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Stabilization of an ODE–Schrödinger Cascade

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ABSTRACT

We consider the problem of stabilization of a linear ODE with input dynamics governed by the linearized Schrödinger equation. The interconnection between the ODE and Schrödinger equation is bi-directional at a single point. We construct an explicit feedback law that compensates the Schrödinger dynamics at the inputs of the ODE and stabilizes the overall system. Our design is based on a two-step backstepping transformation by introducing an intermediate system and an intermediate control. By adopting the Riesz basis approach, the exponential stability of the closed-loop system is built with the pre-designed decay rate and the spectrum-determined growth condition is obtained. A numerical simulation is provided to illustrate the effectiveness of the proposed design.

1. Introduction

The backstepping approach, which was originally developed for parabolic PDEs [1,2], has recently been applied to control problems for PDE–ODE cascades [3–7] etc., with applications of interest including fluids, structures, thermal, chemically-reacting, and plasma systems. This method uses an invertible Volterra integral transformation together with the boundary feedback to convert the unstable plant into a well-damped target system. The kernel of this transformation satisfies a certain PDE and ODE, which turn out to be solvable in closed form. In [3], the backstepping method was applied to a system which consists of the first-order hyperbolic PDE coupled with a second-order (in space) ODE. This system resembles the Korteweg–de Vries equation which describes shallow water waves and ion acoustic waves in plasma. In [5], predictor-like feedback laws and observers have been developed for diffusion PDE–ODE cascades. In [6], an explicit feedback law was developed that compensates the wave PDE dynamics at the input of an linear time-invariant ODE and stabilizes the overall system. In [7], PDE–ODE cascades considered were extended from Dirichlet type interconnections to Neumann type interconnections. For PDE–ODE cascades considered above, the interconnection between the PDE and the ODE is one directional. For example, in [5], the dynamics at the input of the ODE is governed by a heat equation, whereas the ODE does not act on the PDE. In some cases, the interconnection between them could be bi-directional, i.e., the PDE and ODE are coupled with each other as discussed in [8].

In this paper, we consider a problem of stabilization of an ODE–Schrödinger equation cascade as in (1)–(4), where the interconnection between them is bi-directional at a single point:

$$\dot{X}(t) = AX(t) + Bu(0,t), \quad t > 0, \tag{1}$$

$$u_t(x,t) = -iu_{xx}(x,t), \quad x \in (0,1),\ t > 0, \tag{2}$$

$$u_x(0,t) = CX(t), \tag{3}$$

$$u(1,t) = U(t) \tag{4}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times 1}$, $C \in \mathbb{C}^{1\times n}$; $X(t) \in \mathbb{C}^{n\times 1}$ is the state of ordinary differential equation; $u(x,t) \in \mathbb{C}$ is the state of Schrödinger equation; and $U(t) \in \mathbb{C}$ is the control force to the entire system. The whole system is depicted in Fig. 1. Both states of ODE and Schrödinger equation are complex valued. The motivation for this kind of problem can be provided in the context of various applications in quantum mechanics, chemical process control, and other areas.

The stabilization of the linearized Schrödinger equation using boundary control was investigated in [9–11]. For exact controllability and observability results, see [12–14]. The interconnection of Schrödinger–heat equation with boundary coupling was considered in [15] and the Schrödinger equation has the Gevrey regularity under the compensation of the heat equation. As in [11,16], the Schrödinger equation is usually considered as a complex-valued

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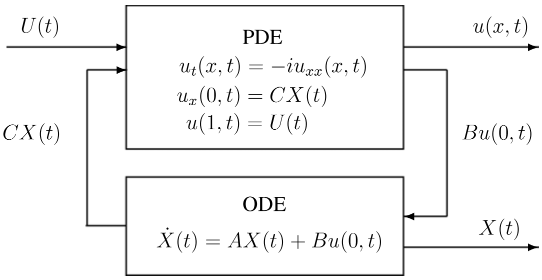

Fig. 1. Block diagram for the coupled ODE–PDE system.

heat equation such that the backstepping method developed for parabolic PDEs could be applied. Motivated by two-step backstepping transformations in [5], which was adopted to improve performance and achieve exponential stability with arbitrarily fast decay rate, we adopt two-step backstepping control design to make the system stable in this paper.

Compared with our previous works using backstepping designs, the main contributions of the results in this paper lie in:

(i) Instead of one-step backstepping control, which results in difficulty in finding the kernels for the stabilization of the ODE–Schrödinger cascade considered in this paper, the design developed in this paper is based on a two-step backstepping transformation by introducing an intermediate system and an intermediate control.
(ii) Instead of the Lyapunov synthesis, the Riesz basis approach is adopted in this paper, through which the exponential stability of the closed-loop system was built with the pre-designed decay rate and the spectrum-determined growth condition was obtained.

The paper is organized as follows. In Section 2, the two-step backstepping design is developed using the bounded and invertible operators. First, we design the comprehensive control to convert the original system into the intermediate system. Secondly, we design the intermediate control to convert the intermediate system into the final target system. Section 3 is devoted to the spectral analysis of the target system. In Section 4, Riesz spectral method is adopted for the stability analysis of the closed-loop system. Finally, a simulation example and the concluding remarks are provided in Section 5.

In this paper, we consider the following energy Hilbert space

$$\mathcal{H} = \mathbb{C}^n \times L^2(0,1) \tag{5}$$

with inner product

$$\langle f_1, f_2\rangle_{\mathcal{H}} = \overline{X_1^T}X_2 + \int_0^1 \overline{z_1(x)}z_2(x)dx,$$
$$f_i = (X_i, z_i) \in \mathcal{H}, \quad i = 1, 2, \tag{6}$$

and $\mathcal{H}$-norm

$$\|f_i\|_{\mathcal{H}} = \left(\|X_i\|^2_{\mathbb{C}^n} + \|z_i\|^2_{L^2(0,1)}\right)^{\frac{1}{2}}. \tag{7}$$

2. Backstepping design

The control objective is to make the system (1)–(4) exponentially stable. To achieve this, a new cascaded ODE–PDE target system is introduced in the form:

$$\dot{X}(t) = (A + BK)X(t) + Bz(0,t), \tag{8}$$
$$z_t(x,t) = -iz_{xx}(x,t) - cz(x,t), \quad c > 0, \tag{9}$$
$$z_x(0,t) = 0, \tag{10}$$
$$z(1,t) = 0 \tag{11}$$

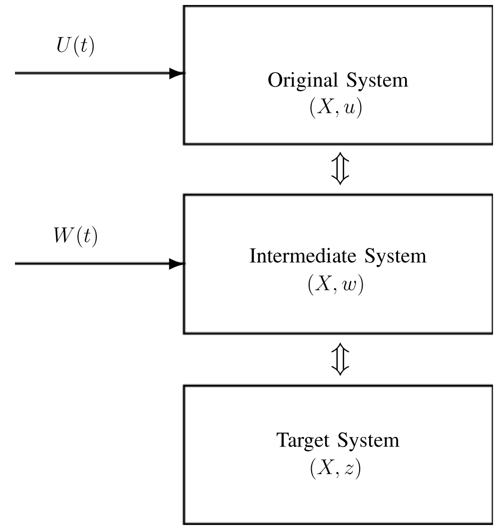

Fig. 2. Block diagram for two-step backstepping control design.

where $z(x,t) \in \mathbb{C}, c > 0$ is an arbitrary pre-defined decay rate, and we assume that the pair (A, B) is stabilizable and take $K \in \mathbb{C}^{1\times n}$ to be a known vector such that $A + BK$ is Hurwitz.

In this section, we seek the boundary controller $U(t)$ in (4) to exponentially stabilize the system (X, u) (1)–(4) into the target system (X, z) (8)–(11) using the backstepping control design for parabolic PDEs [1,2]. The method uses invertible Volterra integral transformation together with the boundary feedback to convert the unstable plant (X, u) (1)–(4) into a well-damped target system (X, z) (8)–(11). For this we use a change of variables based on a Volterra series. However, the one-step backstepping transformation for $(X, u) \longmapsto (X, z)$ results in difficulty in finding the kernels. To avoid this, we introduce an intermediate system (X, w):

$$\dot{X}(t) = (A + BK)X(t) + Bw(0,t), \tag{12}$$
$$w_t(x,t) = -iw_{xx}(x,t), \tag{13}$$
$$w_x(0,t) = 0, \tag{14}$$
$$w(1,t) = W(t) \tag{15}$$

where $w(x,t) \in \mathbb{C}$, and $W(t) \in \mathbb{C}$ is the intermediate control.

The main idea is using two-step backstepping control design as shown in Fig. 2: (i) design the comprehensive control $U(t)$ to convert the original system into the intermediate system; and (ii) design the intermediate control $W(t)$ to convert the intermediate system (X, w) into the final target system (X, z).

2.1. Design for system (X, u) to (X, w)

In this subsection, we look for the backstepping control $U(t)$ and the transformation $(X, u) \longmapsto (X, w)$ between systems (1)–(4) and (12)–(15).

We postulate the transformation $(X, u) \longmapsto (X, w)$ in the form

$$X(t) = X(t) \tag{16}$$

$$w(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)dy - \gamma(x)X(t),$$
$$0 \le y \le x \le 1 \tag{17}$$

where kernels $q(x, y)$ and $\gamma(x)$ are to be derived.

Differentiating (17) once and twice with respect to x, we get, respectively,

$$w_x(x,t) = u_x(x,t) - q(x,x)u(x,t) - \int_0^x q_x(x,y)u(y,t)dy - \gamma'(x)X(t) \tag{18}$$

$$w_{xx}(x,t) = u_{xx}(x,t) - q(x,x)u_x(x,t) - \left(q'(x,x) + q_x(x,x)\right)u(x,t) - \int_0^x q_{xx}(x,y)u(y,t)dy - \gamma''(x)X(t) \quad (19)$$

where $q'(x,x) = q_x(x,x) + q_y(x,x)$.

The first derivative of $w(x,t)$ with respect to t is

$$\begin{aligned} w_t(x,t) &= u_t(x,t) - \int_0^x q(x,y)u_t(y,t)dy \\ &\quad - \gamma(x)(AX(t) + Bu(0,t)) \\ &= -iu_{xx}(x,t) + iq(x,x)u_x(x,t) - iq_y(x,x)u(x,t) \\ &\quad + i\int_0^x q_{yy}(x,y)u(y,t)dy + \big(iq_y(x,0) \\ &\quad - \gamma(x)B\big)u(0,t) - (\gamma(x)A + iq(x,0)C)X(t). \end{aligned} \quad (20)$$

Let us now examine the expressions:

$$w(0,t) = u(0,t) - \gamma(0)X(t) \quad (21)$$

$$w_x(0,t) = (C - \gamma'(0))X(t) - q(0,0)u(0,t) \quad (22)$$

$$\begin{aligned} &w_t(x,t) + iw_{xx}(x,t) \\ &= -2iq'(x,x)u(x,t) - (iq(x,0)C + \gamma(x)A + i\gamma''(x))X(t) \\ &\quad + (iq_y(x,0) - \gamma(x)B)u(0,t) \\ &\quad + i\int_0^x (q_{yy}(x,y) - q_{xx}(x,y))u(y,t)dy. \end{aligned} \quad (23)$$

A sufficient condition for (12)–(15) to hold for any continuous functions $u(x,t)$ and $X(t)$ is that $\gamma(x)$ and $q(x,y)$ satisfy

$$\begin{cases} \gamma''(x) - i\gamma(x)A + q(x,0)C = 0, \quad x \in (0,1) \\ \gamma(0) = K, \\ \gamma'(0) = C, \end{cases} \quad (24)$$

which represents a second-order ODE in x, and

$$\begin{cases} q_{xx}(x,y) = q_{yy}(x,y), \quad 0 < y < x < 1 \\ q(x,x) = 0, \\ q_y(x,0) = -i\gamma(x)B, \end{cases} \quad (25)$$

which is a second-order hyperbolic PDE. A direct computation gives the solution of (25):

$$q(x,y) = \int_0^{x-y} i\gamma(\sigma)Bd\sigma. \quad (26)$$

Substituting (26) into (24) leads to

$$\gamma''(x) - i\gamma(x)A + \int_0^x i\gamma(\sigma)Bd\sigma C = 0. \quad (27)$$

Differentiating (27) once, we get the third order ODE

$$\begin{cases} \gamma^{(3)}(x) - i\gamma'(x)A + i\gamma(x)BC = 0, \\ \gamma''(0) = iKA, \quad \gamma'(0) = C, \quad \gamma(0) = K. \end{cases} \quad (28)$$

Define

$$\Gamma(x) = [\gamma(x)\ \gamma'(x)\ \gamma''(x)], \qquad D = \begin{bmatrix} 0 & 0 & -iBC \\ I & 0 & iA \\ 0 & I & 0 \end{bmatrix}.$$

Then (28) is written into

$$\Gamma'(x) = \Gamma(x)D. \quad (29)$$

Its solution is

$$\Gamma(x) = \Gamma(0)e^{Dx}, \quad \Gamma(0) = [K\ \ C\ \ iKA].$$

Hence, the solution to the ODE (24) is

$$\gamma(x) = [K\ \ C\ \ iKA]e^{\begin{bmatrix} 0 & 0 & -iBC \\ I & 0 & iA \\ 0 & I & 0 \end{bmatrix}x} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}. \quad (30)$$

The transformation $(X,u) \longmapsto (X,w)$ (16)–(17) is invertible, and the inverse transformation $(X,w) \longmapsto (X,u)$ is postulated in the following form

$$X(t) = X(t) \quad (31)$$

$$u(x,t) = w(x,t) + \int_0^x \iota(x,y)w(y,t)dy + \psi(x)X(t) \quad (32)$$

where kernels $\iota(x,y)$ and $\psi(x)$ to be derived.

In a similar manner to finding the kernels $q(x,y)$ and $\gamma(x)$ of the direct transformation, the derivatives u_x, u_{xx} and u_t are computed, and a sufficient condition for (1)–(3) to hold is that $\iota(x,y)$ and $\psi(x)$ satisfy

$$\begin{cases} \psi''(x) - i\psi(x)(A + BK) = 0, \\ \psi(0) = K, \\ \psi'(0) = C, \end{cases} \quad (33)$$

and

$$\begin{cases} \iota_{xx}(x,y) = \iota_{yy}(x,y), \\ \iota(x,x) = 0, \\ \iota_y(x,0) = -i\psi(x)B. \end{cases} \quad (34)$$

This cascade system can be solved explicitly. The solution to the ODE (33) is

$$\psi(x) = [K\ \ C]e^{\begin{bmatrix} 0 & i(A+BK) \\ I & 0 \end{bmatrix}x} \begin{bmatrix} I \\ 0 \end{bmatrix} \quad (35)$$

and the explicit solution to the PDE (34) is

$$\iota(x,y) = \int_0^{x-y} i\psi(\sigma)Bd\sigma. \quad (36)$$

Thus, the control law is obtained by setting $x = 1$ in (17):

$$U(t) = u(1,t) = W(t) + \int_0^1 q(1,y)u(y,t)dy + \gamma(1)X(t) \quad (37)$$

where $q(x,y)$ and $\gamma(x)$ are defined in (26) and (30) respectively, and the intermediate control $W(t)$ to be determined in Section 2.2 later.

From the above computations, it is noted that the transformations (16)–(17) and (31)–(32) are invertible between (X,u) and (X,w) in $\mathcal{H}$. For convenience, we introduce the following lemma.

Lemma 1. *Let $\mathcal{F}_1 : \mathcal{H} \to \mathcal{H}$ be a linear operator defined by*

$$\left[\mathcal{F}_1 \begin{pmatrix} X \\ u \end{pmatrix}\right](x) = \begin{pmatrix} X \\ u(x) - \int_0^x q(x,y)u(y)dy - \gamma(x)X \end{pmatrix}, \quad (X,u) \in \mathcal{H},\ x \in (0,1), \quad (38)$$

where $q(x,y)$ and $\gamma(x)$ are given by (26) and (30) respectively.

Then $\mathcal{F}_1$ is bounded and invertible on $\mathcal{H}$. Moreover, $\mathcal{G}_1 = (\mathcal{F}_1)^{-1}$ has the following expression:

$$\left[\mathcal{G}_1 \begin{pmatrix} X \\ w \end{pmatrix}\right](x) = \begin{pmatrix} X \\ w(x) + \int_0^x \iota(x,y)w(y)dy + \psi(x)X \end{pmatrix}, \quad (X,w) \in \mathcal{H}, x \in (0,1), \quad (39)$$

where $\psi(x)$ and $\iota(x,y)$ are given by (35) and (36) respectively.

Proof. It is noted that the transformations (16)–(17) and (31)–(32) are invertible between (X, u) and (X, w) in $\mathcal{H}$. We have that $\mathcal{F}_1\mathcal{G}_1 = \mathcal{G}_1\mathcal{F}_1 = I$ and $(\mathcal{F}_1)^{-1} = \mathcal{G}_1$.

Let $(X, w) = \mathcal{F}_1(X, u)$ be given by (38). Denote

$$\Theta(s) = \int_0^s \gamma(\sigma)Bd\sigma, \tag{40}$$

$$\Psi(s) = \int_0^s \psi(\sigma)Bd\sigma, \tag{41}$$

and the convolution operator

$$(f * g)(x) = \int_0^x f(x-y)u(y)dy, \tag{42}$$

for functions $f(x), g(x) \in L^2(0,1)$. As $\Theta, \gamma \in L^2(0,1)$, $\|\Theta\|_{L^2(0,1)} \le \|B\|_\infty \|\gamma\|_{L^2(0,1)}$ and $(X, w) = \mathcal{F}_1(X, u)$ are given by (38), a direct calculation gives that

$$\begin{aligned}\|w\|^2_{L^2(0,1)} &= \int_0^1 w(x)\overline{w(x)}dx \\ &= \int_0^1 [u(x) - (\Theta * u)(x) - \gamma(x)X] \\ &\quad \times \overline{[u(x) - (\Theta * u)(x) - \gamma(x)X]}dx \\ &\le 3\int_0^1 \left[|u(x)|^2 + |(\Theta * u)(x)|^2 + |\gamma(x)X|^2\right]dx \\ &\le \alpha_1\|u\|^2_{L^2(0,1)} + \alpha_2\|X\|^2_{\mathbb{C}^n}\end{aligned}$$

where Cauchy–Schwarz Inequality is used and

$$\alpha_1 = 3(1 + \|\Theta\|^2_{L^2(0,1)}), \qquad \alpha_2 = 3\|\gamma\|^2_{L^2(0,1)}. \tag{43}$$

Hence, there is an $M = \max\{\alpha_1, \alpha_2 + 1\}$ such that

$$\begin{aligned}\|(X, w)\|^2_{\mathcal{H}} &= \|w\|^2_{L^2(0,1)} + \|X\|^2_{\mathbb{C}^n} \\ &\le \alpha_1\|u\|^2_{L^2(0,1)} + (1+\alpha_2)\|X\|^2_{\mathbb{C}^n} \\ &\le M\|(X, u)\|^2_{\mathcal{H}},\end{aligned}$$

which indicates that $\mathcal{F}_1$ is bounded on $\mathcal{H}$. Similarly we can show that $\mathcal{G}_1$ is bounded on $\mathcal{H}$. The proof is complete. □

2.2. *Design for system (X, w) to (X, z)*

In this subsection, we look for the intermediate control $W(t)$ and the transformation $(X, w) \longmapsto (X, z)$ between the intermediate system (12)–(15) and the final target system (8)–(11). As in [11,16], the Schrödinger equation is usually considered as a complex-valued heat equation such that the backstepping method developed for parabolic PDEs [1] could be applied.

Taking the diffusion coefficient as $-i$ in the reaction–advection–diffusion equation [1] (with the advection and reaction coefficients being zero) and following the control design developed there, we use the transformation $(X, w) \longmapsto (X, z)$ in the form

$$X(t) = X(t) \tag{44}$$

$$z(x,t) = w(x,t) - \int_0^x \kappa(x,y)w(y,t)dy \tag{45}$$

along with the intermediate controller $W(t)$ at $x = 1$:

$$W(t) = w(1,t) = \int_0^1 \kappa(1,y)w(y,t)dy \tag{46}$$

where $\kappa(x, y)$ is a complex-valued control gain.

Using the same reasoning as above, recalling that z satisfies (8)–(11), we get the gain kernel PDE in the form

$$\begin{cases}\kappa_{xx}(x,y) - \kappa_{yy}(x,y) = ci\kappa(x,y), & 0 < y < x < 1, \\ \kappa_y(x,0) = 0, \\ \kappa(x,x) = -\dfrac{ci}{2}x.\end{cases} \tag{47}$$

The explicit solution to the PDE (47) is given in [1] by:

$$\begin{aligned}\kappa(x,y) &= -cix\frac{I_1\left(\sqrt{ci(x^2-y^2)}\right)}{\sqrt{ci(x^2-y^2)}} \\ &= -cix - cix\sum_{m=1}^{\infty}\frac{(ci(x^2-y^2))^m}{4^m m!(m+1)!}, \quad 0 \le y \le x \le 1,\end{aligned} \tag{48}$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind. In a similar manner to finding the kernel $\kappa(x, y)$, the inverse of the transformation (44), (45) can be found as follows

$$X(t) = X(t) \tag{49}$$

$$w(x,t) = z(x,t) + \int_0^x p(x,y)z(y,t)dy \tag{50}$$

where

$$\begin{aligned}p(x,y) &= -cix\frac{J_1\left(\sqrt{ci(x^2-y^2)}\right)}{\sqrt{ci(x^2-y^2)}} \\ &= -cix - cix\sum_{m=1}^{\infty}\frac{(-1)^m(ci(x^2-y^2))^m}{4^m m!(m+1)!}, \\ &\quad 0 \le y \le x \le 1,\end{aligned} \tag{51}$$

and J_1 is a Bessel function of the first kind.

Thus, the intermediate controller $W(t)$ is given by substituting (48) into (46)

$$\begin{aligned}W(t) &= w(1,t) = \int_0^1 \kappa(1,y)w(y,t)dy \\ &= -ci\int_0^1 \frac{I_1\left(\sqrt{ci(1-y^2)}\right)}{\sqrt{ci(1-y^2)}}w(y,t)dy.\end{aligned} \tag{52}$$

Similar to the Lemma 1, we have the following lemma for the transformations (44)–(45) and (49)–(50) between (X, w) and (X, z) in $\mathcal{H}$.

Lemma 2. *Let $\mathcal{F}_2 : \mathcal{H} \to \mathcal{H}$ be a linear operator defined by*

$$\left[\mathcal{F}_2\begin{pmatrix}X \\ w\end{pmatrix}\right](x) = \begin{pmatrix}X \\ w(x) - \displaystyle\int_0^x \kappa(x,y)w(y)dy\end{pmatrix}, \quad (X, w) \in \mathcal{H},\ x \in (0,1), \tag{53}$$

where $\kappa(x, y)$, $0 \le y \le x \le 1$, is given by (48).

Then $\mathcal{F}_2$ is bounded and invertible on $\mathcal{H}$. Moreover, $\mathcal{G}_2 = (\mathcal{F}_2)^{-1}$ has the following expression:

$$\left[\mathcal{G}_2\begin{pmatrix}X \\ z\end{pmatrix}\right](x) = \begin{pmatrix}X \\ z(x) + \displaystyle\int_0^x p(x,y)z(y)dy\end{pmatrix}, \quad (X, z) \in \mathcal{H}, x \in (0,1), \tag{54}$$

where $p(x, y)$, $0 \le y \le x \le 1$, is given by (51).

3. Stability analysis for the target system

In this section, we consider the target system (8)–(11). Define the system operator of (8)–(11) by

$$\begin{cases}\mathcal{A}_z(X,z) = \left((A+BK)X + Bz(0),\ -iz'' - cz\right), \\ \quad \forall\ (X,z) \in D(\mathcal{A}_z), \\ D(\mathcal{A}_z) = \left\{(X,z) \in \mathbb{C}^n \times H^2(0,1) | z(1) = z'(0) = 0\right\}.\end{cases} \tag{55}$$

Then (8)–(11) can be written as an evolution equation in $\mathcal{H}$:

$$\begin{cases}\dfrac{dY_z(t)}{dt} = \mathcal{A}_z Y_z(t), \quad t > 0, \\ Y_z(0) = Y_{z0}.\end{cases} \tag{56}$$

where $Y_z(t) = (X(t), z(\cdot, t))$.

Theorem 1. *Let $\mathcal{A}_z$ be given by (55). Then $\mathcal{A}_z^{-1}$ exists and is compact on $\mathcal{H}$ and hence $\sigma(\mathcal{A}_z)$, the spectrum of $\mathcal{A}_z$, consists of isolated eigenvalues of finitely algebraic multiplicity only.*

Proof. For any given $(X_1, z_1) \in \mathcal{H}$, solve

$$\mathcal{A}_z(X,z) = \left((A+BK)X + Bz(0),\ -iz'' - cz\right) = (X_1, z_1).$$

We get

$$\begin{cases}(A+BK)X + Bz(0) = X_1, \\ -iz''(x) - cz(x) = z_1(x), \\ z(1) = z'(0) = 0,\end{cases}$$

with the solution

$$\begin{cases}X = (A+BK)^{-1}(X_1 - Bz(0)), \\ z(x) = \varphi(x) - \dfrac{\varphi(1)}{\cosh(\sqrt{ci})}\cosh(\sqrt{ci}x), \\ \varphi(x) = \sqrt{c^{-1}i}\displaystyle\int_0^x \sinh(\sqrt{ci}(x-\tau))z_1(\tau)d\tau, \\ \qquad z(0) = -\dfrac{\varphi(1)}{\cosh(\sqrt{ci})}.\end{cases} \tag{57}$$

Hence, we get the unique $(X,z) \in D(\mathcal{A}_z)$. Hence, $\mathcal{A}_z^{-1}$ exists and is compact on $\mathcal{H}$ by the Sobolev embedding theorem [17, p. 85]. Therefore, $\sigma(\mathcal{A}_z)$ consists of isolated eigenvalues of finite algebraic multiplicity only. □

Now we are in a position to consider the eigenvalue problem of $\mathcal{A}_z$. Let $\mathcal{A}_z Y_z = \lambda Y_z$, where $Y_z = (X,z)$. Then we have

$$\begin{cases}(A+BK)X + Bz(0) = \lambda X, \\ -iz'' - cz = \lambda z, \\ z'(0) = z(1) = 0.\end{cases} \tag{58}$$

Denote

$$\lambda_m^p = -c + \left(m + \frac{1}{2}\right)^2 \pi^2 i, \quad m \in \mathbb{N}. \tag{59}$$

Theorem 2. *Let $\mathcal{A}_z$ be given by (55), let $\lambda_j^o, j = 1,2,\ldots,n$ be the simple eigenvalue of $A+BK$ with the corresponding eigenvector X_j, and assume that*

$$\lambda_j^o \notin \left\{\lambda_m^p, m \in \mathbb{N}\right\}, \quad j = 1,2,\ldots,n, \tag{60}$$

where λ_m^p is defined in (59). Then the eigenvalues of $\mathcal{A}_z$ are

$$\left\{\lambda_j^o,\ j = 1,2,\ldots,n\right\} \cup \left\{\lambda_m^p,\ m = 0,1,2,\ldots\right\}, \tag{61}$$

and the eigenfunctions corresponding to λ_j^o and λ_m^p are respectively

$$Z_j = (X_j, 0), \quad j = 1,2,\ldots, \tag{62}$$

and

$$Z_m(x) = \left([\lambda_m^p - (A+BK)]^{-1}B,\ z_m(x)\right), \quad m \in \mathbb{N}, \tag{63}$$

where

$$z_m(x) = \cos\left(m + \frac{1}{2}\right)\pi x, \quad m \in \mathbb{N}, \tag{64}$$

which forms an orthogonal basis for $L^2(0,1)$.

Proof. Since $A+BK$ is Hurwitz, we have

$$\mathrm{Re}\lambda_j^0 < 0, \quad j = 1,2,\ldots,n. \tag{65}$$

A simple computation shows that the eigenvalue problem

$$\begin{cases}-iz'' - cz = \lambda z, \\ z'(0) = z(1) = 0\end{cases} \tag{66}$$

has the nontrivial solutions

$$\left(\lambda_m^p, z_m(x)\right), \quad m \in \mathbb{N} \tag{67}$$

where λ_m^p and $z_m(x)$ are given by (59) and (64) respectively.

Next we look for the eigenvalues for (58). Let $\lambda = \lambda_j^o, j = 1,2,\ldots,n$. Since $B \neq 0$ and

$$(A+BK)X_j = \lambda_j^o X_j,$$

$(A+BK)X_j + Bz(0) = \lambda_j^o X_j$ yields $z(0) \equiv 0$. Moreover,

$$\begin{cases}-iz'' - cz = \lambda_j^o z, \\ z(0) = z'(0) = z(1) = 0\end{cases}$$

only has trivial solutions. So we get that $\lambda_j^o, j = 1,2,\ldots,n$ is the eigenvalues of (58) and has the corresponding eigenfunctions $(X_j, 0)$, which is (62).

On the other hand, when $\lambda = \lambda_m^p$, $(\lambda_m^p, z_m(x))$ satisfies the second and third equations of (58) and $z_m(0) = 1 \neq 0$. By the first equation of (58), we have

$$X_m^p = [\lambda_m^p - (A+BK)]^{-1}B.$$

So λ_m^p, $m \in \mathbb{N}$, is the eigenvalue of (58) and has the corresponding eigenfunction

$$\left([\lambda - (A+BK)]^{-1}B,\ \cos\left(m + \frac{1}{2}\right)\pi x\right).$$

This is (64). The proof is complete. □

Since (A,B) is controllable, the eigenvalues of $A+BK$ can be designed at any points in complex plane. For simplicity, for each eigenvalue λ of $A+BK$, we assume that $\mathrm{Re}(\lambda) < -c$, which guarantee the condition (60).

Theorem 3. *Let $\mathcal{A}_z$ be given by (55), let $\lambda_j^o, j = 1,2,\ldots,n$ be the simple eigenvalue of $A+BK$ with the corresponding eigenvector X_j, and let $\mathrm{Re}\lambda_j^o < -c$. Then, there is a sequence of eigenfunctions of $\mathcal{A}_z$ which forms a Riesz basis for $\mathcal{H}$. Moreover, the following conclusions are true:*

(i) *$\mathcal{A}_z$ generates a C_0-semigroup $e^{\mathcal{A}_z t}$ on $\mathcal{H}$.*
(ii) *The spectrum-determined growth condition $\omega(\mathcal{A}_z) = s(\mathcal{A}_z)$ holds true for $e^{\mathcal{A}_z t}$, where $\omega(\mathcal{A}_z) = \lim_{t\to\infty}\frac{1}{t}\|e^{\mathcal{A}_z t}\|$ is the growth bound of $e^{\mathcal{A}_z t}$, and $s(\mathcal{A}_z) = \sup\{Re\,\lambda | \lambda \in \sigma(A_z)\}$ is the spectral bound of $\mathcal{A}_z$ [18].*
(iii) *The C_0-semigroup $e^{\mathcal{A}_z t}$ is exponentially stable in the sense*

$$\|e^{\mathcal{A}_z t}\| \le M_1 e^{-ct}, \tag{68}$$

where $M_1 > 0$ and c is an arbitrary pre-designed decay rate.

Proof. It is noted that $\{X_j, j = 1, 2, \dots, n\}$ is an orthogonal basis in $\mathbb{C}^n$ and $\{z_m(x) = \cos\left(m + \frac{1}{2}\right)\pi x, m \in \mathbb{N}\}$ given by (64) forms an orthogonal basis in $L^2(0, 1)$. We have

$\{F_j, F_m(x),\ j = 1, 2, \dots, n,\ m \in \mathbb{N}\}$

which forms an orthogonal basis in $\mathcal{H}$ with $F_j = (X_j, 0)$ and $F_m(x) = (0, z_m(x))$. It follows from (62), (63) that

$$\sum_{j=1}^{n} \|Z_j - F_j\|^2 + \sum_{m=0}^{\infty} \|Z_m(x) - F_m(x)\|^2 = \sum_{m=0}^{\infty} \left\|[\lambda_m^p - (A + BK)]^{-1}B\right\|_{\mathbb{C}^n}^2, \tag{69}$$

where $\|\cdot\|_{\mathbb{C}^n}$ denotes the norm in $\mathbb{C}^n$. A simple computation gives

$$\begin{aligned}
&\left\|[\lambda_m^p - (A + BK)]^{-1}B\right\|_{\mathbb{C}^n}^2 \\
&= \overline{([\lambda_m^p - (A + BK)]^{-1}B)^T}[\lambda_m^p - (A + BK)]^{-1}B \\
&= \overline{B^T}\ \overline{([\lambda_m^p - (A + BK)]^{-1})^T}[\lambda_m^p - (A + BK)]^{-1}B \\
&= \frac{1}{\left|\lambda_m^p\right|^2}\overline{B^T}\ \overline{\left(\left[I - \frac{1}{\lambda_m^p}(A + BK)\right]^{-1}\right)^T}\left[I - \frac{1}{\lambda_m^p}(A + BK)\right]^{-1}B.
\end{aligned}$$

It follows from (59) that when $m \to \infty$, $\lambda_m^p \to \infty$. So there is a positive number N such that for $m > N$, we have

$$\left[I - \frac{1}{\lambda_m^p}(A + BK)\right]^{-1} = I + \mathcal{O}\left(\frac{1}{|\lambda_m^p|}\right), \quad m > N.$$

Thus,

$$\left\|[\lambda_m^p - (A + BK)]^{-1}B\right\|_{\mathbb{C}^n}^2 = \frac{\|B\|_{\mathbb{C}^n}^2}{\left|\lambda_m^p\right|^2}\left(1 + \mathcal{O}\left(\frac{1}{|\lambda_m^p|}\right)\right).$$

Hence, it follows from (59) and (69) that

$$\begin{aligned}
&\sum_{j=1}^{n} \|Z_j - F_j\|^2 + \sum_{m=0}^{\infty} \|Z_m(x) - F_m(x)\|^2 \\
&= \sum_{m=0}^{\infty} \left\|[\lambda_m^p - (A + BK)]^{-1}B\right\|_{\mathbb{C}^n}^2 \\
&= \sum_{m=0}^{N} \left\|[\lambda_m^p - (A + BK)]^{-1}B\right\|_{\mathbb{C}^n}^2 \\
&\quad + \sum_{m=N+1}^{\infty} \frac{\|B\|_{\mathbb{C}^n}^2}{\left|\lambda_m^p\right|^2}\left(1 + \mathcal{O}\left(\frac{1}{|\lambda_m^p|}\right)\right) < \infty.
\end{aligned}$$

Therefore, by Bari's theorem [19, p. 38],

$\{Z_j, Z_m(x),\ j = 1, 2, \dots, n,\ m = 0, 1, \dots\}$

forms a Riesz basis for $\mathcal{H}$. Moreover, by Theorem 1, $\mathcal{A}_z$ generates a C_0-semigroup $e^{\mathcal{A}_z t}$ on $\mathcal{H}$ and the spectrum-determined growth condition $\omega(\mathcal{A}_z) = s(\mathcal{A}_z)$ holds true for $e^{\mathcal{A}_z t}$. Finally, by the eigenvalues of $\mathcal{A}_z$ given by (61) and (ii) and (iii), there is a positive constant $M_1 > 0$ such that

$$\|e^{\mathcal{A}_z t}\| \le M_1 e^{-ct} \quad \forall t \ge 0.$$

The proof is complete. □

4. Stability analysis for the closed-loop system

In this section, we will investigate the stability of the closed-loop systems for both (X, w) and (X, u).

4.1. Closed-loop system for (X, w)

In this subsection, we present the closed-loop system for (X, w) and establish the exponential stability. Under the backstepping feedback control (52), we have the closed-loop system for (12)–(15):

$$\begin{cases}
\dot{X}(t) = (A + BK)X(t) + Bw(0, t), \\
w_t(x, t) = -iw_{xx}(x, t), \\
w_x(0, t) = 0, \\
w(1, t) = \displaystyle\int_0^1 \kappa(1, y)w(y, t)dy,
\end{cases} \tag{70}$$

where $\kappa(x, y)$ is given by (48). Define a linear operator $\mathcal{A}_w$ for the closed-loop system (70) by:

$$\begin{cases}
\mathcal{A}_w(X, w) = \left((A + BK)X + Bw(0),\ -iw''\right), \\
\qquad \forall (X, w) \in D(\mathcal{A}_w), \\
D(\mathcal{A}_w) = \Big\{(X, w) \in \mathbb{C}^n \times H^2(0, 1)\ \big|\, w'(0) = 0, w(1) \\
\qquad = \displaystyle\int_0^1 \kappa(1, y)w(y)dy\Big\}.
\end{cases} \tag{71}$$

With $\mathcal{A}_w$ at hand, the closed-loop system (70) can be written as an evolution equation in $\mathcal{H}$:

$$\begin{cases}
\dfrac{dY_w(t)}{dt} = \mathcal{A}_w Y_w(t), \quad t > 0, \\
Y_w(0) = Y_{w0}.
\end{cases} \tag{72}$$

where $Y_w(t) = (X(t), w(\cdot, t))$.

Theorem 4. *Let $\mathcal{A}_w$ be given by (72), let $\lambda_j^o, j = 1, 2, \dots, n$ be the simple eigenvalue of $A + BK$ with the corresponding eigenvector X_j, and let $\mathrm{Re}\lambda_j^o < -c$. Then, there is a sequence of eigenfunctions of $\mathcal{A}_w$, which forms a Riesz basis for $\mathcal{H}$. Moreover, the following conclusions are true:*

(i) *The eigenvalues of $\mathcal{A}_w$ are*

$$\left\{\lambda_j^o,\ j = 1, 2, \dots, n\right\} \cup \left\{\lambda_m^p,\ m = 0, 1, 2, \dots\right\}, \tag{73}$$

where λ_m^p is defined in (59).

(ii) *$\mathcal{A}_w$ generates a C_0-semigroup $e^{\mathcal{A}_w t}$ on $\mathcal{H}$.*

(iii) *The spectrum-determined growth condition $\omega(\mathcal{A}_w) = s(\mathcal{A}_w)$ holds true for $e^{\mathcal{A}_w t}$, where $\omega(\mathcal{A}_w)$ is the growth bound of $e^{\mathcal{A}_w t}$, and $s(\mathcal{A}_w)$ is the spectral bound of $\mathcal{A}_w$.*

(iv) *The C_0-semigroup $e^{\mathcal{A}_w t}$ is exponential in the sense*

$$\|e^{\mathcal{A}_w t}\| \le M_2 e^{-ct}, \tag{74}$$

where $M_2 > 0$ and c is an arbitrary pre-designed decay rate.

Proof. Since the transformation defined by (44)–(45) is invertible between system (70) and the target system (8)–(11), and its invertible transformation is given by

$$\begin{cases}
X(t) = X(t), \\
w(x, t) = z(x, t) + \displaystyle\int_0^x p(x, y)z(y, t)dy,
\end{cases}$$

it is easy to see that the system operators $\mathcal{A}_w$ and $\mathcal{A}_z$ defined by (55) and (71) respectively, have the relationship:

$$\mathcal{A}_z\mathcal{F}_2 = \mathcal{F}_2\mathcal{A}_w, \tag{75}$$

where $\mathcal{F}_2$ is given by (53). From Lemma 2, we know that $\mathcal{F}_2$ is bounded and invertible with $\mathcal{F}_2^{-1} = \mathcal{G}_2$ given (54). Hence

$$\mathcal{A}_w = \mathcal{F}_2^{-1}\mathcal{A}_z\mathcal{F}_2 = \mathcal{G}_2\mathcal{A}_z F_2,$$

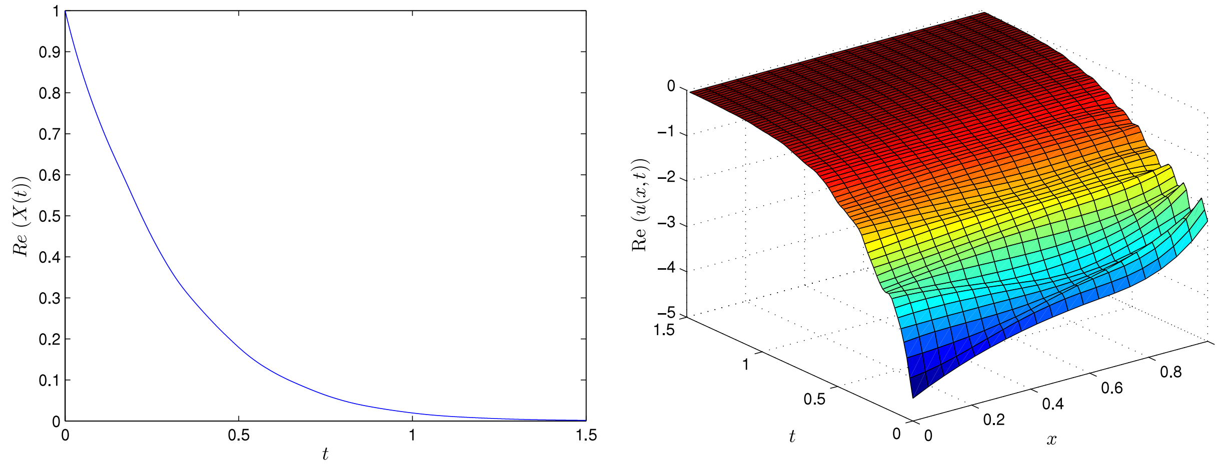

Fig. 3. The closed-loop response of the ODE (left) and Schrödinger equation (right). Only the real part of the state is shown.

and (λ, Y_z) is an eigenpair of $\mathcal{A}_z$ if and only if $(\lambda, \mathcal{F}_2^{-1}Y_z)$ is an eigenpair of $\mathcal{A}_w$. So, from the eigenvalues of $\mathcal{A}_z$ given by (59)–(61), we get the eigenvalues of $\mathcal{A}_w$ as assertion (i). Moreover, since $\mathcal{F}_2$ is bounded in $\mathcal{H}$, it follows that from Theorem 3, the eigenfunctions of $\mathcal{A}_w$ forms a Riesz basis in $\mathcal{H}$. Moreover, by Theorem 1, $\mathcal{A}_z$ generates a C_0-semigroup $e^{\mathcal{A}_z t}$ on $\mathcal{H}$ and the spectrum-determined growth condition $\omega(\mathcal{A}_z) = s(\mathcal{A}_z)$ holds true for $e^{\mathcal{A}_z t}$. Finally, by the relation

$$e^{\mathcal{A}_w t} = e^{\mathcal{F}_2 \mathcal{A}_z \mathcal{F}_2^{-1} t} = \mathcal{F}_2 e^{\mathcal{A}_z t} \mathcal{F}_2^{-1},$$

the assertion (iv) is then concluded. □

4.2. Closed-loop system for (X, u)

In this subsection, we present the closed-loop system for (X, u). By the controls (37) and (52), we have the backstepping feedback control

$$\begin{aligned} U(t) &= u(1,t) = W(t) + \int_0^1 q(1,y)u(y,t)dy + \gamma(1)X(t) \\ &= \int_0^1 \left[\kappa(1,y) + q(1,y) - \int_y^1 \kappa(1,\xi)q(\xi,y)d\xi\right] u(y,t)dy \\ &\quad + \left[\gamma(1) - \int_0^1 \kappa(1,y)\gamma(y)dy\right] X(t), \end{aligned}$$

where $q(x, y)$ is given by (26), $\kappa(x, y)$ is given by (48) and $\gamma(x)$ is given by (30). Then the closed-loop system becomes

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0,t), \\ u_t(x,t) = -iu_{xx}(x,t), \quad x \in (0,1), \\ u_x(0,t) = CX(t), \\ u(1,t) = \displaystyle\int_0^1 \left[\kappa(1,y) + q(1,y) - \int_y^1 \kappa(1,\xi)q(\xi,y)d\xi\right] u(y,t)dy \\ \qquad + \left[\gamma(1) - \displaystyle\int_0^1 \kappa(1,y)\gamma(y)dy\right] X(t). \end{cases} \tag{76}$$

Define a linear operator $\mathcal{A}_u$ for the closed-loop system (76) by:

$$\begin{cases} \mathcal{A}_u(X,u) = \left(AX + Bu(0), \ -iu''\right), \quad \forall\, (X,u) \in D(\mathcal{A}_u), \\ D(\mathcal{A}_u) = \Big\{(X,u) \in \mathbb{C}^n \times H^2(0,1) | u'(0) = CX \\ u(1) = \left[\gamma(1) - \displaystyle\int_0^1 \kappa(1,y)\gamma(y)dy\right] \\ \qquad + \displaystyle\int_0^1 \left[\kappa(1,y) + q(1,y) - \int_y^1 \kappa(1,\xi)q(\xi,y)d\xi\right] u(y)dy \Big\}. \end{cases} \tag{77}$$

With $\mathcal{A}_u$ at hand, the closed-loop system (76) can be written as an evolution equation in $\mathcal{H}$:

$$\begin{cases} \dfrac{dY_u(t)}{dt} = \mathcal{A}_u Y_u(t), \quad t > 0, \\ Y_u(0) = Y_{u0}, \end{cases} \tag{78}$$

where $Y_u(t) = (X(t), u(\cdot, t))$.

Theorem 5. *Let $\mathcal{A}_u$ be given by (77), let $\lambda_j^o, j = 1, 2, \ldots, n$ be the simple eigenvalue of $A + BK$ with the corresponding eigenvector X_j, and let $\mathrm{Re}\lambda_j^o < -c$. Then, there is a sequence of eigenfunctions of $\mathcal{A}_u$, which forms a Riesz basis for $\mathcal{H}$. Moreover, the following conclusions are true:*

(i) *The eigenvalues of $\mathcal{A}_u$ are*

$$\left\{\lambda_j^o,\ j = 1, 2, \ldots, n\right\} \cup \left\{\lambda_m^p,\ m = 0, 1, 2, \ldots\right\}, \tag{79}$$

where λ_m^p is defined in (59).

(ii) *$\mathcal{A}_u$ generates a C_0-semigroup $e^{\mathcal{A}_u t}$ on $\mathcal{H}$.*

(iii) *The spectrum-determined growth condition $\omega(\mathcal{A}_u) = s(\mathcal{A}_u)$ holds true for $e^{\mathcal{A}_u t}$, where $\omega(\mathcal{A}_u)$ is the growth bound of $e^{\mathcal{A}_u t}$, and $s(\mathcal{A}_u)$ is the spectral bound of $\mathcal{A}_u$.*

(iv) *The C_0-semigroup $e^{\mathcal{A}_u t}$ is exponential in the sense*

$$\|e^{\mathcal{A}_u t}\| \leq Me^{-ct}, \tag{80}$$

where $M > 0$ and c is an arbitrary pre-designed decay rate.

Proof. The proof is similar to that of Theorem 3. Since the transformation defined by (16)–(17) is invertible between system (70) and the closed-loop system (76), and its invertible transformation is given by

$$\begin{cases} X(t) = X(t), \\ u(x,t) = w(x,t) + \displaystyle\int_0^x \iota(x,y)w(y,t)dy + \psi(x)X(t), \end{cases}$$

it is easy to see that the system operators $\mathcal{A}_w$ and $\mathcal{A}_u$ defined by (71) and (77) respectively, have the relationship:

$$\mathcal{A}_w \mathcal{F}_1 = \mathcal{F}_1 \mathcal{A}_u \tag{81}$$

where $\mathcal{F}_1$ is given by (38). From Lemma 1, we know that $\mathcal{F}_1$ is bounded and invertible with $\mathcal{F}_1^{-1} = G_1$ given (39). Hence

$$\mathcal{A}_u = \mathcal{F}_1^{-1} \mathcal{A}_w \mathcal{F}_1 = \mathcal{G}_1 \mathcal{A}_w F_1,$$

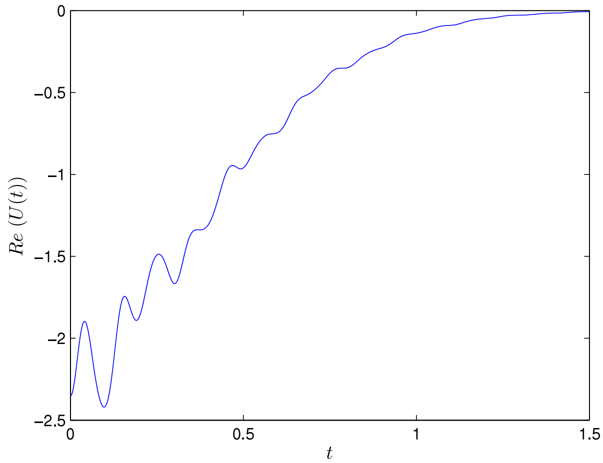

Fig. 4. Control effort. Only the real part is shown.

and (λ, Y_w) is an eigenpair of $\mathcal{A}_w$ if and only if $(\lambda, \mathcal{F}_1^{-1}Y_w)$ is an eigenpair of $\mathcal{A}_u$. The rest of the proof is similar to Theorem 3. The details are omitted. □

5. Simulation results and concluding remarks

In this section, we present the results of numerical simulations for the system (1)–(4) with boundary control (37) and (52). The scalar case is considered here, with the parameters of the system taken as $A = 1, B = 1, C = 1, K = -5, c = 3$. The response of the closed-loop of the plant (1)–(4) are shown in Fig. 3. Both the ODE and Schrödinger equation are stabilized. Fig. 4 shows the control effort (only the real part is shown).

In conclusion, we have designed an explicit feedback law for a linear ODE coupled with a linearized Schrödinger equation. Our design is based on two-step backstepping transformation by introducing an intermediate system and an intermediate control. Using the Riesz spectral method we proved exponential stability of the target system and the closed-loop systems.

References

[1] A. Smyshlyaev, M. Krstic, Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations, IEEE Transactions on Automatic Control 49 (12) (2004) 2185–2202.
[2] A. Smyshlyaev, M. Krstic, Backstepping observers for a class of parabolic PDEs, Systems & Control Letters 54 (2005) 613–625.
[3] A. Smyshlyaev, M. Krstic, Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays, Systems & Control Letters 57 (2008) 750–758.
[4] M. Krstic, Delay Compensation for Nonlinear, Adaptive, and PDE Systems, Birkhauser, 2009.
[5] M. Krstic, Compensating actuator and sensor dynamics governed by diffusion PDEs, Systems & Control Letters 58 (2009) 372–377.
[6] M. Krstic, Compensating a string PDE in the actuation or sensing path of an unstable ODE, IEEE Transactions on Automatic Control 54 (6) (2009) 1362–1368.
[7] G.A. Susto, M. Krstic, Control of PDE–ODE cascades with Neumann interconnections, Journal of The Franklin Institute 347 (2010) 284–314.
[8] S. Tang, C. Xie, State and output feedback boundary control for a coupled PDE–ODE system, Systems & Control Letters 60 (2011) 540–545.
[9] B.-Z. Guo, Z.C. Shao, Regularity of a Schrödinger equation with Dirichlet control and colocated observation, Systems & Control Letters 54 (2005) 1135–1142.
[10] E. Machtyngier, E. Zuazua, Stabilization of the Schrödinger equation, Portugaliae Mathematica 51 (1994) 243–256.
[11] M. Krstic, B.-Z. Guo, A. Smyshlyaev, Boundary controllers and observers for the linearized Schrödinger equation, SIAM Journal on Control and Optimization 49 (4) (2011) 1479–1497.
[12] I. Lasiecka, R. Triggiani, Optimal regularity, exact controllability and uniform stabilization of Schrödinger equations with Dirichlet control, Differential Integral Equations 5 (1992) 521–535.
[13] E. Machtyngier, Exact controllability for the Schrödinger equation, SIAM Journal on Control and Optimization 32 (1994) 24–34.
[14] K.-D. Phung, Observability and control of Schrödinger equation, SIAM Journal on Control and Optimization 40 (2001) 211–230.
[15] J.-M. Wang, B. Ren, M. Krstic, Stabilization and Gevrey regularity of a Schrödinger equation in boundary feedback with a heat equation, IEEE Transactions on Automatic Control 57 (2012) 179–185.
[16] M. Krstic, A. Smyshlyaev, Boundary Control of PDEs: A Course on Backstepping Designs, SIAM, 2008.
[17] R.A. Adams, J.J.F. Fournier, Sobolev Spaces, second ed., in: Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.
[18] Z.-H. Luo, B.-Z. Guo, O. Morgul, Stability and Stabilization of Infinite Dimensional Systems with Applications, in: Communications and Control Engineering Series, Springer-Verlag London, Ltd., London, 1999.
[19] R.M. Young, An Introduction to Non-Harmonic Fourier Series, Revised ed., Academic Press, Inc., San Diego, CA, 2001.