3}dx + \rho_2\int_0^L V_4\overline{f_3}dx - \frac{b}{\lambda^2}\int_0^L V_{4,x}\overline{f_{3,x}}dx \\ + \frac{ib}{\lambda}\int_0^L f_{3,x}\overline{V_{3,x}}dx + \frac{i\delta}{\lambda}\int_0^L V_5\overline{f_{3,x}}dx + \rho_2\int_0^L |V_4|^2\,dx. \end{aligned} \tag{87}$$

Multiplying (68) by $\overline{V_1}$, then, by $V_{1,x}(0) = V_{1,x}(L) = V_3(0) = V_3(L) = 0$, we have:

$$(i\lambda+\alpha)\rho_1\int_0^L V_2\underbrace{\overline{V_1}}_{I_{35}}\,dx + k\int_0^L (V_{1,x}+V_3)\overline{V_{1,x}}dx = \rho_1\int_0^L f_2\overline{V_1}dx.$$

From (67), we have $\overline{V_1} = \frac{i}{\lambda}\overline{V_2} + \frac{i}{\lambda}\overline{f_1}$. Substituting $\overline{V_1}$ into I_{35}, results in

$$\begin{aligned} \left(\frac{i\alpha}{\lambda} - 1\right)\rho_1\int_0^L |V_2|^2\,dx + k\int_0^L (V_{1,x}+V_3)\overline{V_{1,x}}dx \\ = \rho_1\int_0^L f_2\overline{V_1}dx + \rho_1\int_0^L V_2\overline{f_1}dx - \frac{i\alpha\rho_1}{\lambda}\int_0^L V_2\overline{f_1}dx. \end{aligned} \tag{88}$$

Again, from Cauchy-Schwarz inequality, we have:

$$\begin{aligned} \int_0^L V_{5,x}\overline{V_4}dx \le C\|V_{5,x}\|\|V_4\| \le C(\|V_{5,x}\|^2 + \|V_4\|^2) \\ \le C\|V\|_{\mathcal{H}_2}\|F\|_{\mathcal{H}_2}. \end{aligned} \tag{89}$$

Adding (87) and (88), then taking the real part and using (89) with it, we have:

$$k\int_0^L |V_{1,x} + V_3|^2\,dx + \rho_2\int_0^L |V_4|^2\,dx \le C\|V\|_{\mathcal{H}_2}\|F\|_{\mathcal{H}_2}.$$

This completes the proof.

Lemma III.5. Suppose that $\tau = 0$ and the resolvent equation $V = (i\lambda - \mathcal{A}_2)^{-1}F$ is given by (66). Then, for positive constants b, there exists a positive constant C independent of F, such that

$$b\int_0^L |V_{3,x}|^2 \le C\|V\|_{\mathcal{H}_2}\|F\|_{\mathcal{H}_2}.$$

Proof. Multiplying (70) by $\overline{V_3}$, we have:

$$\begin{aligned} i\lambda\rho_2\int_0^L \underbrace{V_4}_{I_{38}}\overline{V_3}dx + k\int_0^L (V_{1,x}+V_3)\overline{V_3}dx \\ - b\int_0^L V_{3,xx}\overline{V_3}dx + \delta\int_0^L V_{5,x}\overline{V_3}dx = \rho_2\int_0^L f_4\overline{V_3}dx. \end{aligned}$$

By (69), we have $V_4 = i\lambda V_3 – f_3$. Substitute V_4 into I_{38}, and by $V_3(0) = V_3(L) = 0$, we have:

$$\begin{aligned} -\lambda^2\rho_2\int_0^L |V_3|^2\,dx + k\int_0^L (V_{1,x}+V_3)\overline{V_3}dx + b\int_0^L |V_{3,x}|^2\,dx \\ + \delta\int_0^L V_{5,x}\overline{V_3}dx = \rho_2\int_0^L f_4\overline{V_3}dx + i\lambda\rho_2\int_0^L f_3\overline{V_3}dx. \end{aligned} \tag{90}$$

Again, from Cauchy-Schwarz inequality, we have:

$$\begin{aligned} \int_0^L V_{5,x}\overline{V_3}dx \le C\|V_{5,x}\|\|V_3\| \le C(\|V_{5,x}\|^2 + \|V_3\|^2) \\ \le C\|V\|_{\mathcal{H}_2}\|F\|_{\mathcal{H}_2}. \end{aligned} \tag{91}$$

Then, by adding (88) and (90) together, taking the real part, and using Poincaré's inequality, by (91) we have $b\int_0^L |V_{3,x}|^2 \le C\|V\|_{\mathcal{H}_2}\|F\|_{\mathcal{H}_2}$. This completes the proof.

Now, we prove Theorem III.1.

Proof of Theorem III.1. We use Theorem II.1 to prove the exponential stability. First, from Lemma III.2, the resolvent set $\rho(\mathcal{A}_2)$ of $\mathcal{A}_2$ contains the imaginary axis, which satisfies Condition (*i*) of Theorem II.1. By the resolvent equation $V = (i\lambda - \mathcal{A}_2)^{-1}F$ given by (66), we have V and F, where $V = (V_1, V_2, V_3, V_4, V_5)^T$ and $F = (f_1, f_2, f_3, f_4, f_5)^T$ satisfy (67)–(71). Then, from Lemma III.3, Lemma III.4 and Lemma

III.5, we see that there exists $C > 0$, independent of λ (and F, V), such that

$$\rho_1 \|V_2\|^2_{\mathcal{H}_2} + \rho_3 \|V_5\|^2_{\mathcal{H}_2} \leq C \|V\|_{\mathcal{H}_2} \|F\|_{\mathcal{H}_2}, \tag{92}$$

$$k \|V_{1,x} + V_3\|^2_{\mathcal{H}_2} + \rho_2 \|V_4\|^2_{\mathcal{H}_2} \leq C \|V\|_{\mathcal{H}_2} \|F\|_{\mathcal{H}_2}, \tag{93}$$

$$b \|V_{3,x}\|^2_{\mathcal{H}_2} \leq C \|V\|_{\mathcal{H}_2} \|F\|_{\mathcal{H}_2}. \tag{94}$$

Adding (92)–(94) together, then there exists a positive C, such that

$$\|V\|^2_{\mathcal{H}_2} \leq C \|F\|^2_{\mathcal{H}_2}.$$

This completes the proof.

IV. CONCLUSIONS

A one-dimensional damped hyperbolic Timoshenko beam that is coupled with a heat equation is considered. With internal dampings being introduced and the help of the frequency domain method, Timoshenko beam under Cattaneo's and Fourier's law is showed to be exponentially stable, respectively.

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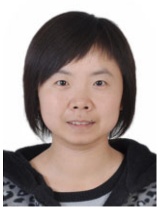

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