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ORIGINAL ARTICLE

Stabilization of the Euler–Bernoulli equation via boundary connection with heat equation

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Abstract In this paper, we are concerned with the stabilization of a coupled system of Euler–Bernoulli beam or plate with heat equation, where the heat equation (or vice versa the beam equation) is considered as the controller of the whole system. The dissipative damping is produced in the heat equation via the boundary connections only. The one-dimensional problem is thoroughly studied by Riesz basis approach: The closed-loop system is showed to be a Riesz spectral system and the spectrum-determined growth condition holds. As the consequences, the boundary connections with dissipation only in heat equation can stabilize exponentially the whole system, and the solution of the system has the Gevrey regularity. The exponential stability is proved for a two dimensional system with additional dissipation in the boundary of the plate part. The study gives rise to a different design in control of distributed parameter systems through weak connections with subsystems where the controls are imposed.

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1 Introduction

In the past few decades, there has been increasing interest in the control of coupled systems described by the partial differential equations (PDEs). Examples can be found in fluid–structure interactions, composite laminates in smart materials and structures, structural-acoustic systems, and many other interactive physical process modeled by PDEs cascades or interconnected PDEs. The stability and controllability analysis for a heat-wave system, arising from the fluid–structure interaction, are treated in [50,51] and a physical problem coupling a heat equation together with an elastic structure is presented in [10,27]. The feedback controllers for several classes of coupled PDEs and structural-acoustic models are studied in [2,25], where the beam is used as the noise controller in wave-beam coupled system. The energy decay and exponential stability for a linear thermoelastic bar and plate are considered in [18]. The stabilization of a transmission wave/plate equation is investigated in [1], where boundary feedback controls are applied on both wave and plate. The stability and Riesz basis property of the composite laminates and the sandwich beam with boundary controls are analyzed in [43,44]. Backstepping approach is adopted to stabilize ODEs cascaded with PDEs in [22,23] and the references therein.

In this paper, we investigate a coupled PDEs in a very different way, where one subsystem plays the role of the controller. For the motivation, let us consider a controlled ODE:

$$\dot{x}(t) = Ax(t) + Bu(t-\tau), \tau > 0, \tag{1.1}$$

where A is an $n \times n$ matrix and B is the appropriate-sized control matrix and τ is the time delay in control. Let

$$z(x,t) = u(t + \tau(x-1)),\ 0 < x < 1.$$

Then z satisfies

$$\tau z_t(x,t) = z_x(x,t),\ 0 < x < 1, t > 0. \tag{1.2}$$

So the control problem (1.1) can be formulated as the following coupled ODEs-PDEs control system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bz(0,t), & t > 0, \\ \tau z_t(x,t) = z_x(x,t), & 0 < x < 1,\ t > 0, \\ z(1,t) = u(t), \end{cases} \tag{1.3}$$

where the PDE part is considered as the controller and the original control plant ODEs is connected with PDEs through boundary output of the PDEs. It is seen that the time delay disappears in the state of the new formulated system (1.3). This point of view is completely different to the treatment of [8, p.53] and clearly shows the infinite-dimensional nature of delay systems.

This method first appeared in [48] and subsequently in [15,16] and many others [36]. The systematic application of this approach is attributed to [5,20–23]. Very recently, this idea is applied to the stabilization of beam and Schrödinger equations through connection with heat equation in [45] and [46], respectively, where all connections and feedbacks are performed through boundaries. The remarkable characterization of these researches is that the original system is connected with the heat equation through some weak boundary connections in the sense that after taking the derivative of the energy function with time, the only dissipative term left is the term on the "heat part". Nevertheless, it is proved in [45,46] that the closed-loop system in this way has even better performance than the direct control for the original beam or Schrödinger equation. Actually, the Gevrey regularities are verified for these systems. Moreover, the exponential stability holds true for both positive and negative feedback gains.

This approach is expected to solve many process control problems in engineering where the heat control is the major control strategy. In this paper, we continue to apply this idea to study the stabilization of an Euler–Bernoulli beam and plate equations through different type connections with heat equation. This paper is composed into two parts. The first part is on the 1-d problem that includes Sects. 2 through 4, and the second part is on the 2-d problem that includes Sects. 5 and 6. In Sect. 2, the spectral analysis is presented. The completeness of the root subspace is proved in Sect. 3. Section 4 is devoted to the Riesz basis generation, exponential stability, and the Gevrey regularity as well. In Sect. 5, the semigroup generation for the 2-d problem is presented. The exponential stability for the 2-d problem is presented in Sect. 6. For the 2-d problem, the exponential stability requires the dissipation from both "plate part" and "heat part", in sharp contrast with the 1-d problem where only the heat dissipation is required. This proposes a completely new problem for this coupled multi-dimensional PDEs.

2 Spectral analysis of the 1-d problem

In this section, we consider a controlled Euler–Bernoulli beam with the connection of the heat equation through the common boundary $x = 1$:

$$\begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) = 0, & 0 < x < 1, t > 0, \\ v_t(x,t) - v_{xx}(x,t) = 0, & 1 < x < 2, t > 0, \\ y_{xxx}(1,t) = -v_x(1,t), & t \geq 0, \\ y_t(1,t) = v(1,t), & t \geq 0, \\ y_{xx}(1,t) = v(2,t) = 0, & t \geq 0, \\ y_{xx}(0,t) = y_x(0,t), & t \geq 0, \\ y_{xxx}(0,t) = -y(0,t) - ky_t(0,t), & t \geq 0, \\ y(x,0) = y_0(x), \ \ y_t(x,0) = y_1(x), & 0 \leq x \leq 1, \\ v(x,0) = v_0(x), & 1 \leq x \leq 2 \end{cases}$$

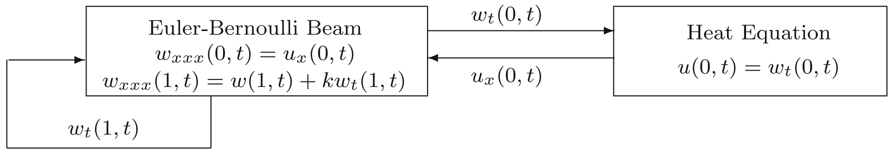

Fig. 1 An Euler–Bernoulli beam via boundary connected with a heat equation

where $k \geq 0$ is a feedback gain. Set

$$\begin{cases} w(x,t) = y(1-x,t), & x \in (0,1), t > 0 \\ u(x,t) = v(1+x,t), & x \in (0,1), t > 0. \end{cases}$$

Then we transform the above system into the following system that is connected through the boundary $x = 0$ only (see Fig. 1).

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, & 0 < x < 1, t > 0, \\ u_t(x,t) - u_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ w_{xxx}(0,t) = u_x(0,t), & t \geq 0, \\ w_t(0,t) = u(0,t), & t \geq 0, \\ w_{xx}(0,t) = u(1,t) = 0, & t \geq 0, \\ w_{xx}(1,t) = -w_x(1,t), & t \geq 0, \\ w_{xxx}(1,t) = w(1,t) + kw_t(1,t), & t \geq 0, \\ w(x,0) = w_0(x), \;\; w_t(x,0) = w_1(x), & 0 \leq x \leq 1, \\ u(x,0) = u_0(x), & 0 \leq x \leq 1. \end{cases} \tag{2.1}$$

By this equivalence, we consider, throughout the first part, the system (2.1) only.

The energy function for (2.1) is given by

$$E(t) = \frac{1}{2}\int_0^1 \left[w_t^2(x,t) + w_{xx}^2(x,t) + u^2(x,t)\right] \mathrm{d}x + \frac{1}{2}\left[w^2(1,t) + w_x^2(1,t)\right]. \tag{2.2}$$

Formally, it is found that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -kw_t^2(1,t) - \int_0^1 u_x^2(x,t)dx \leq 0. \tag{2.3}$$

So $E(t)$ is non-increasing. A remarkable property is that when $k = 0$, there is no explicit damping term for the "beam part" on the right-hand side of (2.3). We are specially interested in the stability for the system (2.1) with $k = 0$. In this sense, we consider the heat equation as the controller of the whole system.

This control design is motivated from our earlier works in [14,31] that the above connections represent the same order feedbacks from the spectral point of view, and recently in [45] where the boundary exchange of the heat equation and an Euler–Bernoulli beam makes the coupled system exponentially stable. In this paper, we consider somehow the heat equation as the controller of an Euler–Bernoulli beam with the boundary connections indicated in Fig. 1 that the shear force of the beam is (dimensionless) equal to the heat flux $w_{xxx}(0,t) = u_x(0,t)$ and the velocity is equal to the temperature $w_t(0,t) = u(0,t)$.

We consider system (2.1) in the energy state space $\mathcal{H} = H^2(0,1) \times (L^2(0,1))^2$ with the norm induced by the following inner product:

$$\begin{aligned}\langle X_1, X_2\rangle &= \int_0^1 \left[f_1''(x)\overline{f_2''(x)} + g_1(x)\overline{g_2(x)} + h_1(x)\overline{h_2(x)}\right]\mathrm{d}x \\ &\quad + f_1(1)\overline{g_1(1)} + f_1'(1)\overline{g_1'(1)}, \\ &\quad \forall\, X_i = (f_i, g_i, h_i) \in \mathcal{H}, i = 1, 2.\end{aligned} \tag{2.4}$$

Define the system operator for (2.1) by

$$\begin{cases} \mathcal{A}(f,g,h) = (g, -f^{(4)}, h''),\ \forall\,(f,g,h) \in D(\mathcal{A}), \\ D(\mathcal{A}) = \left\{ (f,g,h) \in (H^4 \times H^2 \times H^2) \,\middle|\, \begin{array}{l} h(1) = f''(0) = 0, \\ f'''(0) = h'(0), g(0) = h(0), \\ f''(1) = -f'(1), f'''(1) = f(1) + kg(1) \end{array} \right\}. \end{cases} \tag{2.5}$$

Then (2.1) can be written as an evolution equation in $\mathcal{H}$:

$$\begin{cases} \dfrac{\mathrm{d}X(t)}{\mathrm{d}t} = \mathcal{A}X(t), \quad t > 0, \\ X(0) = X_0. \end{cases} \tag{2.6}$$

where $X(t) = (w(\cdot,t), w_t(\cdot,t), u(\cdot,t))$ and $X_0 = (w_0, w_1, u_0)$.

Theorem 1 *Let $\mathcal{A}$ be given by (2.5). Then $\mathcal{A}^{-1}$ exists and is compact. Hence, $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues of finite algebraic multiplicity only. Moreover, $\mathcal{A}$ is dissipative in $\mathcal{H}$ and generates a C_0-semigroup of contractions $e^{\mathcal{A}t}$ on $\mathcal{H}$.*

Proof For any given $(f_1, g_1, h_1) \in \mathcal{H}$, solving

$$\mathcal{A}(f,g,h) = (g, -f^{(4)}, h'') = (f_1, g_1, h_1)$$

gives $g(x) = f_1(x)$ with h satisfying

$$\begin{cases} h''(x) = h_1(x), \\ h(1) = 0,\ h(0) = g(0) = f_1(0). \end{cases}$$

The solution of the above ODE has the form

$$h(x) = \left[f_1(0) + \int_0^1 (1-\xi)h_1(\xi)\mathrm{d}\xi\right](1-x) - \left[\int_0^x (1-x)h_1(\xi)\mathrm{d}\xi + \int_x^1 (1-\xi)h_1(\xi)\mathrm{d}\xi\right]. \tag{2.7}$$

In addition, f satisfies

$$\begin{cases} f^{(4)}(x) = -g_1(x), \\ f''(0) = 0,\ f'''(0) = h'(0),\ f''(1) = -f'(1),\ \ f'''(1) = f(1) + kf_1(1). \end{cases}$$

A direct computation gives the solution of the above ODE:

$$\begin{cases} f(x) = \varphi'(1) - kf_1(1) + \varphi(1)(1-x) + \int\limits_x^1 \varphi(s)(s-x)\mathrm{d}s, \\ \varphi(x) = f''(x) = -\left[f_1(0) + \int\limits_0^1 (1-\xi)h_1(\xi)d\xi \right] x - \int\limits_0^x g_1(s)(x-s)\mathrm{d}s. \end{cases} \tag{2.8}$$

By (2.7), (2.8) and $g(x) = f_1(x)$, we get the unique $(f, g, h) \in D(\mathcal{A})$. Hence, $\mathcal{A}^{-1}$ exists and is compact on $\mathcal{H}$ by the Sobolev embedding theorem. Therefore, $\sigma(\mathcal{A})$ consists of the isolated eigenvalues of finite algebraic multiplicity only.

Now we show that $\mathcal{A}$ is dissipative in $\mathcal{H}$. Given any $X = (f, g, h) \in D(\mathcal{A})$. We have

$$\langle \mathcal{A}X, X \rangle = -k|g(1)|^2 + \int\limits_0^1 g''\overline{f''}\mathrm{d}x - \int\limits_0^1 f''\overline{g''}\mathrm{d}x - \int\limits_0^1 |h'|^2\mathrm{d}x$$

and hence

$$\mathcal{R}e\langle \mathcal{A}X, X \rangle = -k|g(1)|^2 - \int\limits_0^1 |h'|^2\mathrm{d}x \le 0. \tag{2.9}$$

So $\mathcal{A}$ is dissipative and generates a C_0-semigroup $\mathrm{e}^{\mathcal{A}t}$ of contractions in $\mathcal{H}$ by the Lumer–Philips theorem ([39, p.14]). □

Remark 1 It is noted that when $k = 0$, the right-hand side of (2.9) contains no dissipation term of the "beam part".

Let us now consider the eigenvalue problem of $\mathcal{A}$. It is seen that $\mathcal{A}X = \lambda X$, where $X = (f, g, h) \in D(\mathcal{A}), \lambda \in \mathbb{C}$, if and only if $g(x) = \lambda f(x)$, and f, h satisfy the following eigenvalue problem:

$$\begin{cases} f^{(4)}(x) + \lambda^2 f(x) = 0, \\ h''(x) - \lambda h(x) = 0, \\ f''(0) = h(1) = 0, \\ f'''(0) = h'(0), \\ \lambda f(0) = h(0), \\ f''(1) = -f'(1), \\ f'''(1) = f(1) + k\lambda f(1). \end{cases} \tag{2.10}$$

Lemma 1 *Let $\mathcal{A}$ be defined by (2.5). Then $\mathcal{R}e\lambda < 0$ for every $\lambda \in \sigma(\mathcal{A})$.*

Proof By Theorem 1, any $\lambda \in \sigma(\mathcal{A})$ must satisfy $\mathrm{Re}\lambda \le 0$. So we only need to show that there is no eigenvalue on the imaginary axis. When $\lambda = 0$, by Theorem 1, we have that $\mathcal{A}^{-1}$ exists and hence $\lambda \in \rho(\mathcal{A})$. Let $\lambda = i\mu^2 \in \sigma(\mathcal{A})$ with $\mu \in \mathbb{R}^+$ and $X = (f, g, h) \in D(\mathcal{A})$ be its associated eigenfunction of $\mathcal{A}$. By (2.9), we have

$$\mathcal{R}e\langle \mathcal{A}X, X\rangle = \mathcal{R}e\left(i\mu^2\langle X, X\rangle\right) = -k|g(1)|^2 - \int_0^1 |h'|^2 \mathrm{d}x = 0$$

and hence $h'(x) = 0$. Since $h(1) = 0$, it must have $h \equiv 0$. Furthermore, $\mathcal{A}X = i\mu^2 X$ gives that $g = i\mu^2 f$ with f satisfying

$$\begin{cases} f^{(4)}(x) - \mu^4 f(x) = 0, \\ f(0) = f''(0) = f'''(0) = 0,\ \ f''(1) = -f'(1),\ \ f'''(1) = (1 + k\lambda) f(1). \end{cases}$$

A direct computation shows that the above equation admits only zero solution no matter $k = 0$ or not. Hence $X = (f, g, h) = 0$. Therefore, there is no eigenvalue on the imaginary axis. □

Owing to Lemma 1 and the fact that the eigenvalues are symmetric about the real axis, we consider only those λ which are located in the second quadrant of the complex plane:

$$\lambda := i\rho^2,\ \ \rho \in \mathcal{S} := \left\{\rho \in \mathbb{C} \mid 0 \le \arg\rho \le \frac{\pi}{4}\right\}. \tag{2.11}$$

Note that for any $\rho \in \mathcal{S}$, we have

$$\mathrm{Re}(-\rho) \le \mathrm{Re}(i\rho) \le \mathrm{Re}(-i\rho) \le \mathrm{Re}(\rho), \tag{2.12}$$

and

$$\begin{cases} \mathrm{Re}(-\rho) = -|\rho|\cos(\arg\rho) \le -\frac{\sqrt{2}}{2}|\rho| < 0, \\ \mathrm{Re}(i\rho) = -|\rho|\sin(\arg\rho) \le 0. \end{cases} \tag{2.13}$$

Set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ with

$$\begin{cases} \mathcal{S}_1 := \{\rho \in \mathbb{C} \mid \frac{\pi}{8} < \arg\rho \le \frac{\pi}{4}\}, \\ \mathcal{S}_2 := \{\rho \in \mathbb{C} \mid 0 \le \arg\rho \le \frac{\pi}{8}\}. \end{cases} \tag{2.14}$$

Then

$$\begin{cases} \mathrm{Re}(i\rho) = -|\rho|\sin(\arg\rho) \le -|\rho|\sin\left(\frac{1}{8}\pi\right) < 0, \quad \forall \rho \in \mathcal{S}_1, \\ \mathrm{Re}(-\sqrt{i}\rho) = -|\rho|\cos\left(\frac{\pi}{4} + \arg\rho\right) \le -|\rho|\cos\left(\frac{3}{8}\pi\right) < 0, \quad \forall \rho \in \mathcal{S}_2. \end{cases} \tag{2.15}$$

Substituting $\lambda = i\rho^2$ with $\rho \neq 0$ into (2.10), gives the eigenvalue system of (2.1) in ρ:

$$\begin{cases} f^{(4)}(x) - \rho^4 f(x) = 0, \\ h''(x) - i\rho^2 h(x) = 0, \\ f''(0) = 0, \\ f''(1) = -f'(1), \\ f'''(1) = f(1) + ki\rho^2 f(1), \\ h(1) = 0, \\ i\rho^2 f(0) = h(0), \\ f'''(0) = h'(0). \end{cases} \tag{2.16}$$

Let

$$f(x) = c_1 e^{\rho x} + c_2 e^{-\rho x} + c_3 e^{i\rho x} + c_4 e^{-i\rho x}, \quad h(x) = d_1 e^{\sqrt{i}\rho x} + d_2 e^{-\sqrt{i}\rho x}, \tag{2.17}$$

where c_s, $s = 1, 2, 3, 4$, d_1, d_2 are constants, and $\sqrt{i} = e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. Substitute these into the boundary conditions of (2.16), to get

$$\begin{cases} c_1 + c_2 - c_3 - c_4 = 0, \\ c_1(1+\rho)e^{\rho} - c_2(1-\rho)e^{-\rho} + c_3(i-\rho)e^{i\rho} - c_4(i+\rho)e^{-i\rho} = 0, \\ c_1(1 + ki\rho^2 - \rho^3)e^{\rho} + c_2(1 + ki\rho^2 + \rho^3)e^{-\rho} \\ \quad + c_3(1 + ki\rho^2 + i\rho^3)e^{i\rho} + c_4(1 + ki\rho^2 - i\rho^3)e^{-i\rho} = 0, \\ d_1 e^{\sqrt{i}\rho} + d_2 e^{-\sqrt{i}\rho} = 0, \\ c_1 i\rho^2 + c_2 i\rho^2 + c_3 i\rho^2 + c_4 i\rho^2 - d_1 - d_2 = 0, \\ c_1\rho^2 - c_2\rho^2 - c_3 i\rho^2 + c_4 i\rho^2 - d_1\sqrt{i} + d_2\sqrt{i} = 0. \end{cases} \tag{2.18}$$

(2.16) has non-zero solution if and only if the characteristic determinant $\det \Delta(\rho) = 0$, where

$$\Delta(\rho) = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \nu_1 e^{\rho} & \nu_2 e^{-\rho} & \nu_3 e^{i\rho} & \nu_4 e^{-i\rho} & 0 & 0 \\ \nu_5 e^{\rho} & \nu_6 e^{-\rho} & \nu_7 e^{i\rho} & \nu_8 e^{-i\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho} & e^{-\sqrt{i}\rho} \\ i\rho^2 & i\rho^2 & i\rho^2 & i\rho^2 & -1 & -1 \\ \rho^2 & -\rho^2 & -i\rho^2 & i\rho^2 & -\sqrt{i} & \sqrt{i} \end{bmatrix} \tag{2.19}$$

and

$$\begin{cases} \nu_1 = 1 + \rho, \nu_2 = -(1-\rho), \nu_3 = i - \rho, \nu_4 = -(i+\rho), \\ \nu_5 = 1 + ki\rho^2 - \rho^3, \nu_6 = 1 + ki\rho^2 + \rho^3, \nu_7 = 1 + ki\rho^2 + i\rho^3, \nu_8 = 1 + ki\rho^2 - i\rho^3. \end{cases} \tag{2.20}$$

Lemma 2 *Let $\lambda = i\rho^2$ with $\rho \in \mathcal{S}$ and let $\Delta(\rho)$ be given by (2.19). Then the following asymptotic expansion holds:*

$$-\frac{1}{2}\rho^{-6} e^{-\rho} e^{i\rho} e^{-\sqrt{i}\rho} \det \Delta(\rho) = D_1(\rho) + \mathcal{O}(\rho^{-2}) \tag{2.21}$$

where

$$D_1(\rho)=a_1\mathrm{e}^{i2\rho}+a_2\mathrm{e}^{i2\rho}\mathrm{e}^{-2\sqrt{i}\rho}+a_3+a_4\mathrm{e}^{-2\sqrt{i}\rho}$$
$$+\frac{1}{\rho}[a_5(k+1)\mathrm{e}^{i2\rho}+a_6(k+1)\mathrm{e}^{i2\rho}\mathrm{e}^{-2\sqrt{i}\rho}+a_7(k-1)+a_8(k-1)\mathrm{e}^{-2\sqrt{i}\rho}] \tag{2.22}$$

and

$$\begin{cases} a_1=(1+\sqrt{2})i,\ a_2=(\sqrt{2}-1)i,\ a_5=(i+1)(\sqrt{2}+1),\ a_6=(i+1)(\sqrt{2}-1),\\ a_3=\sqrt{2}+i,\ a_4=\sqrt{2}-i,\ a_7=1-\sqrt{2}-i(1+\sqrt{2}),\ a_8=-\sqrt{2}-1-i(\sqrt{2}-1). \end{cases} \tag{2.23}$$

Moreover, when $\rho\in\mathcal{S}_1$ or $\rho\in\mathcal{S}_2$, $\det\Delta(\rho)$ *has more accurate asymptotic expansions: for $\rho\in\mathcal{S}_1$,*

$$-\frac{1}{2}\rho^{-6}\mathrm{e}^{-\rho}\mathrm{e}^{i\rho}\mathrm{e}^{-\sqrt{i}\rho}\det\Delta(\rho)=a_3+a_4\mathrm{e}^{-2\sqrt{i}\rho}+\rho^{-1}[a_7(k-1)$$
$$+a_8(k-1)\mathrm{e}^{-2\sqrt{i}\rho}]+\mathcal{O}(\rho^{-2}) \tag{2.24}$$

and for $\rho\in\mathcal{S}_2$,

$$-\frac{1}{2}\rho^{-6}\mathrm{e}^{-\rho}\mathrm{e}^{i\rho}\mathrm{e}^{-\sqrt{i}\rho}\det\Delta(\rho)=a_1\mathrm{e}^{i2\rho}+a_3+\rho^{-1}[a_5(k+1)\mathrm{e}^{i2\rho}$$
$$+a_7(k-1)]+\mathcal{O}(\rho^{-2}). \tag{2.25}$$

Proof Starting from (2.19), with a straightforward computation, we obtain

$$\begin{aligned}\det\Delta(\rho)&=-\rho^2G_1(\rho)\begin{vmatrix}\mathrm{e}^{\sqrt{i}\rho} & \mathrm{e}^{-\sqrt{i}\rho}\\ 1 & 1\end{vmatrix}+i\sqrt{i}\rho^2G_2(\rho)\begin{vmatrix}\mathrm{e}^{\sqrt{i}\rho} & \mathrm{e}^{-\sqrt{i}\rho}\\ 1 & -1\end{vmatrix}\\ &=-\rho^2[\mathrm{e}^{\sqrt{i}\rho}-\mathrm{e}^{-\sqrt{i}\rho}]G_1(\rho)-i\sqrt{i}\rho^2[\mathrm{e}^{\sqrt{i}\rho}+\mathrm{e}^{-\sqrt{i}\rho}]G_2(\rho),\end{aligned}$$

where

$$G_1(\rho)=\begin{vmatrix} 1 & 1 & -1 & -1\\ (1+\rho)\mathrm{e}^{\rho} & -(1-\rho)\mathrm{e}^{-\rho} & (i-\rho)\mathrm{e}^{i\rho} & -(i+\rho)\mathrm{e}^{-i\rho}\\ (1+ki\rho^2-\rho^3)\mathrm{e}^{\rho} & (1+ki\rho^2+\rho^3)\mathrm{e}^{-\rho} & (1+ki\rho^2+i\rho^3)\mathrm{e}^{i\rho} & (1+ki\rho^2-i\rho^3)\mathrm{e}^{-i\rho}\\ 1 & -1 & -i & i \end{vmatrix} \tag{2.26}$$

and

$$G_2(\rho)=\begin{vmatrix} 1 & 1 & -1 & -1\\ (1+\rho)\mathrm{e}^{\rho} & -(1-\rho)\mathrm{e}^{-\rho} & (i-\rho)\mathrm{e}^{i\rho} & -(i+\rho)\mathrm{e}^{-i\rho}\\ (1+ki\rho^2-\rho^3)\mathrm{e}^{\rho} & (1+ki\rho^2+\rho^3)\mathrm{e}^{-\rho} & (1+ki\rho^2+i\rho^3)\mathrm{e}^{i\rho} & (1+ki\rho^2-i\rho^3)\mathrm{e}^{-i\rho}\\ 1 & 1 & 1 & 1 \end{vmatrix}. \tag{2.27}$$

A further direct computation yields

$$G_1(\rho)=\rho^4 e^{\rho}\left[\begin{vmatrix} 0 & 1 & -1 & -1 \\ 1+\frac{1}{\rho} & 0 & (-1+\frac{i}{\rho})e^{i\rho} & -(1+\frac{i}{\rho})e^{-i\rho} \\ -1+\frac{ki}{\rho} & 0 & (i+\frac{ki}{\rho})e^{i\rho} & (-i+\frac{ki}{\rho})e^{-i\rho} \\ 0 & -1 & -i & i \end{vmatrix}+\mathcal{O}(\rho^{-2}e^{-i\rho})\right]$$
$$=2\rho^4 e^{\rho}e^{-i\rho}\left[\left[i+(1+i)(k+1)\frac{1}{\rho}\right]e^{i2\rho}+\left[i+(1-i)(k-1)\frac{1}{\rho}\right]+\mathcal{O}(\rho^{-2})\right]$$

and

$$G_2(\rho)=\rho^4 e^{\rho}\left[\begin{vmatrix} 0 & 1 & -1 & -1 \\ 1+\frac{1}{\rho} & 0 & (-1+\frac{i}{\rho})e^{i\rho} & -(1+\frac{i}{\rho})e^{-i\rho} \\ -1+\frac{ki}{\rho} & 0 & (i+\frac{ki}{\rho})e^{i\rho} & (-i+\frac{ki}{\rho})e^{-i\rho} \\ 0 & 1 & 1 & 1 \end{vmatrix}+\mathcal{O}(\rho^{-2}e^{-i\rho})\right]$$
$$=2\rho^4 e^{\rho}e^{-i\rho}\left[\left[1-i-2i(k+1)\frac{1}{\rho}\right]e^{i2\rho}+\left[-i-1+2i(k-1)\frac{1}{\rho}\right]+\mathcal{O}(\rho^{-2})\right].$$

Hence,

$$\begin{aligned}
&\det\Delta(\rho)\\
&=-2\rho^6 e^{\rho}e^{-i\rho}e^{\sqrt{i}\rho}\Big\{[1-e^{-2\sqrt{i}\rho}]\Big[\Big[i+(1+i)(k+1)\frac{1}{\rho}\Big]e^{i2\rho}+\Big[i+(1-i)(k-1)\frac{1}{\rho}\Big]\Big]\\
&\quad+i\sqrt{i}\Big[1+e^{-2\sqrt{i}\rho}\Big]\Big[\Big[1-i-2i(k+1)\frac{1}{\rho}\Big]e^{i\rho}+\Big[-i-1+2i(k-1)\frac{1}{\rho}\Big]\Big]+\mathcal{O}(\rho^{-2})\Big\}\\
&=-2\rho^6 e^{\rho}e^{-i\rho}e^{\sqrt{i}\rho}\Big\{a_1e^{i2\rho}+a_2e^{i2\rho}e^{-2\sqrt{i}\rho}+a_3+a_4e^{-2\sqrt{i}\rho}\\
&\quad+\frac{1}{\rho}[a_5(k+1)e^{i2\rho}+a_6(k+1)e^{i2\rho}e^{-2\sqrt{i}\rho}+a_7(k-1)+a_8(k-1)e^{-2\sqrt{i}\rho}]+\mathcal{O}(\rho^{-2})\Big\},
\end{aligned}$$

where $a_i, i=1,2,\ldots,8$, are given by (2.23). Moreover, when $\rho\in\mathcal{S}_1$ and $\rho\in\mathcal{S}_2$, it follows from (2.15) that

$$\begin{cases} e^{-i\rho}\to\infty \quad \text{as } |\rho|\to\infty,\ \rho\in\mathcal{S}_1,\\ e^{\sqrt{i}\rho}\to\infty \quad \text{as } |\rho|\to\infty,\ \rho\in\mathcal{S}_2,\end{cases}$$

and hence $\det\Delta(\lambda)$ has more accurate asymptotic expressions given by (2.24) and (2.25) in $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. □

Theorem 2 *Let $\mathcal{A}$ be defined by (2.5). The spectrum $\sigma(\mathcal{A})$ has two families:*

$$\sigma(\mathcal{A})=\{\lambda_{1n}, n\in\mathbb{N}\}\cup\{\lambda_{2n},\overline{\lambda}_{2n},\quad n\in\mathbb{N}\},\tag{2.28}$$

where λ_{1n} *and* λ_{2n} *have the following asymptotic expansions:*

$$\begin{cases} \lambda_{1n} = -\left[n\pi + \frac{1}{2}\theta_1\right]^2 + \mathcal{O}(n^{-1}), \\ \lambda_{2n} = -2k - \left[n\pi + \frac{1}{2}\theta_2\right]|\ln r| + 2i + \frac{1}{4}\left[(2n\pi + \theta_2)^2 - (\ln r)^2\right]i + \mathcal{O}(n^{-1}), \end{cases} \tag{2.29}$$

where

$$\theta_1 = \pi - \arctan 2\sqrt{2},\ \ \theta_2 = -\pi - \arctan\sqrt{2},\ \ r = \frac{\sqrt{3}}{1+\sqrt{2}} < 1, \quad \ln r < 0. \tag{2.30}$$

Therefore,

$$\mathcal{R}e\lambda_{1n}, \mathcal{R}e\lambda_{2n} \to -\infty \text{ as } n \to \infty. \tag{2.31}$$

Proof Let $\det \Delta(\rho) = 0$. By (2.24), $\rho \in \mathcal{S}_1$ satisfies

$$a_3 + a_4 e^{-2\sqrt{i}\rho} + \rho^{-1}\left[a_7(k-1) + a_8(k-1)e^{-2\sqrt{i}\rho}\right] + \mathcal{O}(\rho^{-2}) = 0, \tag{2.32}$$

which can be rewritten as

$$a_3 + a_4 e^{-2\sqrt{i}\rho} + \mathcal{O}(\rho^{-1}) = 0. \tag{2.33}$$

By (2.23), $a_3 e^{\sqrt{i}\rho} + a_4 e^{-\sqrt{i}\rho} = 0$ is equivalent to

$$e^{2\sqrt{i}\rho} = -\frac{a_4}{a_3} = -\frac{\sqrt{2}-i}{\sqrt{2}+i} = \frac{-1+2\sqrt{2}i}{3} = e^{i\theta_1}, \tag{2.34}$$

where θ_1 is given by (2.30). So, the roots of $a_3 e^{\sqrt{i}\rho} + a_4 e^{-\sqrt{i}\rho} = 0$ are found to be

$$\tilde{\rho}_{1n} = \left[n\pi + \frac{1}{2}\theta_1\right]\sqrt{i},\ \ n = 1, 2, \ldots.$$

By Rouché's theorem, the roots of (2.33) have the following asymptotic expression

$$\rho_{1n} = \tilde{\rho}_{1n} + \mathcal{O}\left(\rho_{1n}^{-1}\right) = \left[n\pi + \frac{1}{2}\theta_1\right]\sqrt{i} + \mathcal{O}\left(\rho_{1n}^{-1}\right), \quad n > N_1, \tag{2.35}$$

where N_1 is a sufficiently large positive integer. Substitute ρ_{1n} into (2.32) to get

$$a_3 + a_4 e^{-2\sqrt{i}\left[\tilde{\rho}_{1n} + \mathcal{O}(\rho_{1n}^{-1})\right]} + \rho_{1n}^{-1}\left[a_7(k-1) + a_8(k-1)e^{-2\sqrt{i}\left[\tilde{\rho}_{1n} + \mathcal{O}(\rho_{1n}^{-1})\right]}\right] + \mathcal{O}(\rho_{1n}^{-2}) = 0.$$

By $a_3 e^{\sqrt{i}\tilde{\rho}_{1n}} + a_4 e^{-\sqrt{i}\tilde{\rho}_{1n}} = 0$, we have

$$1 - e^{-2\sqrt{i}\mathcal{O}(\rho_{1n}^{-1})} + \rho_{1n}^{-1}[a_3^{-1}a_7(k-1) - a_4^{-1}a_8(k-1)e^{-2\sqrt{i}\mathcal{O}(\rho_{1n}^{-1})}] + \mathcal{O}(\rho_{1n}^{-2}) = 0.$$

This together with Taylor's expansion gives

$$\mathcal{O}(\rho_{1n}^{-1}) = -\frac{1}{2\sqrt{i}}\frac{k-1}{\left[n\pi+\frac{1}{2}\theta_1\right]\sqrt{i}}[a_3^{-1}a_7 - a_4^{-1}a_8] + \mathcal{O}(n^{-2}) = \mathcal{O}(n^{-2}),$$

where we used the fact $a_3^{-1}a_7 = a_4^{-1}a_8$. It then follows from (2.35) that

$$\rho_{1n} = \left[n\pi + \frac{1}{2}\theta_1\right]\sqrt{i} + \mathcal{O}(n^{-2}), \quad n > N_1. \tag{2.36}$$

Moreover, applying $\lambda_{1n} = i\rho_{1n}^2$, we get λ_{1n} that is expressed in the first part of (2.29). Now we show the second part of (2.29). Similarly, by (2.25), it follows that $\rho \in \mathcal{S}_2$ satisfies

$$a_1 e^{i2\rho} + a_3 + \rho^{-1}[a_5(k+1)e^{i2\rho} + a_7(k-1)] + \mathcal{O}(\rho^{-2}) = 0, \tag{2.37}$$

which is equivalent to

$$a_1 e^{i2\rho} + a_3 + \mathcal{O}(\rho^{-1}) = 0. \tag{2.38}$$

By (2.23), $a_1 e^{i\rho} + a_3 e^{-i\rho} = 0$ is equivalent to

$$e^{2i\rho} = -\frac{a_3}{a_1} = -\frac{\sqrt{2}+i}{(1+\sqrt{2})i} = \frac{\sqrt{2}i-1}{1+\sqrt{2}} = re^{i\theta_2}, \tag{2.39}$$

where θ_2 and r are given by (2.30). Hence, the roots of $a_1 e^{i\rho} + a_3 e^{-i\rho} = 0$ are given by

$$\tilde{\rho}_{2n} = \frac{1}{2i}\left[\ln r + (2n\pi + \theta_2)i\right], \ n = 1, 2, \ldots .$$

By Rouché's theorem again, the roots of (2.38) have the following asymptotic expression

$$\rho_{2n} = \tilde{\rho}_{2n} + \mathcal{O}(\rho_{2n}^{-1}) = \frac{1}{2i}\left[\ln r + (2n\pi + \theta_2)i\right] + \mathcal{O}(\rho_{2n}^{-1}), \quad n > N_2, \tag{2.40}$$

where N_2 is a sufficiently large positive integer. Substitute ρ_{2n} into (2.37) to get

$$a_1 e^{i2[\tilde{\rho}_{2n}+\mathcal{O}(\rho_{2n}^{-1})]} + a_3 + \rho_{2n}^{-1}[a_5(k+1)e^{i2[\tilde{\rho}_{2n}+\mathcal{O}(\rho_{2n}^{-1})]} + a_7(k-1)] + \mathcal{O}(\rho_{2n}^{-2}) = 0.$$

By $a_1 e^{i\tilde{\rho}_{2n}} + a_3 e^{-i\tilde{\rho}_{2n}} = 0$, we have

$$e^{i2\mathcal{O}(\rho_{2n}^{-1})} - 1 + \rho_{2n}^{-1}\left[a_1^{-1}a_5(k+1)e^{i2\mathcal{O}(\rho_{2n}^{-1})} - a_3^{-1}a_7(k-1)\right] + \mathcal{O}(\rho_{2n}^{-2}) = 0.$$

Using Taylor's expansion, the above expression gives

$$\mathcal{O}(\rho_{2n}^{-1}) = -\frac{a_1^{-1}a_5(k+1) - a_3^{-1}a_7(k-1)}{[\ln r + (2n\pi + \theta_2)i]} + \mathcal{O}(\rho_{2n}^{-2}) = -\frac{2k-2i}{[\ln r + (2n\pi + \theta_2)i]} + \mathcal{O}(\rho_{2n}^{-2}),$$

where we used the fact $a_1^{-1}a_5 = 1 - i$ and $a_3^{-1}a_7 = -(1+i)$. Once again, it follows from (2.40) that

$$\rho_{2n} = \frac{1}{2i}\,[\ln r + (2n\pi + \theta_2)i] - \frac{2k-2i}{[\ln r + (2n\pi + \theta_2)i]} + \mathcal{O}(\rho_{2n}^{-2}), \quad n > N_2, \tag{2.41}$$

Finally, applying $\lambda_{2n} = i\rho_{2n}^2$, we get λ_{2n} that is given by the second part of (2.29). □

The asymptotic behavior of the eigenfunctions is expressed in the following theorem.

Theorem 3 *Let $\mathcal{A}$ be defined by (2.5), let $\sigma(\mathcal{A}) = \{\lambda_{1n}, n \in \mathbb{N}\} \cup \{\lambda_{2n}, \overline{\lambda}_{2n}, n \in \mathbb{N}\}$ be the spectrum of $\mathcal{A}$, and let $\lambda_{jn} := i\rho_{jn}^2$, $j = 1, 2$, with ρ_{1n}, ρ_{2n} given by (2.35) and (2.40), respectively. Then there are two families of approximate normalized eigenfunctions of $\mathcal{A}$:*

(i) One family $\{\Phi_{1n} = (f_{1n}, \lambda_{1n} f_{1n}, h_{1n}), n \in \mathbb{N}\}$, where Φ_{1n} is the eigenfunction of $\mathcal{A}$ corresponding to the eigenvalue λ_{1n}, has the following asymptotic expression:

$$\begin{pmatrix} f_{1n}''(x) \\ \lambda_{1n} f_{1n}(x) \\ h_{1n}(x) \end{pmatrix} = \begin{pmatrix} \sqrt{2}i\,[\varphi_{1n1}(x) - \varphi_{1n2}(x)] \\ \sqrt{2}\,[\varphi_{1n1}(x) + \varphi_{1n2}(x)] \\ a_3\varphi_{1n3}(x) + a_4\varphi_{1n4}(x) \end{pmatrix} + \mathcal{O}(n^{-1}), \tag{2.42}$$

where $\varphi_{1n1}(x), \varphi_{1n2}(x), \varphi_{1n3}(x)$ and $\varphi_{1n4}(x)$ have the following forms:

$$\begin{cases} \varphi_{1n1}(x) = \mathrm{e}^{i\rho_{1n}x} = \mathrm{e}^{i\sqrt{i}\left[n\pi + \frac{1}{2}\theta_1\right]x + \mathcal{O}(n^{-1})} = \mathrm{e}^{\left[-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right]\left[n\pi + \frac{1}{2}\theta_1\right]x + \mathcal{O}(n^{-1})}, \\ \varphi_{1n2}(x) = \mathrm{e}^{-\rho_{1n}x} = \mathrm{e}^{-\sqrt{i}\left[n\pi + \frac{1}{2}\theta_1\right]x + \mathcal{O}(n^{-1})} = \mathrm{e}^{\left[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right]\left[n\pi + \frac{1}{2}\theta_1\right]x + \mathcal{O}(n^{-1})} \\ \varphi_{1n3}(x) = \mathrm{e}^{\sqrt{i}\rho_{1n}x} = \mathrm{e}^{i\left[n\pi + \frac{1}{2}\theta_1\right]x + \mathcal{O}(n^{-1})}, \\ \varphi_{1n4}(x) = \mathrm{e}^{-\sqrt{i}\rho_{1n}x} = \mathrm{e}^{-i\left[n\pi + \frac{1}{2}\theta_1\right]x + \mathcal{O}(n^{-1})}, \end{cases} \tag{2.43}$$

a_3, a_4, θ_1 are constants given by (2.23) and (2.30) respectively. Furthermore, $\Phi_{1n} = (f_{1n}, \lambda_{1n} f_{1n}, h_{1n})$ are approximately normalized in $\mathcal{H}$ in the sense that there exist positive constants b_1 and b_2 independent of n, such that

$$b_1 \le \|\Phi_{1n}\| = \|f_{1n}''\|_{L^2(0,1)} + \|\lambda_{1n} f_{1n}\|_{L^2(0,1)} + \|h_{1n}\|_{L^2(0,1)} \le b_2, \quad \forall n \in \mathbb{N}. \tag{2.44}$$

(ii) The another family $\{\Phi_{2n} = (f_{2n}, \lambda_{2n} f_{2n}, h_{2n}), \overline{\Phi}_{2n} = (\overline{f}_{2n}, \overline{\lambda}_{2n}\overline{f}_{2n}, \overline{h}_{2n}), n \in \mathbb{N}\}$, where Φ_{2n} and $\overline{\Phi}_{2n}$ are the eigenfunctions of $\mathcal{A}$ corresponding to the complex conjugate eigenvalues λ_{2n} and $\overline{\lambda}_{2n}$, respectively, has the following asymptotic expression:

$$\begin{pmatrix} f''_{2n}(x) \\ \lambda_{2n} f_{2n}(x) \\ h_{2n}(x) \end{pmatrix} = \begin{pmatrix} -i\varphi_{2n1}(x) - \varphi_{2n2}(x) + \varepsilon_n \varphi_{2n3}(x) + (1+i)\varphi_{2n4}(x) \\ -\varphi_{2n1}(x) + i\varphi_{2n2}(x) + i\varepsilon_n \varphi_{2n3}(x) + (i-1)\varphi_{2n4}(x) \\ -i\varepsilon_n \varphi_{2n5}(x) \end{pmatrix} + \mathcal{O}\left(\frac{1}{n}\right) \tag{2.45}$$

where ε_n *and* $\varphi_{2nj}(x)$, $j = 1, 2, 3, 4, 5$ *have the following expressions*

$$\begin{cases} \varepsilon_n = (-1)^n \left[i\sqrt{r} e^{\frac{1}{2} i\theta_2} + (\sqrt{r})^{-1} e^{-\frac{1}{2} i\theta_2} \right], \\ \varphi_{2n1}(x) = e^{i\rho_{2n}(1-x)} = e^{\frac{1}{2}[\ln r + (2n\pi+\theta_2)i](1-x) + \mathcal{O}(n^{-1})}, \\ \varphi_{2n2}(x) = e^{-i\rho_{2n}(1-x)} = e^{-\frac{1}{2}[\ln r + (2n\pi+\theta_2)i](1-x) + \mathcal{O}(n^{-1})}, \\ \varphi_{2n3}(x) = e^{-\rho_{2n}x} = e^{\frac{1}{2}[i\ln r - (2n\pi+\theta_2)]x + \mathcal{O}(n^{-1})}, \\ \varphi_{2n4}(x) = e^{-\rho_{2n}(1-x)} = e^{\frac{1}{2}[i\ln r - (2n\pi+\theta_2)](1-x) + \mathcal{O}(n^{-1})}, \\ \varphi_{2n5}(x) = e^{-\sqrt{i}\rho_{2n}x} = e^{\frac{1}{2}\sqrt{i}[i\ln r - (2n\pi+\theta_2)]x + \mathcal{O}(n^{-1})}, \end{cases} \tag{2.46}$$

θ_2, r *are constants given by (2.30). Furthermore,* $\Phi_{2n} = (f_{2n}, \lambda_{2n} f_{2n}, h_{2n})$ *are approximately normalized in* $\mathcal{H}$ *in the sense that there exist positive constants* b_3 *and* b_4 *independent of* n*, such that*

$$b_3 \le \|\Phi_{2n}\| = \|f''_{2n}\|_{L^2(0,1)} + \|\lambda_{2n} f_{2n}\|_{L^2(0,1)} + \|h_{2n}\|_{L^2(0,1)} \le b_4, \quad \forall n \in \mathbb{N}. \tag{2.47}$$

Proof We first look for eigenfunction Φ_{1n} corresponding to λ_{1n}. For notational simplicity, we write $[a]_1 = a + \mathcal{O}(\rho^{-1})$ in what follows: from (2.13)–(2.17), (2.19), and some linear algebra calculations, for $\rho \in \mathcal{S}_1$, $h_1(x)$ is given by

$$h_1(x) = \begin{vmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \rho e^{\rho}[1]_1 & \rho e^{-\rho}[1]_1 & \rho e^{i\rho}[-1]_1 & \rho e^{-i\rho}[-1]_1 & 0 & 0 \\ \rho^3 e^{\rho}[-1]_1 & \rho^3 e^{-\rho}[1]_1 & \rho^3 e^{i\rho}[i]_1 & \rho^3 e^{-i\rho}[-i]_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho x} & e^{-\sqrt{i}\rho x} \\ i\rho^2 & i\rho^2 & i\rho^2 & i\rho^2 & -1 & -1 \\ \rho^2 & -\rho^2 & -i\rho^2 & i\rho^2 & -\sqrt{i} & \sqrt{i} \end{vmatrix}$$

$$= \rho^6 e^{\rho} e^{-i\rho} \left[\begin{vmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho x} & e^{-\sqrt{i}\rho x} \\ 0 & i & i & 0 & -\rho^{-2} & -\rho^{-2} \\ 0 & -\rho^2 & -i\rho^2 & 0 & -\sqrt{i} & \sqrt{i} \end{vmatrix} + \mathcal{O}(\rho^{-1}) \right]$$

$$= -(1+i)\rho^6 e^{\rho} e^{-i\rho} [-2i\sqrt{i}[e^{\sqrt{i}\rho x} + e^{-\sqrt{i}\rho x}] + (1+i)[e^{\sqrt{i}\rho x} - e^{-\sqrt{i}\rho x}] + \mathcal{O}(\rho^{-1})]$$

$$= -2\rho^6 e^{\rho} e^{-i\rho} [a_3 e^{\sqrt{i}\rho x} + a_4 e^{-\sqrt{i}\rho x} + \mathcal{O}(\rho^{-1})].$$

Similarly,

$$
\begin{aligned}
f_1(x) &= \begin{vmatrix}
1 & 1 & -1 & -1 & 0 & 0 \\
\rho e^{\rho}[1]_1 & \rho e^{-\rho}[1]_1 & \rho e^{i\rho}[-1]_1 & \rho e^{-i\rho}[-1]_1 & 0 & 0 \\
\rho^3 e^{\rho}[-1]_1 & \rho^3 e^{-\rho}[1]_1 & \rho^3 e^{i\rho}[i]_1 & \rho^3 e^{-i\rho}[-i]_1 & 0 & 0 \\
e^{\rho x} & e^{-\rho x} & e^{i\rho x} & e^{-i\rho x} & 0 & 0 \\
i\rho^2 & i\rho^2 & i\rho^2 & i\rho^2 & -1 & -1 \\
\rho^2 & -\rho^2 & -i\rho^2 & i\rho^2 & -\sqrt{i} & \sqrt{i}
\end{vmatrix} \\
&= -2\sqrt{i}\rho^4 e^{\rho} e^{-i\rho} \left[\begin{vmatrix}
0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1 \\
-1 & 0 & 0 & -i \\
e^{-\rho(1-x)} & e^{-\rho x} & e^{i\rho x} & e^{i\rho(1-x)}
\end{vmatrix} + \mathcal{O}(\rho^{-1}) \right] \\
&= 2\sqrt{2} i \rho^4 e^{\rho} e^{-i\rho} [e^{i\rho x} + e^{-\rho x} + \mathcal{O}(\rho^{-1})]
\end{aligned}
$$

and

$$
f_1''(x) = -2\sqrt{2} i \rho^6 e^{\rho} e^{-i\rho} [e^{i\rho x} - e^{-\rho x} + \mathcal{O}(\rho^{-1})].
$$

By setting

$$
\Phi_{1n} = \begin{pmatrix} f_{1n}(x) \\ \lambda_{1n} f_{1n}(x) \\ h_{1n}(x) \end{pmatrix} = -\frac{1}{2} \rho_{1n}^{-6} e^{-\rho_{1n}} e^{i\rho_{1n}} \begin{pmatrix} f_1(x, \rho_{1n}) \\ i\rho_{1n}^2 f_1(x, \rho_{1n}) \\ h_1(x, \rho_{1n}) \end{pmatrix}, \tag{2.48}
$$

we get

$$
\begin{pmatrix} f_{1n}''(x) \\ \lambda_{1n} f_{1n}(x) \\ h_{1n}(x) \end{pmatrix} = \begin{pmatrix} \sqrt{2} i e^{i\rho_{1n}x} - \sqrt{2} i e^{-\rho_{1n}x} \\ \sqrt{2} e^{i\rho_{1n}x} + \sqrt{2} e^{-\rho_{1n}x} \\ a_3 e^{\sqrt{i}\rho_{1n}x} + a_4 e^{-\sqrt{i}\rho_{1n}x} \end{pmatrix} + \mathcal{O}(n^{-1}), \tag{2.49}
$$

where a_3, a_4 are given by (2.23). Substituting ρ_{1n} given by (2.35) into (2.49) yields (2.42). By (2.43), we have

$$
\begin{cases}
\|\varphi_{1n1}(x)\|^2_{L^2(0,1)} = \mathcal{O}(n^{-1}), \quad \|\varphi_{1n2}(x)\|^2_{L^2(0,1)} = \mathcal{O}(n^{-1}), \\
\|\varphi_{1n3}(x)\|^2_{L^2(0,1)} = 1 + \mathcal{O}(n^{-1}), \quad \|\varphi_{1n4}(x)\|^2_{L^2(0,1)} = 1 + \mathcal{O}(n^{-1}), \\
\|a_3\varphi_{1n3}(x) + a_4\varphi_{1n4}(x)\|^2_{L^2(0,1)} = |a_3|^2 + |a_4|^2 + \mathcal{O}(n^{-1}).
\end{cases} \tag{2.50}
$$

These together with (2.42), (2.43) yield (2.44). Now we look for Φ_{2n}. Since $e^{\sqrt{i}\rho} \to \infty$ as $|\rho| \to \infty$ for $\rho \in \mathcal{S}_2$, we can get, similar to $h_1(x)$, that

$$h_2(x) = \begin{vmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \rho e^{\rho}[1]_1 & \rho e^{-\rho}[1]_1 & \rho e^{i\rho}[-1]_1 & \rho e^{-i\rho}[-1]_1 & 0 & 0 \\ \rho^3 e^{\rho}[-1]_1 & \rho^3 e^{-\rho}[1]_1 & \rho^3 e^{i\rho}[i]_1 & \rho^3 e^{-i\rho}[-i]_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho} & e^{-\sqrt{i}\rho} \\ i\rho^2 & i\rho^2 & i\rho^2 & i\rho^2 & -1 & -1 \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho x} & e^{-\sqrt{i}\rho x} \end{vmatrix}$$
$$= i\rho^6 e^{\rho}[e^{\sqrt{i}\rho(1-x)} - e^{-\sqrt{i}\rho(1-x)}][(1-i)e^{i\rho} - (1+i)e^{-i\rho} + \mathcal{O}(\rho^{-1}e^{-i\rho})]$$
$$= -\frac{2i}{1-i}\rho^6 e^{\rho}[e^{\sqrt{i}\rho(1-x)} - e^{-\sqrt{i}\rho(1-x)}][ie^{i\rho} + e^{-i\rho} + \mathcal{O}(\rho^{-1}e^{-i\rho})],$$

where $G_2(\rho)$ is given by (2.27). Moreover,

$$f_2(x) = \begin{vmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ \rho e^{\rho}[1]_1 & \rho e^{-\rho}[1]_1 & \rho e^{i\rho}[-1]_1 & \rho e^{-i\rho}[-1]_1 & 0 & 0 \\ \rho^3 e^{\rho}[-1]_1 & \rho^3 e^{-\rho}[1]_1 & \rho^3 e^{i\rho}[i]_1 & \rho^3 e^{-i\rho}[-i]_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho} & e^{-\sqrt{i}\rho} \\ i\rho^2 & i\rho^2 & i\rho^2 & i\rho^2 & -1 & -1 \\ e^{\rho x} & e^{-\rho x} & e^{i\rho x} & e^{-i\rho x} & 0 & 0 \end{vmatrix}$$

$$= -\rho^4 e^{\rho} e^{\sqrt{i}\rho}\left[\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & e^{i\rho} & e^{-i\rho} \\ -1 & 0 & -ie^{i\rho} & ie^{-i\rho} \\ e^{-\rho(1-x)} & e^{-\rho x} & -e^{i\rho x} & -e^{-i\rho x} \end{vmatrix} + \mathcal{O}(\rho^{-1})\right]$$

$$= -\rho^4 e^{\rho} e^{\sqrt{i}\rho}[-2ie^{-\rho(1-x)} + [(1-i)e^{i\rho} - (1+i)e^{-i\rho}]e^{-\rho x}$$
$$-(1+i)e^{-i\rho(1-x)} + (1-i)e^{i\rho(1-x)} + \mathcal{O}(\rho^{-1})]$$
$$= \frac{2}{1-i}\rho^4 e^{\rho} e^{\sqrt{i}\rho}[(1+i)e^{-\rho(1-x)} + [ie^{i\rho} + e^{-i\rho}]e^{-\rho x}$$
$$+ e^{-i\rho(1-x)} + ie^{i\rho(1-x)} + \mathcal{O}(\rho^{-1})]$$

and

$$f_2''(x) = \frac{2}{1-i}\rho^6 e^{\rho} e^{\sqrt{i}\rho}[(1+i)e^{-\rho(1-x)} + [ie^{i\rho} + e^{-i\rho}]e^{-\rho x}$$
$$-e^{-i\rho(1-x)} - ie^{i\rho(1-x)} + \mathcal{O}(\rho^{-1})].$$

By setting

$$\Phi_{2n} = \begin{pmatrix} f_{2n}(x) \\ \lambda_{2n} f_{2n}(x) \\ h_{2n}(x) \end{pmatrix} = \frac{1-i}{2}\rho_{2n}^{-6} e^{-\rho_{2n}} e^{-\sqrt{i}\rho_{2n}} \begin{pmatrix} f_2(x, \rho_{2n}) \\ i\rho_{2n}^2 f_2(x, \rho_{2n}) \\ h_2(x, \rho_{2n}) \end{pmatrix}, \quad (2.51)$$

we obtain

$$\begin{pmatrix} f''_{2n}(x) \\ \lambda_{2n} f_{2n}(x) \\ h_{2n}(x) \end{pmatrix} = \begin{pmatrix} (1+i)e^{-\rho_{2n}(1-x)} + \left[ie^{i\rho_{2n}} + e^{-i\rho_{2n}}\right]e^{-\rho_{2n}x} - e^{-i\rho_{2n}(1-x)} - ie^{i\rho_{2n}(1-x)} \\ (i-1)e^{-\rho_{2n}(1-x)} + i\left[ie^{i\rho_{2n}} + e^{-i\rho_{2n}}\right]e^{-\rho_{2n}x} + ie^{-i\rho_{2n}(1-x)} - e^{i\rho_{2n}(1-x)} \\ -ie^{-\sqrt{i}\rho_{2n}x}\left[ie^{i\rho_{2n}} + e^{-i\rho_{2n}}\right] \end{pmatrix} + \mathcal{O}\left(\rho_{2n}^{-1}\right). \tag{2.52}$$

By (2.39) and (2.40), it follows that

$$e^{i\rho_{2n}} = (-1)^n \sqrt{r} e^{\frac{1}{2}i\theta_2} + \mathcal{O}\left(n^{-1}\right), \; e^{-\sqrt{i}\rho_{2n}} = \mathcal{O}\left(n^{-1}\right),$$

and hence by (2.46),

$$\begin{pmatrix} f''_{2n}(x) \\ \lambda_{2n} f_{2n}(x) \\ h_{2n}(x) \end{pmatrix} = \begin{pmatrix} (1+i)e^{-\rho_{2n}(1-x)} + \varepsilon_n e^{-\rho_{2n}x} - e^{-i\rho_{2n}(1-x)} - ie^{i\rho_{2n}(1-x)} \\ (i-1)e^{-\rho_{2n}(1-x)} + i\varepsilon_n e^{-\rho_{2n}x} + ie^{-i\rho_{2n}(1-x)} - e^{i\rho_{2n}(1-x)} \\ -i\varepsilon_n e^{-\sqrt{i}\rho_{2n}x} \end{pmatrix} + \mathcal{O}\left(n^{-1}\right).$$

Substituting ρ_{2n} given by (2.40) into the above equation, we get (2.45). By virtue of (2.46), we have

$$\begin{cases} \|\varphi_{2n1}(x)\|^2_{L^2(0,1)} = [r-1][\ln r]^{-1} + \mathcal{O}(n^{-1}), \\ \|\varphi_{2n2}(x)\|^2_{L^2(0,1)} = \left[1 - \frac{1}{r}\right][\ln r]^{-1} + \mathcal{O}(n^{-1}), \\ \|\varphi_{2n3}(x)\|^2_{L^2(0,1)} = \|\varphi_{2n4}(x)\|^2_{L^2(0,1)} = \|\varphi_{2n5}(x)\|^2_{L^2(0,1)} = \mathcal{O}(n^{-1}). \end{cases} \tag{2.53}$$

These together with (2.45), (2.46) yield (2.47). The proof is complete. □

To end this section, we point out that the same process can be used to produce asymptotic expansions for the eigenpairs of $\mathcal{A}^*$, the adjoint operator of $\mathcal{A}$,

$$\begin{cases} \mathcal{A}^*(f,g,h) = (-g, f^{(4)}, h''), \; \forall\, (f,g,h) \in D(\mathcal{A}^*), \\ D(\mathcal{A}^*) = \left\{ (f,g,h) \in (H^4 \times H^2 \times H^2) \left| \begin{array}{l} h(1) = f''(0) = 0, \\ f'''(0) = -h'(0), g(0) = h(0) \\ f''(1) = -f'(1), f'''(1) = f(1) - kg(1) \end{array} \right. \right\}. \end{cases} \tag{2.54}$$

This is because when $\mathcal{A}$ is a discrete operator (that is $\mathcal{A}^{-1}$ is compact), and so is for $\mathcal{A}^*$ ([11, p.2354]); and due to the symmetry of eigenvalues on the real axis, $\mathcal{A}^*$ has the same eigenvalues as $\mathcal{A}$ ([30, p.26]) with the same algebraic multiplicity for the conjugate eigenvalues ([11, p.2354] or [12, p.20]). Moreover, we can also get the asymptotic eigenfunctions of $\mathcal{A}^*$. Actually, from $\mathcal{A}^*X = \lambda X$, where $X = (f, g, h)$ is the eigenfunction of $\mathcal{A}^*$ corresponding to the eigenvalue λ, we have $g = -\lambda f$ and that f, h satisfy the following equation:

$$\begin{cases} f^{(4)}(x) + \lambda^2 f(x) = 0, \\ h''(x) - \lambda h(x) = 0, \\ f''(0) = h(1) = 0, \\ f'''(0) = -h'(0), \\ \lambda f(0) = -h(0), \\ f''(1) = -f'(1), \\ f'''(1) = f(1) + k\lambda f(1), \end{cases}$$

which is exactly the same as (2.10). So the eigenfunctions of $\mathcal{A}^*$ can be obtained from Theorem 3.

Theorem 4 *Let $\mathcal{A}^*$ be defined by (2.54), let $\sigma(\mathcal{A}^*) = \sigma(\mathcal{A}) = \{\lambda_{1n}, n \in \mathbb{N}\} \cup \{\lambda_{2n}, \overline{\lambda}_{2n}, n \in \mathbb{N}\}$, and let $\lambda_{jn} := i\rho_{jn}^2$, $j = 1, 2$, with ρ_{1n}, ρ_{2n} given by (2.35) and (2.40), respectively. Then there are two families of approximate normalized eigenfunctions of $\mathcal{A}^*$:*

(i) One family $\{\Psi_{1n} = (f_{1n}, -\lambda_{1n} f_{1n}, h_{1n}), n \in \mathbb{N}\}$, where Ψ_{1n} is the eigenfunction of $\mathcal{A}^*$ corresponding to the eigenvalue λ_{1n}, has the following asymptotic expression:

$$\begin{pmatrix} f''_{1n}(x) \\ -\lambda_{1n} f_{1n}(x) \\ h_{1n}(x) \end{pmatrix} = \begin{pmatrix} \sqrt{2}i\,[\varphi_{1n1}(x) - \varphi_{1n2}(x)] \\ -\sqrt{2}\,[\varphi_{1n1}(x) + \varphi_{1n2}(x)] \\ a_3\varphi_{1n3}(x) + a_4\varphi_{1n4}(x) \end{pmatrix} + \mathcal{O}(n^{-1}), \tag{2.55}$$

where $\varphi_{1nj}(x)$, $j = 1, 2, 3, 4$, are given by (2.43) and a_3, a_4 are constants given by (2.23). Moreover, $\Psi_{1n} = (f_{1n}, -\lambda_{1n} f_{1n}, h_{1n})$ are approximately normalized in $\mathcal{H}$.

(ii) The another family $\{\Psi_{2n} = (f_{2n}, -\lambda_{2n} f_{2n}, h_{2n}), \overline{\Psi}_{2n} = (\overline{f}_{2n}, -\overline{\lambda}_{2n}\overline{f}_{2n}, \overline{h}_{2n}), n \in \mathbb{N}\}$, where Ψ_{2n} and $\overline{\Psi}_{2n}$ are the eigenfunctions of $\mathcal{A}^*$ corresponding to the complex conjugate eigenvalues λ_{2n} and $\overline{\lambda}_{2n}$ respectively, has the following asymptotic expression:

$$\begin{pmatrix} f''_{2n}(x) \\ -\lambda_{2n} f_{2n}(x) \\ h_{2n}(x) \end{pmatrix} = \begin{pmatrix} -i\varphi_{2n1}(x) - \varphi_{2n2}(x) + \varepsilon_n\varphi_{2n3}(x) + (1+i)\varphi_{2n4}(x) \\ \varphi_{2n1}(x) - i\varphi_{2n2}(x) - i\varepsilon_n\varphi_{2n3}(x) - (i-1)\varphi_{2n4}(x) \\ -i\varepsilon_n\varphi_{2n5}(x) \end{pmatrix} + \mathcal{O}\left(\frac{1}{n}\right) \tag{2.56}$$

where ε_n and $\varphi_{2nj}(x)$, $j = 1, 2, 3, 4, 5$, are given by (2.46). Moreover, $\Psi_{2n} = (f_{2n}, -\lambda_{2n} f_{2n}, h_{2n})$ are approximately normalized in $\mathcal{H}$.

Remark 2 For the sequences of $\{\Psi_{1n}, \Psi_{2n}, \overline{\Psi_{2n}}, n \in \mathbb{N}\}$, which are the eigenfunctions of $\mathcal{A}^*$ obtained in Theorem 4, and $\{\Phi_{1n}, \Phi_{2n}, \overline{\Phi_{2n}}, n \in \mathbb{N}\}$, which are the eigenfunctions of $\mathcal{A}$ obtained in Theorem 3, as $n \to \infty$, we have the following estimates:

(i)

$$\langle\Phi_{1n},\Psi_{1n}\rangle=\int_0^1\Big\{[\sqrt{2}i[\varphi_{1n1}(x)-\varphi_{1n2}(x)]]\overline{[\sqrt{2}i[\varphi_{1n1}(x)-\varphi_{1n2}(x)]]}$$
$$-[\sqrt{2}[\varphi_{1n1}(x)+\varphi_{1n2}(x)]]\overline{[\sqrt{2}[\varphi_{1n1}(x)+\varphi_{1n2}(x)]]}$$
$$+[a_3\varphi_{1n3}(x)+a_4\varphi_{1n4}(x)]\overline{[a_3\varphi_{1n3}(x)+a_4\varphi_{1n4}(x)]}\Big\}\mathrm{d}x$$
$$=\int_0^1\{2|\varphi_{1n1}(x)-\varphi_{1n2}(x)|^2-2|\varphi_{1n1}(x)+\varphi_{1n2}(x)|^2+|a_3\varphi_{1n3}(x)$$
$$+a_4\varphi_{1n4}(x)|^2\}\mathrm{d}x$$

and hence

$$\langle\Phi_{1n},\Psi_{1n}\rangle=|a_3|^2+|a_4|^2+\mathcal{O}(n^{-1}). \tag{2.57}$$

(ii)

$$\langle\Phi_{2n},\overline{\Psi_{2n}}\rangle=\int_0^1\Big\{[-i\varphi_{2n1}(x)-\varphi_{2n2}(x)+\varepsilon_n\varphi_{2n3}(x)+(1+i)\varphi_{2n4}(x)]^2$$
$$-[-\varphi_{2n1}(x)+i\varphi_{2n2}(x)+i\varepsilon_n\varphi_{2n3}(x)+(i-1)\varphi_{2n4}(x)]^2+[-i\varepsilon_n\varphi_{2n5}(x)]^2\Big\}\mathrm{d}x$$
$$=\int_0^1\Big\{[i\varphi_{2n1}(x)+\varphi_{2n2}(x)-\varepsilon_n\varphi_{2n3}(x)-(1+i)\varphi_{2n4}(x)]^2$$
$$+[i\varphi_{2n1}(x)+\varphi_{2n2}(x)+\varepsilon_n\varphi_{2n3}(x)+(1+i)\varphi_{2n4}(x)]^2-\varepsilon_n^2\varphi_{2n5}^2(x)\Big\}\mathrm{d}x$$
$$=4i\int_0^1\varphi_{2n1}(x)\varphi_{2n2}(x)\mathrm{d}x+\mathcal{O}(n^{-1})$$

and hence

$$\langle\Phi_{2n},\overline{\Psi_{2n}}\rangle=4i+\mathcal{O}(n^{-1}). \tag{2.58}$$

Moreover, we have

$$\langle\overline{\Phi_{2n}},\Psi_{2n}\rangle=\overline{\langle\Phi_{2n},\overline{\Psi_{2n}}\rangle}=-4i+\mathcal{O}(n^{-1}). \tag{2.59}$$

(iii) From (2.57)–(2.59), it follows that there are $M_1, M_2>0$ independent of n such that

$$M_1\le|\langle\Phi_{1n},\Psi_{1n}\rangle|,\ |\langle\Phi_{2n},\overline{\Psi_{2n}}\rangle|,\ |\langle\overline{\Phi_{2n}},\Psi_{2n}\rangle|\le M_2. \tag{2.60}$$

3 Completeness of the root subspace for the 1-d problem

In this section, we show that the root subspace of the system (2.6) is complete in $\mathcal{H}$.

Lemma 3 *Let $\mathcal{A}$ be defined by (2.5) and for $x \in [0, 1]$ and $\rho \in \mathbb{C}$, let*

$$\begin{cases} Q_1(x,\xi) = \frac{1}{8}\operatorname{sign}(x-\xi)\rho^{-3}\left[e^{\rho(x-\xi)} - e^{-\rho(x-\xi)} + ie^{i\rho(x-\xi)} - ie^{-i\rho(x-\xi)}\right], \\ Q_2(x,\xi) = -i\frac{1}{4}\operatorname{sign}(x-\xi)\rho^{-1}\left[\sqrt{i}e^{\sqrt{i}\rho(x-\xi)} - \sqrt{i}e^{-\sqrt{i}\rho(x-\xi)}\right]. \end{cases} \tag{3.1}$$

For any $\lambda = i\rho^2 \in \rho(\mathcal{A})$ with $\lambda \neq 0$ and $(\phi, \psi, \chi) \in \mathcal{H}$, let $R(\lambda, \mathcal{A}) = (\lambda - \mathcal{A})^{-1}$ be the resolvent operator of $\mathcal{A}$ and let

$$F_0(x,\rho) = \int_0^1 Q_1(x,\xi)[i\rho^2\phi(\xi) + \psi(\xi)]d\xi, \quad H_0(x,\rho) = -\int_0^1 Q_2(x,\xi)\chi(\xi)d\xi. \tag{3.2}$$

Then the solution of the resolvent equation $R(\lambda, \mathcal{A})(\phi, \psi, \chi) = (f, g, h)$ is given by

$$f(x) = \frac{F(x,\rho)}{\det\Delta(\rho)}, \quad g(x) = \lambda f(x) - \phi(x), \quad h(x) = \frac{H(x,\rho)}{\det\Delta(\rho)}, \quad f''(x) = \frac{F''_x(x,\rho)}{\det\Delta(\rho)} \tag{3.3}$$

where $\Delta(\rho)$ is defined by (2.19),

$$F(x,\rho) = \begin{vmatrix} e^{\rho x} & e^{-\rho x} & e^{i\rho x} & e^{-i\rho x} & 0 & 0 & F_0(x,\rho) \\ 1 & 1 & -1 & -1 & 0 & 0 & F_1 \\ \nu_1 e^{\rho} & \nu_2 e^{-\rho} & \nu_3 e^{i\rho} & \nu_4 e^{-i\rho} & 0 & 0 & F_2 \\ \nu_5 e^{\rho} & \nu_6 e^{-\rho} & \nu_7 e^{i\rho} & \nu_8 e^{-i\rho} & 0 & 0 & F_3 - k\phi(1) \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho} & e^{-\sqrt{i}\rho} & H_4 \\ i\rho^2 & i\rho^2 & i\rho^2 & i\rho^2 & -1 & -1 & F_5 - H_5 - \phi(0) \\ \rho^2 & -\rho^2 & -i\rho^2 & i\rho^2 & -\sqrt{i} & \sqrt{i} & F_6 - H_6 \end{vmatrix}, \tag{3.4}$$

$$H(x,\rho) = \begin{vmatrix} 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho x} & e^{-\sqrt{i}\rho x} & H_0(x,\rho) \\ 1 & 1 & -1 & -1 & 0 & 0 & F_1 \\ \nu_1 e^{\rho} & \nu_2 e^{-\rho} & \nu_3 e^{i\rho} & \nu_4 e^{-i\rho} & 0 & 0 & F_2 \\ \nu_5 e^{\rho} & \nu_6 e^{-\rho} & \nu_7 e^{i\rho} & \nu_8 e^{-i\rho} & 0 & 0 & F_3 - k\phi(1) \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho} & e^{-\sqrt{i}\rho} & H_4 \\ i\rho^2 & i\rho^2 & i\rho^2 & i\rho^2 & -1 & -1 & F_5 - H_5 - \phi(0) \\ \rho^2 & -\rho^2 & -i\rho^2 & i\rho^2 & -\sqrt{i} & \sqrt{i} & F_6 - H_6 \end{vmatrix}. \tag{3.5}$$

Here ν_j, $j = 1, 2, \ldots, 8$ *are given by (2.20), and* F_j, $j = 1, 2, 3, 5, 6$, H_s, $s = 4, 5, 6$ *are constants given by*

$$\begin{cases} F_1 = -\frac{1}{8}\rho^{-3}\int\limits_0^1 \left[e^{-\rho\xi} - e^{\rho\xi} - ie^{-i\rho\xi} + ie^{i\rho\xi}\right]\left[i\rho^2\phi(\xi) + \psi(\xi)\right]d\xi, \\ F_2 = \frac{1}{8}\rho^{-2}\int\limits_0^1 \left[e^{\rho(1-\xi)} - e^{-\rho(1-\xi)} - ie^{i\rho(1-\xi)} + ie^{-i\rho(1-\xi)}\right]\left[i\rho^2\phi(\xi) + \psi(\xi)\right]d\xi \\ \quad + \frac{1}{8}\rho^{-3}\int\limits_0^1 \left[e^{\rho(1-\xi)} + e^{-\rho(1-\xi)} - e^{i\rho(1-\xi)} - e^{-i\rho(1-\xi)}\right]\left[i\rho^2\phi(\xi) + \psi(\xi)\right]d\xi, \\ F_3 = \frac{1}{8}\rho^{-3}(1+ki\rho^2)\int_0^1 \left[e^{\rho(1-\xi)} - e^{-\rho(1-\xi)} + ie^{i\rho(1-\xi)} - ie^{-i\rho(1-\xi)}\right]\left[i\rho^2\phi(\xi) + \psi(\xi)\right]d\xi, \\ \quad - \frac{1}{8}\int\limits_0^1 \left[e^{\rho(1-\xi)} + e^{-\rho(1-\xi)} - e^{i\rho(1-\xi)} - e^{-i\rho(1-\xi)}\right]\left[i\rho^2\phi(\xi) + \psi(\xi)\right]d\xi, \\ F_5 = -i\frac{1}{8}\rho^{-1}\int\limits_0^1 \left[e^{-\rho\xi} - e^{\rho\xi} + ie^{-i\rho\xi} - ie^{i\rho\xi}\right]\left[i\rho^2\phi(\xi) + \psi(\xi)\right]d\xi, \\ F_6 = -\frac{1}{8}\rho^{-1}\int\limits_0^1 \left[e^{-\rho\xi} + e^{\rho\xi} + e^{-i\rho\xi} + e^{i\rho\xi}\right]\left[i\rho^2\phi(\xi) + \psi(\xi)\right]d\xi, \end{cases} \tag{3.6}$$

$$\begin{cases} H_4 = i\frac{1}{4}\rho^{-1}\int_0^1 \left[\sqrt{i}e^{\sqrt{i}\rho(1-\xi)} - \sqrt{i}e^{-\sqrt{i}\rho(1-\xi)}\right]\chi(\xi)d\xi, \\ H_5 = i\frac{1}{4}\rho^{-1}\int_0^1 \left[\sqrt{i}e^{-\sqrt{i}\rho\xi} - \sqrt{i}e^{\sqrt{i}\rho\xi}\right]\chi(\xi)d\xi, \\ H_6 = -\frac{1}{4}\rho^{-1}\int_0^1 \left[e^{-\sqrt{i}\rho\xi} + e^{\sqrt{i}\rho\xi}\right]\chi(\xi)d\xi. \end{cases} \tag{3.7}$$

Proof For any $(\phi, \psi, \chi) \in \mathcal{H}$ and any $\lambda = i\rho^2 \in \rho(\mathcal{A})$ with $\rho \neq 0$, solving the resolvent equation

$$(\lambda - \mathcal{A})(f, g, h) = (\phi, \psi, \chi)$$

yields $g = \lambda f - \phi$ with f, h satisfying

$$\begin{cases} f^{(4)}(x) - \rho^4 f(x) = i\rho^2\phi(x) + \psi(x), \\ h''(x) - i\rho^2 h(x) = -\chi(x), \\ f''(0) = 0, \\ f''(1) = -f'(1), \\ f(1) + ki\rho^2 f(1) - f'''(1) = k\phi(1), \\ h(1) = 0, \\ i\rho^2 f(0) - h(0) = \phi(0), \\ f'''(0) = h'(0). \end{cases} \tag{3.8}$$

Note that $f^{(4)}(x)-\rho^4 f(x)=i\rho^2\phi(x)+\psi(x)$ and $h''(x)-i\rho^2 h(x)=-\chi(x)$ have the general solutions respectively

$$\begin{cases} f(x)=c_1\mathrm{e}^{\rho x}+c_2\mathrm{e}^{-\rho x}+c_3\mathrm{e}^{i\rho x}+c_4\mathrm{e}^{-i\rho x}+F_0(x,\rho),\\ h(x)=d_1\mathrm{e}^{\sqrt{i}\rho x}+d_2\mathrm{e}^{-\sqrt{i}\rho x}+H_0(x,\rho), \end{cases} \tag{3.9}$$

where $F_0(x,\rho)$ and $H_0(x,\rho)$ are given by (3.2). Hence, by the boundary conditions of (3.8), c_j, $j=1,2,3,4$ and d_1,d_2 satisfy the following algebraic equations:

$$\begin{cases} c_1+c_2-c_3-c_4=-F_1,\\ c_1(1+\rho)\mathrm{e}^{\rho}-c_2(1-\rho)\mathrm{e}^{-\rho}+c_3(i-\rho)\mathrm{e}^{i\rho}-c_4(i+\rho)\mathrm{e}^{-i\rho}=-F_2,\\ c_1(1+ki\rho^2-\rho^3)\mathrm{e}^{\rho}+c_2(1+ki\rho^2+\rho^3)\mathrm{e}^{-\rho}\\ \quad +c_3(1+ki\rho^2+i\rho^3)\mathrm{e}^{i\rho}+c_4(1+ki\rho^2-i\rho^3)\mathrm{e}^{-i\rho}=-F_3+k\phi(1),\\ d_1\mathrm{e}^{\sqrt{i}\rho}+d_2\mathrm{e}^{-\sqrt{i}\rho}=-H_4,\\ c_1i\rho^2+c_2i\rho^2+c_3i\rho^2+c_4i\rho^2-d_1-d_2=-F_5+H_5+\phi(0),\\ c_1\rho^2-c_2\rho^2-c_3i\rho^2+c_4i\rho^2-d_1\sqrt{i}+d_2\sqrt{i}=-F_6+H_6. \end{cases} \tag{3.10}$$

Since as $\lambda=i\rho^2\in\rho(\mathcal{A})$, $\det\Delta(\rho)\neq 0$, it follows that (3.10) admits a unique solution. Moreover, the solution $f(x,\rho)$ and of $h(x,\rho)$ of (3.9) can be written in (3.3). □

Proposition 1 *Let $\mathcal{A}$ be defined by (2.5). Then all $\lambda=i\rho^2\in\sigma(\mathcal{A})$ with sufficiently large moduli are algebraically simple.*

Proof We only prove the case of $\rho\in\mathcal{S}$ since the proof for $\overline{\lambda}=-i\overline{\rho}^2$ is similar. From Lemma 3, the order of each $\lambda\in\sigma(\mathcal{A})$, as a pole of $R(\lambda,\mathcal{A})$, with sufficiently large modulus is less than or equal to the multiplicity of λ as a zero of the entire function $\det(\Delta(\rho))$ with respect to ρ. Since it is easy to see that λ is geometrically simple and from (2.32) and (2.37) all zeros of $\det(\Delta(\rho))=0$ with large moduli are simple in $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, the result then follows from the formula: $m_a\le p\cdot m_g$ (see e.g. [32, p.148]), where p denotes the order of the pole of the resolvent operator and m_a, m_g denote the algebraic and geometric multiplicities, respectively. □

To estimate the norm of the resolvent operator, we recall the Lemma 1.2 of [41] (see also [24]).

Lemma 4 *Let*

$$D(\lambda)=1+\sum_{i=1}^{n}Q_i(\lambda)e^{\alpha_i\lambda},$$

where Q_i are polynomials of λ, α_i are some complex numbers, and n is a positive integer. Then for all λ outside those circles centered at the roots of $D(\cdot)$ with radius $\epsilon>0$, one has

$$|D(\lambda)|\ge C(\epsilon)>0$$

for some constant $C(\epsilon)$ that depends only on ϵ.

Theorem 5 *Let $\mathcal{A}$ be defined by (2.5) and for $\lambda \in \rho(\mathcal{A})$, let $R(\lambda, \mathcal{A}) = (\lambda - \mathcal{A})^{-1}$ be the resolvent operator of $\mathcal{A}$. Then there exists a constant $M > 0$ independent of λ such that*

$$\|R(\lambda, \mathcal{A})\| \leq M(1 + |\lambda|^3) \tag{3.11}$$

for all $\lambda = i\rho^2$ with $\rho \in \mathbb{C}$ lying outside all circles centered at the zeros of $\det(\Delta(\rho))$ *with radius $\epsilon > 0$.*

Proof We first consider those $\lambda = i\rho^2$ with $\rho \in \mathcal{S}$. Let $\rho \in \mathcal{S}$ with $\rho \neq 0$. For $(\phi, \psi, \chi) \in \mathcal{H}$, $(f, g, h) = R(\lambda, \mathcal{A})(\phi, \psi, \chi)$ has the expression given by (3.3). To estimate $R(\lambda, \mathcal{A})$, we notice the fact that in sector $\mathcal{S}$, it follows from (2.12), (2.13) and (2.15) that

$$\mathcal{R}e(-\rho) \leq \mathcal{R}e(i\rho) \leq 0 \ \text{ and } \ \mathrm{Re}(-\sqrt{i}\rho) \leq 0.$$

Then we need to use the transformation of the determinant to make the elements F_j, $j = 1, 2, \ldots, 6$ given by (3.6) and (3.7) stable. So, for $F(x, \rho)$ and $H(x, \rho)$ given by (3.4) and (3.5), multiply

$$\begin{cases} \text{the first column by } -\frac{1}{8}\rho^{-3}\int_0^1 e^{-\rho\xi}\left[i\rho^2\phi(\xi) + \psi(\xi)\right]d\xi, \\ \text{the second column by } -\frac{1}{8}\rho^{-3}\int_0^1 e^{\rho\xi}\left[i\rho^2\phi(\xi) + \psi(\xi)\right]d\xi, \\ \text{the third column by } \frac{1}{8}\rho^{-3}\int_0^1 ie^{-i\rho\xi}\left[i\rho^2\phi(\xi) + \psi(\xi)\right]d\xi, \\ \text{the fourth column by } \frac{1}{8}\rho^{-3}\int_0^1 ie^{i\rho\xi}\left[i\rho^2\phi(\xi) + \psi(\xi)\right]d\xi, \\ \text{the fifth column by } -\frac{1}{4}\rho^{-1}\int_0^1 i\sqrt{i}e^{-\sqrt{i}\rho\xi}\chi(\xi)d\xi, \\ \text{the sixth column by } -\frac{1}{4}\rho^{-1}\int_0^1 i\sqrt{i}e^{\sqrt{i}\rho\xi}\chi(\xi)d\xi, \end{cases}$$

and add these columns to the last column of $F(x, \rho)$ and $H(x, \rho)$, respectively, to obtain

$$F(x,\rho) = \rho^{-3}\begin{vmatrix} e^{\rho x} & e^{-\rho x} & e^{i\rho x} & e^{-i\rho x} & 0 & 0 & \widetilde{F}_0(x,\rho) \\ 1 & 1 & -1 & -1 & 0 & 0 & \widetilde{F}_1 \\ \nu_1 e^{\rho} & \nu_2 e^{-\rho} & \nu_3 e^{i\rho} & \nu_4 e^{-i\rho} & 0 & 0 & \rho\widetilde{F}_2 \\ \nu_5 e^{\rho} & \nu_6 e^{-\rho} & \nu_7 e^{i\rho} & \nu_8 e^{-i\rho} & 0 & 0 & \rho^3[\widetilde{F}_3 - k\phi(1)] \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho} & e^{-\sqrt{i}\rho} & \rho^2\widetilde{H_4} \\ i\rho^2 & i\rho^2 & i\rho^2 & i\rho^2 & -1 & -1 & \rho^2[\widetilde{F}_5 - \widetilde{H}_5 - \phi(0)] \\ \rho^2 & -\rho^2 & -i\rho^2 & i\rho^2 & -\sqrt{i} & \sqrt{i} & \rho^2[\widetilde{F}_6 - \widetilde{H}_6] \end{vmatrix}$$

and

$$H(x,\rho)=\rho^{-1}\begin{vmatrix} 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho x} & e^{-\sqrt{i}\rho x} & \widetilde{H_0}(x,\rho) \\ 1 & 1 & -1 & -1 & 0 & 0 & \rho^{-2}\widetilde{F}_1 \\ \nu_1 e^{\rho} & \nu_2 e^{-\rho} & \nu_3 e^{i\rho} & \nu_4 e^{-i\rho} & 0 & 0 & \rho^{-1}\widetilde{F}_2 \\ \nu_5 e^{\rho} & \nu_6 e^{-\rho} & \nu_7 e^{i\rho} & \nu_8 e^{-i\rho} & 0 & 0 & \rho[\widetilde{F}_3 - k\phi(1)] \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho} & e^{-\sqrt{i}\rho} & \widetilde{H_4} \\ i\rho^2 & i\rho^2 & i\rho^2 & i\rho^2 & -1 & -1 & \widetilde{F}_5 - \widetilde{H_5} - \phi(0) \\ \rho^2 & -\rho^2 & -i\rho^2 & i\rho^2 & -\sqrt{i} & \sqrt{i} & \widetilde{F}_6 - \widetilde{H_6} \end{vmatrix},$$

where

$$\begin{aligned} \widetilde{F}_0(x,\rho) &= \frac{1}{4}\int_0^x [-e^{-\rho(x-\xi)} + i e^{i\rho(x-\xi)}][i\rho^2\phi(\xi)+\psi(\xi)]d\xi \\ &\quad + \frac{1}{4}\int_x^1 [-e^{-\rho(\xi-x)} + i e^{i\rho(\xi-x)}][i\rho^2\phi(\xi)+\psi(\xi)]d\xi \end{aligned} \tag{3.12}$$

$$\widetilde{H_0}(x,\rho) = \frac{1}{2}\frac{1}{\sqrt{i}}\left[\int_0^x e^{-\sqrt{i}\rho(x-\xi)}\chi(\xi)d\xi + \int_x^1 e^{-\sqrt{i}\rho(\xi-x)}\chi(\xi)d\xi\right], \tag{3.13}$$

and

$$\begin{cases} \widetilde{F}_1 = -\frac{1}{4}\int_0^1 \left[e^{-\rho\xi} + i e^{i\rho\xi}\right]\left[i\rho^2\phi(\xi)+\psi(\xi)\right]d\xi, \\ \widetilde{F}_2 = -\frac{1}{4}\int_0^1 \left[[1]_1 e^{-\rho(1-\xi)} + [i]_1 e^{i\rho(1-\xi)}\right]\left[i\rho^2\phi(\xi)+\psi(\xi)\right]d\xi, \\ \widetilde{F}_3 = -\frac{1}{4}\int_0^1 \left[[1]_1 e^{-\rho(1-\xi)} - [1]_1 e^{i\rho(1-\xi)}\right]\left[i\rho^2\phi(\xi)+\psi(\xi)\right]d\xi, \\ \widetilde{F}_5 = -\frac{i}{4}\int_0^1 \left[e^{-\rho\xi} - i e^{i\rho\xi}\right]\left[i\rho^2\phi(\xi)+\psi(\xi)\right]d\xi, \\ \widetilde{F}_6 = -\frac{1}{4}\int_0^1 \left[e^{-\rho\xi} + e^{i\rho\xi}\right]\left[i\rho^2\phi(\xi)+\psi(\xi)\right]d\xi, \end{cases} \tag{3.14}$$

$$\begin{cases} \widetilde{H_4} = -\frac{1}{2} i\sqrt{i} \int\limits_0^1 \mathrm{e}^{-\sqrt{i}\rho(1-\xi)} \chi(\xi)\mathrm{d}\xi, \\ \widetilde{H_5} = \frac{1}{2} i\sqrt{i} \int\limits_0^1 \sqrt{i}\mathrm{e}^{-\sqrt{i}\rho\xi} \chi(\xi)\mathrm{d}\xi, \\ \widetilde{H_6} = -\frac{1}{2} \int\limits_0^1 \mathrm{e}^{-\sqrt{i}\rho\xi} \chi(\xi)\mathrm{d}\xi. \end{cases} \tag{3.15}$$

A direct computation gives

$$\begin{aligned} F(x,\rho) &= \rho^3 \mathrm{e}^{\rho(1-i+\sqrt{i})}[\widetilde{F}(x,\rho) + \mathcal{O}(\rho^{-1})], \\ H(x,\rho) &= \rho^5 \mathrm{e}^{\rho(1-i+\sqrt{i})}[\widetilde{H}(x,\rho) + \mathcal{O}(\rho^{-1})] \end{aligned} \tag{3.16}$$

where

$$\widetilde{F}(x,\rho) = \begin{vmatrix} \mathrm{e}^{-\rho(1-x)} & \mathrm{e}^{-\rho x} & \mathrm{e}^{i\rho x} & \mathrm{e}^{i\rho(1-x)} & 0 & 0 & \widetilde{F}_0(x,\rho) \\ e^{-\rho} & 1 & -1 & -\mathrm{e}^{i\rho} & 0 & 0 & \widetilde{F}_1 \\ 1 & \mathrm{e}^{-\rho} & -\mathrm{e}^{i\rho} & -1 & 0 & 0 & \widetilde{F}_2 \\ -1 & \mathrm{e}^{-\rho} & i\mathrm{e}^{i\rho} & -i & 0 & 0 & \widetilde{F}_3 - k\phi(1) \\ 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{-\sqrt{i}\rho} & \rho^2\widetilde{H_4} \\ i\mathrm{e}^{-\rho} & i & i & i\mathrm{e}^{i\rho} & -\rho^{-2}\mathrm{e}^{-\sqrt{i}\rho} & -\rho^{-2} & \widetilde{F}_5 - \widetilde{H_5} - \phi(0) \\ \rho^2\mathrm{e}^{-\rho} & -\rho^2 & -i\rho^2 & i\rho^2\mathrm{e}^{i\rho} & -\sqrt{i}\mathrm{e}^{-\sqrt{i}\rho} & \sqrt{i} & \rho^2[\widetilde{F}_6 - \widetilde{H_6}] \end{vmatrix}$$

and

$$\widetilde{H}(x,\rho) = \begin{vmatrix} 0 & 0 & 0 & 0 & \mathrm{e}^{-\sqrt{i}\rho(1-x)} & \mathrm{e}^{-\sqrt{i}\rho x} & \widetilde{H_0}(x,\rho) \\ e^{-\rho} & 1 & -1 & -\mathrm{e}^{i\rho} & 0 & 0 & \rho^{-2}\widetilde{F}_1 \\ 1 & \mathrm{e}^{-\rho} & -\mathrm{e}^{i\rho} & -1 & 0 & 0 & \rho^{-2}\widetilde{F}_2 \\ -1 & \mathrm{e}^{-\rho} & i\mathrm{e}^{i\rho} & -i & 0 & 0 & \rho^{-2}[\widetilde{F}_3 - k\phi(1)] \\ 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{-\sqrt{i}\rho} & \widetilde{H_4} \\ i\mathrm{e}^{-\rho} & i & i & i\mathrm{e}^{i\rho} & -\rho^{-2}\mathrm{e}^{-\sqrt{i}\rho} & -\rho^{-2} & \rho^{-2}[\widetilde{F}_5 - \widetilde{H_5} - \phi(0)] \\ \rho^2\mathrm{e}^{-\rho} & -\rho^2 & -i\rho^2 & i\rho^2\mathrm{e}^{i\rho} & -\sqrt{i}\mathrm{e}^{-\sqrt{i}\rho} & \sqrt{i} & [\widetilde{F}_6 - \widetilde{H_6}] \end{vmatrix}.$$

Moreover, for $s = 1, 2$,

$$\rho^{-s}\widetilde{F}_x^{(s)}(x,\rho) = \widetilde{\widetilde{F}}_s(x,\rho) + \mathcal{O}(\rho^{-1}) \tag{3.17}$$

where

$$\widetilde{\widetilde{F}}_s(x,\rho)=\begin{vmatrix} e^{-\rho(1-x)} & (-1)^s e^{-\rho x} & i^s e^{i\rho x} & (-i)^s e^{i\rho(1-x)} & 0 & 0 & \widetilde{F_{0}}_x^{(s)}(x,\rho) \\ e^{-\rho} & 1 & -1 & -e^{i\rho} & 0 & 0 & \widetilde{F_1} \\ 1 & e^{-\rho} & -e^{i\rho} & -1 & 0 & 0 & \widetilde{F_2} \\ -1 & e^{-\rho} & ie^{i\rho} & -i & 0 & 0 & \widetilde{F_3}-k\phi(1) \\ 0 & 0 & 0 & 0 & 1 & e^{-\sqrt{i}\rho} & \rho^2\widetilde{H_4} \\ ie^{-\rho} & i & i & ie^{i\rho} & -\rho^{-2}e^{-\sqrt{i}\rho} & -\rho^{-2} & \widetilde{F_5}-\widetilde{H_5}-\phi(0) \\ \rho^2 e^{-\rho} & -\rho^2 & -i\rho^2 & i\rho^2 e^{i\rho} & -\sqrt{i}e^{-\sqrt{i}\rho} & \sqrt{i} & \rho^2[\widetilde{F_6}-\widetilde{H_6}] \end{vmatrix} \tag{3.18}$$

$$\widetilde{F_{0}}_x^{(s)}(x,\rho)=\frac{1}{4}\int_0^1 P_x^{(s)}(x,\xi)\left[i\rho^2\phi(\xi)+\psi(\xi)\right]\mathrm{d}\xi,\quad s=1,2 \tag{3.19}$$

with

$$\begin{cases} P_x'(x,\xi)=\begin{cases} -e^{-\rho(x-\xi)}+ie^{i\rho(x-\xi)}, & x\ge\xi, \\ -e^{-\rho(\xi-x)}+ie^{i\rho(\xi-x)}, & x<\xi, \end{cases} \\ P_{xx}''(x,\xi)=\begin{cases} e^{-\rho(x-\xi)}-e^{i\rho(x-\xi)}, & x\ge\xi, \\ -e^{-\rho(\xi-x)}+e^{i\rho(\xi-x)}, & x<\xi. \end{cases} \end{cases} \tag{3.20}$$

Hence, by (2.11)–(2.15),(3.12)–(3.15), and by expanding the last column for $\widetilde{F}(x,\rho)$, $\widetilde{H}(x,\rho)$ and $\widetilde{\widetilde{F}}_s(x,\rho)$, $s=1,2$ respectively, there is an $M_0>0$ such that for all $\rho\in\mathcal{S}$, we have

$$|\widetilde{F}(x,\rho)|\le M_0\left[\int_0^1[|\rho^2||\phi(\xi)|+|\psi(\xi)|+|\chi(\xi)|]\mathrm{d}\xi+|\phi(0)|+|\phi(1)|\right], \tag{3.21}$$

$$|\widetilde{H}(x,\rho)|\le M_0\left[\int_0^1[|\rho^2||\phi(\xi)|+|\psi(\xi)|+|\chi(\xi)|]d\xi+|\phi(0)|+|\phi(1)|\right] \tag{3.22}$$

and

$$|\widetilde{\widetilde{F}}_s(x,\rho)|\le M_0[\int_0^1[|\rho^2||\phi(\xi)|+|\psi(\xi)|+|\chi(\xi)|]d\xi+|\phi(0)|+|\phi(1)|],\quad s=1,2. \tag{3.23}$$

On the other hand, by (2.19), a direct computation gives that for any $\rho\in\mathcal{S}$, $\det(\Delta(\rho))$ has the analytic expression as follows:

$$
\begin{aligned}
&-\rho^{-2}\mathrm{e}^{-\rho}\mathrm{e}^{i\rho}\mathrm{e}^{-\sqrt{i}\rho}\det\Delta(\rho)\\
&= -[1+i+2i\sqrt{i}]Q_4(\rho)+Q_1(\rho)\mathrm{e}^{-\rho}\mathrm{e}^{i\rho}+[2i\sqrt{i}-1+i]Q_2(\rho)\mathrm{e}^{-2\rho}\\
&\quad +[1+i-2i\sqrt{i}]Q_3(\rho)\mathrm{e}^{-2\rho}\mathrm{e}^{i2\rho}+[1-i+2i\sqrt{i}]Q_5(\rho)\mathrm{e}^{i2\rho}\\
&\quad -[1+i+2i\sqrt{i}]Q_3(\rho)\mathrm{e}^{-2\rho}\mathrm{e}^{i2\rho}\mathrm{e}^{-2\sqrt{i}\rho}-[1-i-2i\sqrt{i}]Q_5(\rho)\mathrm{e}^{i2\rho}\mathrm{e}^{-2\sqrt{i}\rho}\\
&\quad -Q_1(\rho)\mathrm{e}^{-\rho}\mathrm{e}^{i\rho}\mathrm{e}^{-2\sqrt{i}\rho}\\
&\quad +[2i\sqrt{i}+1-i]Q_2(\rho)\mathrm{e}^{-2\rho}\mathrm{e}^{-2\sqrt{i}\rho}+[1+i-2i\sqrt{i}]Q_4(\rho)\mathrm{e}^{-2\sqrt{i}\rho}, \qquad (3.24)
\end{aligned}
$$

where $Q_j(\rho)$, $j=1,2,3,4,5$, are polynomials of order four. Precisely,

$$
\begin{cases}
Q_1(\rho)=-8[i-k\rho^2+i\rho^4], \quad Q_2(\rho)=(i-1)\\
\qquad\qquad +2\rho-k(1+i)\rho^2+2(k+1)i\rho^3+(1-i)\rho^4,\\
Q_3(\rho)=-(1+i)+2\rho+k(1-i)\rho^2-2(1-k)i\rho^3+(1+i)\rho^4,\\
Q_4(\rho)=(1+i)+2\rho-k(1-i)\rho^2+2(k-1)i\rho^3+(1+i)\rho^4,\\
Q_5(\rho)=(1-i)+2\rho+k(1+i)\rho^2+2(k+1)i\rho^3+(i-1)\rho^4.
\end{cases} \qquad (3.25)
$$

So $[1+i+2i\sqrt{i}]^{-1}(1+i)^{-1}\rho^{-2}\mathrm{e}^{-\rho}\mathrm{e}^{i\rho}\mathrm{e}^{-\sqrt{i}\rho}\det\Delta(\rho)$ satisfies the assumption of Lemma 4. Then it follows that there is an $M_1(\epsilon)>0$ such that for any $\rho\in\mathcal{S}$ lying outside all the circles centered at the zeros of $\det\Delta(\rho)$ with radius $\epsilon>0$, we have

$$
\left|[1+i+2i\sqrt{i}]^{-1}(1+i)^{-1}\rho^{-2}\mathrm{e}^{-\rho}\mathrm{e}^{i\rho}\mathrm{e}^{-\sqrt{i}\rho}\det\Delta(\rho)\right|>M_1(\epsilon)>0. \qquad (3.26)
$$

By (3.3), (3.16), (3.17) and (3.21)–(3.26), there is an $M_2(\epsilon)>0$ such that

$$
\begin{aligned}
|f''(x)|&\le M_2|\rho|^3\left[\int_0^1[|\lambda||\phi(\xi)|+|\psi(\xi)|+|\chi(\xi)|]\,d\xi+|\phi(0)|+|\phi(1)|\right],\\
|g(x)|&\le M_2|\rho|^3\left[\int_0^1[|\lambda||\phi(\xi)|+|\psi(\xi)|+|\chi(\xi)|]\,d\xi+|\phi(0)|+|\phi(1)|\right]+|\phi(x)|,\\
|h(x)|&\le M_2|\rho|^3\left[\int_0^1[|\lambda||\phi(\xi)|+|\psi(\xi)|+|\chi(\xi)|]\,d\xi+|\phi(0)|+|\phi(1)|\right]
\end{aligned}
$$

for all $\lambda=i\rho^2$ with $\rho\in\mathcal{S}$ lying outside all the circles centered at the zeros of $\det(\Delta(\rho))$ with radius $\epsilon>0$. Since $|l'(x)|\le\|l''\|_{L^2}+|l'(1)|$ and $|l(x)|\le\|l'\|_{L^2}+|l(1)|\le\|l''\|_{L^2}+|l(1)|+|l'(1)|$ for any $x\in[0,1]$ and $l\in H^2[0,1]$, it follows that $\forall(\phi,\psi,\chi)\in\mathcal{H}$

$$
\begin{cases}
|\lambda|^{-3}|f''(x)|\le\dfrac{M_2}{|\rho|}[\|\phi''\|_{L^2}+|\lambda|^{-1}\|\psi\|_{L^2}+|\lambda|^{-1}\|\chi\|_{L^2}+|\lambda|^{-1}\|\phi''\|_{L^2}+|\phi(1)|+|\phi'(1)|],\\
|\lambda|^{-3}|g(x)|\le\dfrac{M_2}{|\rho|}[\|\phi''\|_{L^2}+|\lambda|^{-1}\|\psi\|_{L^2}+|\lambda|^{-1}\|\chi\|_{L^2}+|\lambda|^{-1}\|\phi''\|_{L^2}+|\phi(1)|+|\phi'(1)|]\\
\qquad\qquad +|\lambda|^{-1}[\|\phi''\|_{L^2}+|\phi(1)|+|\phi'(1)|],\\
|\lambda|^{-3}|h(x)|\le\dfrac{M_2}{|\rho|}[\|\phi''\|_{L^2}+|\lambda|^{-1}\|\psi\|_{L^2}+|\lambda|^{-1}\|\chi\|_{L^2}+|\lambda|^{-1}\|\phi''\|_{L^2}+|\phi(1)|+|\phi'(1)|].
\end{cases}
$$

It is seen from the above that we can find constants $M_3(\epsilon)$, $K > 0$ independent of λ such that

$$\|(f, g, h)\| \le M_3(1 + |\lambda|^3)\|(\phi, \psi, \chi)\|$$

for all $|\lambda| = |\rho^2| > K > 1$ large enough with $\rho \in \mathcal{S}$ lying outside all circles centered at the zeros of $\det(\Delta(\rho))$ with radius $\epsilon > 0$. Moreover, there is an $M > M_3(\epsilon)$ such that for $|\lambda| \le K$, it has $\|(f, g, h)\| \le M\|(\phi, \psi, \chi)\|$. Therefore,

$$\|(f, g, h)\| \le M(1 + |\lambda|^3)\|(\phi, \psi, \chi)\|$$

for all $\lambda = i\rho^2$ with $\rho \in \mathcal{S}$ lying outside all circles centered at the zeros of $\det(\Delta(\rho))$ with radius $\epsilon > 0$.

This result can be extended to all the other ρ's by the exact same arguments of [33, p. 56–60]. □

Now, we are in a position to show the completeness of the root subspace of $\mathcal{A}$. Firstly, let us recall that a non-zero Y is called a generalized eigenvector of a linear operator A in a Hilbert space H, corresponding to an eigenvalue λ (with finite algebraic multiplicity) of A, if there is a positive integer n such that $(\lambda - A)^n Y = 0$. Let $\mathrm{Sp}(A)$ be the root subspace of a linear operator A, which is defined as the closed subspace spanned by all generalized eigenvectors of A.

Theorem 6 *Let $\mathcal{A}$ be defined by (2.5). Then both the root subspaces of $\mathcal{A}$ and $\mathcal{A}^*$ are complete in $\mathcal{H}$, that is,* $\mathrm{Sp}(\mathcal{A}^*) = \mathrm{Sp}(\mathcal{A}) = \mathcal{H}$.

Proof We only show the completeness of the root subspace of $\mathcal{A}$ since the proof for that of $\mathcal{A}^*$ is almost the same. It follows from Lemma 5 of [11, p. 2355] that the following orthogonal decomposition holds:

$$\mathcal{H} = \sigma_\infty(\mathcal{A}^*) \oplus \mathrm{Sp}(\mathcal{A})$$

where $\sigma_\infty(\mathcal{A}^*)$ consists of those $Y \in \mathcal{H}$ so that $R(\lambda, \mathcal{A}^*)Y$ is an analytic function of λ in the whole complex plane. Hence, $\mathrm{Sp}(\mathcal{A}) = \mathcal{H}$ if and only if $\sigma_\infty(\mathcal{A}^*) = \{0\}$. Now suppose that $Y \in \sigma_\infty(\mathcal{A}^*)$. Since $R(\lambda, \mathcal{A}^*)Y$ is an analytic function in λ, it is also analytic in ρ. By the maximum modulus principle (or the Phragmén-Lindelöf theorem) and the fact that $\|R(\lambda, \mathcal{A}^*)\| = \|R(\overline{\lambda}, \mathcal{A})\|$, it follows from Theorem 5 that

$$\|R(\lambda, \mathcal{A}^*)Y\| \le M(1 + |\lambda|^3)\|Y\|, \quad \forall\, \lambda \in \mathbb{C},$$

for some constant $M > 0$. By theorem 1 of [28, p.3], we conclude that $R(\lambda, \mathcal{A}^*)Y$ is a polynomial in λ of degree ≤ 3, i.e., $R(\lambda, \mathcal{A}^*)Y = Y_0 + \lambda Y_1 + \lambda^2 Y_2 + \lambda^3 Y_3$ for some $Y_0, Y_1, Y_2, Y_3 \in \mathcal{H}$. Thus, $Y = (\lambda - \mathcal{A}^*)(Y_0 + \lambda Y_1 + \lambda^2 Y_2 + \lambda^3 Y_3)$. Since $\mathcal{A}^*$ is a closed operator, Y_1 belongs to $D(\mathcal{A}^*)$ and so does Y_0. Therefore,

$$-\mathcal{A}^* Y_0 + \lambda(Y_0 - \mathcal{A}^* Y_1) + \lambda^2(Y_1 - \mathcal{A}^* Y_2) + \lambda^3(Y_2 - \mathcal{A}^* Y_3) + \lambda^4 Y_3 = Y, \quad \forall\, \lambda \in \mathbb{C}.$$

Comparing the coefficients of λ^4, λ^3, λ^2, λ and λ^0 in two sides of the above equation, we get $Y_3 = Y_2 = Y_1 = Y_0 = Y = 0$. □

4 Riesz basis property and exponential stability of the 1-d problem

In this section, we show the Riesz basis generation and exponential stability of the system (2.6). To this purpose, let us recall two lemmas.

Lemma 5 *An approximately normalized sequence $\{e_i\}_{i=1}^{\infty}$ and its approximately normalized bi-orthogonal sequence $\{e_i^*\}_{i=1}^{\infty}$ are Riesz bases for a Hilbert space H if and only if [49, p. 27]*

(a) both $\{e_i\}_{i=1}^{\infty}$ and $\{e_i^*\}_{i=1}^{\infty}$ are complete in H;
(b) both $\{e_i\}_{i=1}^{\infty}$ and $\{e_i^*\}_{i=1}^{\infty}$ are Bessel sequences in H, that is, for any $f \in H$, two sequences $\{\langle f, e_i\rangle\}_{i=1}^{\infty}$, $\{\langle f, e_i^*\rangle\}_{i=1}^{\infty}$ belong to ℓ^2.

Lemma 6 ([41, Lemma 3.2]) *Let $\{\mu_n\}$ be a sequence which has asymptotics*

$$\mu_n = \alpha(n + i\beta \ln n) + \mathcal{O}(1), \quad \alpha \neq 0, \ n = 1, 2, 3, \ldots, \tag{4.1}$$

where β is a real number. If μ_n satisfies $\sup_{n\geq 1} Re\mu_n < \infty$, then the sequence $\{e^{\mu_n x}\}_{n=1}^{\infty}$ is a Bessel sequence in $L^2(0, 1)$.

Lemma 7 *Let $\varphi_{1ns}(x)$ and $\varphi_{2nj}(x)$ $s = 1, 2, 3, 4$, $j = 1, 2, 3, 4, 5$ be given by (2.43) and (2.46), respectively. Then all $\{\varphi_{1ns}(x)\}_{n=1}^{\infty}$ and $\{\varphi_{2nj}(x)\}_{n=1}^{\infty}$, $s = 1, 2, 3, 4$, $j = 1, 2, 3, 4, 5$ are Bessel sequences in $L^2(0, 1)$.*

Proof By (2.43), if we take $\alpha = i\sqrt{i}\pi$, $\beta = 0$ in $\varphi_{1n1}(x)$, $\alpha = -\sqrt{i}\pi$, $\beta = 0$ in $\varphi_{1n2}(x)$, $\alpha = i\pi$, $\beta = 0$ in $\varphi_{1n3}(x)$, and $\alpha = -i\pi$, $\beta = 0$ in $\varphi_{1n4}(x)$, respectively, then it follows from Lemma 6 directly that $\{\varphi_{1ns}(x)\}_{n=1}^{\infty}$ is a Bessel sequence in $L^2(0, 1)$ for any $s = 1, 2, 3, 4$.

Similarly, by (2.46), if we take $\alpha = i\pi$, $\beta = 0$ in $\varphi_{2n1}(x)$, $\alpha = -i\pi$, $\beta = 0$ in $\varphi_{2n2}(x)$, $\alpha = -\pi$, $\beta = 0$ in $\varphi_{2n3}(x)$, $\varphi_{2n4}(x)$, and $\alpha = -\sqrt{i}\pi$, $\beta = 0$ in $\varphi_{2n5}(x)$, respectively, then it follows from Lemma 6 directly that $\varphi_{2nj}(x)$ is a Bessel sequence in $L^2(0, 1)$ for any $j = 1, 2, 3, 4, 5$. □

The following theorem establishes the Riesz basis property for system (2.6).

Theorem 7 *Let $\mathcal{A}$ be defined by (2.5). Then the generalized eigenfunctions of $\mathcal{A}$ form a Riesz basis for $\mathcal{H}$.*

Proof Let $\sigma(\mathcal{A}) = \{\lambda_{1n}, \lambda_{2n}, \overline{\lambda}_{2n}\}_{n=1}^{\infty}$ be the eigenvalues of $\mathcal{A}$. By Theorem 2 and Proposition 1, there exists an integer $N > 0$ such that all $\lambda_{1n}, \lambda_{2n}, \overline{\lambda}_{2n}$ with $n \geq N$, are algebraically simple. For $n \leq N$, if the algebraic multiplicity of each λ_{sn} is m_{sn}, $s = 1, 2$, we can find the highest order generalized eigenfunction $\Phi_{s,n,1}$ from

$$(\mathcal{A} - \lambda_{sn})^{m_{sn}} \Phi_{s,n,1} = 0 \ \text{ and } \ (\mathcal{A} - \lambda_{sn})^{m_{sn}-1} \Phi_{s,n,1} \neq 0, \quad s = 1, 2.$$

The other lower order linearly independent generalized eigenfunctions corresponding to λ_{sn} can be found through $\Phi_{s,n,j} = (\mathcal{A}-\lambda_{sn})^{j-1}\Phi_{s,n,1},\ j = 2, 3, \ldots, m_{sn}$. Assume $\Phi_{s,n}$ is an eigenfunction of $\mathcal{A}$ corresponding to λ_{sn} with $n \geq N$. Then

$$\left\{\left\{\{\Phi_{s,n,j}\}_{j=1}^{m_{sn}}\right\}_{n<N} \cup \left\{\Phi_{s,n}\right\}_{n\geq N}\right\}_{s=1}^{2} \bigcup \left\{\left\{\{\overline{\Phi_{2,n,j}}\}_{j=1}^{m_{2n}}\right\}_{n<N} \cup \left\{\overline{\Phi_{2,n}}\right\}_{n\geq N}\right\} \tag{4.2}$$

are all linearly independent generalized eigenfunctions of $\mathcal{A}$. On the other hand, we also have that

$$\left\{\left\{\{\Psi_{s,n,j}\}_{j=1}^{m_{sn}}\right\}_{n<N} \cup \left\{\Psi_{s,n}\right\}_{n\geq N}\right\}_{s=1}^{2} \bigcup \left\{\left\{\{\overline{\Psi_{2,n,j}}\}_{j=1}^{m_{2n}}\right\}_{n<N} \cup \left\{\overline{\Psi_{2,n}}\right\}_{n\geq N}\right\} \tag{4.3}$$

are all linearly independent generalized eigenfunctions of $\mathcal{A}^*$. Let

$$\begin{cases} \Phi_{1,n,j}^* = \frac{\Psi_{1,n,j}}{\overline{\langle \Phi_{1,n,j}, \Psi_{1,n,j}\rangle}}, & n < N,\ j = 1, 2, \ldots, m_{1n}, \\ \Phi_{1,n}^* = \frac{\Psi_{1,n}}{\overline{\langle \Phi_{1,n}, \Psi_{1,n}\rangle}}, & n \geq N \end{cases} \tag{4.4}$$

and

$$\begin{cases} \Phi_{2,n,j}^* = \frac{\overline{\Psi_{2,n,j}}}{\overline{\langle \Phi_{2,n,j}, \overline{\Psi_{2,n,j}}\rangle}}, & n < N,\ j = 1, 2, \ldots, m_{2n}, \\ \Phi_{2,n}^* = \frac{\overline{\Psi_{2,n}}}{\overline{\langle \Phi_{2,n}, \overline{\Psi_{2,n}}\rangle}}, & n \geq N. \end{cases} \tag{4.5}$$

Then

$$\left\{\left\{\{\Phi_{s,n,j}^*\}_{j=1}^{m_{sn}}\right\}_{n<N} \cup \left\{\Phi_{s,n}^*\right\}_{n\geq N}\right\}_{s=1}^{2} \bigcup \left\{\left\{\{\overline{\Phi_{2,n,j}^*}\}_{j=1}^{m_{2n}}\right\}_{n<N} \cup \left\{\overline{\Phi_{2,n}^*}\right\}_{n\geq N}\right\} \tag{4.6}$$

are all linearly independent generalized eigenfunctions of $\mathcal{A}^*$ and they are bi-orthogonal to the sequence given by (4.2). Actually, from Remark 2, it is easily seen that

$$\begin{cases} \langle \Phi_{i,n,p}, \Phi_{j,s,q}^*\rangle = \delta_{ij} \times \delta_{mn} \times \delta_{pq}, & \text{where } i, j = 1, 2,\ 1 \leq n, s < N,\ 1 \leq p \leq m_{in},\ 1 \leq q \leq m_{js}, \\ \langle \Phi_{i,n}, \Phi_{j,s}^*\rangle = \delta_{ij} \times \delta_{mn}, & \text{where } i, j = 1, 2,\ N \leq n, s < \infty, \\ \langle \Phi_{i,n,p}, \overline{\Phi_{2,s,q}^*}\rangle = 0, & \text{where } i = 1, 2,\ 1 \leq n, s < N,\ 1 \leq p \leq m_{in},\ 1 \leq q \leq m_{2s}, \\ \langle \Phi_{i,n}, \overline{\Phi_{2,s}^*}\rangle = 0, & \text{where } i = 1, 2,\ N \leq n, s < \infty, \end{cases} \tag{4.7}$$

where δ_{ij} is Kronecker delta:

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Hence, the set (4.6) is the bi-orthogonal sequence of (4.2). From Theorem 6, we know that all the sequences given by (4.2), (4.3) and (4.6) are complete in $\mathcal{H}$.

Since both numbers of $\{\{\Phi_{s,n,j}\}_{j=1}^{m_{sn}}\}_{n<N}$ and $\{\{\Phi^*_{s,n,j}\}_{j=1}^{m_{sn}}\}_{n<N}$ are finite, to prove the Riesz basis property of the system, it suffices to show that both $\{\Phi_{s,n}\}_{n\geq N,s=1,2}$ and $\{\Phi^*_{s,n}\}_{n\geq N,s=1,2}$ of $\mathcal{A}$ and $\mathcal{A}^*$, respectively, are Bessel sequences in $\mathcal{H}$. Moreover, it follows from (2.60), (4.4) and (4.5) that $\{\Phi^*_{s,n}\}_{n\geq N,s=1,2}$ is a Bessel sequence if and only if $\{\Psi_{s,n}\}_{n\geq N,s=1,2}$ is a Bessel sequence. So we only need to show that $\{\Phi_{s,n}\}_{n\geq N,s=1,2}$ and $\{\Phi^*_{s,n}\}_{n\geq N,s=1,2}$ are Bessel sequences in $\mathcal{H}$.

We may assume without loss of generality that $\Phi_{s,n} = (f_{sn}, \lambda_{sn} f_{sn}, h_{sn})$ and $\Psi_{s,n} = (f_{sn}, -\lambda_{sn} f_{sn}, h_{sn})$ given by (2.42), (2.45) and (2.55), (2.56), respectively, for all $s = 1, 2, n \geq N$. It then follows from Lemma 7 and the expansions of (2.42), (2.45) and (2.55), (2.56) that all of $\{f''_{sn}\}_{n=N}^{\infty}$, $\{\pm\lambda_{sn} f_{sn}\}_{n=N}^{\infty}$ and $\{h_{sn}\}_{n=N}^{\infty}$, $s = 1, 2$ are Bessel sequences in $L^2(0, 1)$. Hence both of $\{\Phi_{s,n}\}_{n\geq N,s=1,2}$ and $\{\Psi_{s,n}\}_{n\geq N,s=1,2}$ are also Bessel sequences in $\mathcal{H}$. Therefore, both of $\{\Phi_{s,n}\}_{n\geq N,s=1,2}$ and $\{\Phi^*_{s,n}\}_{n\geq N,s=1,2}$ are also Bessel sequences in $\mathcal{H}$. The desired result of the theorem then follows from Lemma 5. □

Theorem 8 *Let $\mathcal{A}$ be defined by (2.5). Then the spectrum-determined growth condition $\omega(\mathcal{A}) = s(\mathcal{A})$ holds true, where $\omega(\mathcal{A})$ is the growth bound of the C_0-semigroup $e^{\mathcal{A}t}$ and*

$$s(\mathcal{A}) := \sup\{\mathcal{R}e\lambda \mid \lambda \in \sigma(\mathcal{A})\}$$

is the spectral bound of $\mathcal{A}$. Moreover, the system (2.6) is exponentially stable, that is, there exist two positive constants M and ω such that the C_0-semigroup $\mathrm{e}^{\mathcal{A}t}$ generated by $\mathcal{A}$ satisfies

$$\|\mathrm{e}^{\mathcal{A}t}\| \leq M\mathrm{e}^{-\omega t}. \tag{4.8}$$

Proof The spectrum-determined growth condition follows directly from Theorem 7. By Lemma 1, $\mathcal{R}e\lambda < 0$ for any $\lambda \in \sigma(\mathcal{A})$. This together with (2.28)–(2.31) and the spectrum-determined growth condition shows that $e^{\mathcal{A}t}$ is exponentially stable. □

Remark 3 It should be pointed out that the exponential stability in Theorem 8 allows the special case of $k = 0$. Actually, from the spectral analysis of (2.29), feedback gain k only increases the damping for the "beam part". Furthermore, from (2.3), we can say in the case of $k = 0$ that the "heat part" plays the role of the controller in the whole system.

Remark 4 We remark that the optimal decay rate ω in (4.8) is related to the optimality of δ in Theorem 9 and Theorem 10 later. Analytically, it is hard to get the lower bound of δ, but the numerical method like the spectral method is always possible to apply to get the optimal δ [47].

To end this section, we show that C_0-semigroup $\mathrm{e}^{\mathcal{A}t}$ generated by $\mathcal{A}$ is of a Gevrey class δ with any $\delta > 2$, which is the semigroup class between the differentiable semigroup class and the analytical ones.

Definition 1 [4,42] A C_0-semigroup $T(t)$ in a Hilbert space H is of a Gevrey class $\delta > 1$ for $t > t_0$ if $T(t)$ is infinitely differentiable for $t > t_0$ and for every compact subset $K \subset (t_0, \infty)$ and each $\theta > 0$, there is a constant $C = C(K, \theta)$ such that

$$\|T^{(n)}(t)\| \leq C\theta^n (n!)^\delta, \ \forall t \in K, \ n = 0, 1, 2, \ldots.$$

It is seen that when $\delta = 1$, $T(t)$ is analytic.

To get the Gevrey regularity of the system (2.6), we need the following theorem established in [37, theorem 13].

Theorem 9 *Let A be an infinitesimal generator of a C_0-semigroup e^{At} and let A be a Riesz-spectral operator in a Hilbert space, that is, the generalized eigenfunctions of A form a Riesz basis in the Hilbert space. Then the following assertions are equivalent:*

1. *e^{At} is of Gevrey class $\delta \geq 1$ for $t > 0$.*
2. *There are $b > 0$ and $a \in \mathbb{R}$ such that*

$$\sigma(A) \subset \left\{\lambda \in \mathbb{C} : \mathcal{R}e\lambda \leq a - b|\text{Im}\lambda|^{1/\delta}\right\}. \tag{4.9}$$

Now, we are in a position to establish the Gevrey regularity for system (2.6).

Theorem 10 *Let $\mathcal{A}$ be defined by (2.5). Then the semigroup $e^{\mathcal{A}t}$, generated by $\mathcal{A}$, is of a Gevrey class $\delta > 2$ with $t_0 = 0$.*

Proof From (2.28), (2.29), (4.9) is satisfied if we take $\delta = 2$, $b = |\ln r|$ and $a > 0$ large enough. Therefore, the semigroup $e^{\mathcal{A}t}$, generated by $\mathcal{A}$, is of a Gevrey class $\delta > 2$ with $t_0 = 0$. □

5 C_0-semigroup for the 2-d problem

We have indicated in Remark 3 that for the 1-d problem, only the boundary connection conditions ($k = 0$) can stabilize exponentially the coupled system. This is an interesting phenomenon because in the derivative of the energy in (2.3), the term on "beam part" disappears, which proposes a challenging problem for the control of multi-dimensional plate under the same connections. This is because the connection for multi-dimensional problem is not as simple as one-dimensional one. It is relevant with the measure of the connected boundary where the two systems are connected. In this section, we try to generalize the stability result to the 2-d problem. Due to lack of information of the plate in the derivative of the energy, the exponential stability relies heavily on the dissipation on both subsystems.

Let Ω be a bounded open connected convex domain in $\mathbb{R}^2$, whose boundary is a curvilinear polygon of class $C^{1,1}$. Suppose $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, where Ω_i, $i = 1, 2$ are bounded convex domains with curvilinearly polygonal boundaries and satisfy $\Omega_1 \cap \Omega_2 = \emptyset$. We denote the interface between Ω_1 and Ω_2 by γ, and the exterior boundary of Ω_i by $\Gamma_i = \partial\Omega_i \backslash \overline{\gamma}$, $i = 1, 2$. Assume that Γ_i has positive measure,

$i = 1, 2$. The system that we are concerned with is described by the following 2-d PDEs:

$$\begin{cases} w_{tt}(x,t) + \Delta^2 w(x,t) = 0 & \text{in } \Omega_1 \times \mathbb{R}^+ \\ u_t(x,t) - \Delta u(x,t) = 0 & \text{in } \Omega_2 \times \mathbb{R}^+ \\ \mathcal{B}_1 w(x,t) = 0,\ \mathcal{B}_2 w(x,t) = \partial_{\nu_2} u(x,t),\ w_t(x,t) = u(x,t) & \text{on } \gamma \times \mathbb{R}^+, \\ \mathcal{B}_1 w(x,t) = -\partial_{\nu_1} w(x,t),\ \mathcal{B}_2 w(x,t) = w(x,t) + k w_t(x,t) & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ u(x,t) = 0 & \text{on } \Gamma_2 \times \mathbb{R}^+, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x) & \text{in } \Omega_1, \\ u(0) = u_0 & \text{in } \Omega_2, \end{cases} \tag{5.1}$$

where $\mathcal{B}_1, \mathcal{B}_2$ are boundary operators:

$$\begin{cases} \mathcal{B}_1 w = \Delta w + (1-\mu)\left(2\nu_{11}\nu_{12}\dfrac{\partial^2 w}{\partial x_1 \partial x_2} - \nu_{11}^2 \dfrac{\partial^2 w}{\partial x_2^2} - \nu_{12}^2 \dfrac{\partial^2 w}{\partial x_1^2}\right), \\ \mathcal{B}_2 w = \partial_{\nu_1}\Delta w + (1-\mu)\partial_{\tau_1}\left(\left(\nu_{11}^2 - \nu_{12}^2\right)\frac{\partial^2 w}{\partial x_1 \partial x_2} + \nu_{11}\nu_{12}\left(\frac{\partial^2 w}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_1^2}\right)\right), \end{cases} \tag{5.2}$$

$\nu_i = (\nu_{i1}, \nu_{i2})$ is the unit outer normal vector of $\partial\Omega_i$, $\tau_i = (-\nu_{i2}, \nu_{i1})$ is the unit tangent vector of $\partial\Omega_i$, $i = 1, 2$, $0 < \mu < \frac{1}{2}$ is the Poisson ratio, k is a non-negative constant. The energy of the system (5.1) is defined by

$$E_1(t) = \frac{1}{2}\left\{\int_{\Omega_1} |w_t(x,t)|^2 \mathrm{d}x + \int_{\Omega_2} |u(x,t)|^2 \mathrm{d}x + a(w(\cdot,t)) + \int_{\Gamma_1} \left[|\partial_{\nu_1} w(x,t)|^2 + |w(x,t)|^2\right] \mathrm{d}\Gamma\right\}, \tag{5.3}$$

where $a(w) = a(w, w)$, and

$$a(w_1, w_2) = \int_{\Omega_1} \left[\frac{\partial^2 w_1}{\partial x_1^2}\frac{\partial^2 w_2}{\partial x_1^2} + \frac{\partial^2 w_1}{\partial x_2^2}\frac{\partial^2 w_2}{\partial x_2^2} + \mu\left(\frac{\partial^2 w_1}{\partial x_1^2}\frac{\partial^2 w_2}{\partial x_2^2} + \frac{\partial^2 w_1}{\partial x_2^2}\frac{\partial^2 w_2}{\partial x_1^2}\right) + 2(1-\mu)\frac{\partial^2 w_1}{\partial x_1 \partial x_2}\frac{\partial^2 w_2}{\partial x_1 \partial x_2}\right] \mathrm{d}x, \quad \forall\, w_1, w_2 \in H^2(\Omega_1). \tag{5.4}$$

Formally, it is found from Lemma 9 later that

$$\frac{\mathrm{d}}{\mathrm{d}t} E_1(t) = -k \int_{\Gamma_1} |w_t|^2 \mathrm{d}\Gamma - \int_{\Omega_1} |\nabla u|^2 \mathrm{d}x \le 0, \tag{5.5}$$

which shows that system (5.1) is dissipative. In what follows, we always assume that $k > 0$ and leave the case of $k = 0$ as an open question.

We consider system (5.1) in the state Hilbert space $\mathcal{H}_1 = H^2(\Omega_1) \times L^2(\Omega_1) \times L^2(\Omega_2)$ with the inner product induced norm:

$$\|Z\|_{\mathcal{H}_1}^2 = a(w) + \|y\|_{L^2(\Omega_1)}^2 + \|u\|_{L^2(\Omega_2)}^2 + \|w\|_{L^2(\Gamma_1)}^2 + \|\partial_{\nu_1} w\|_{L^2(\Gamma_1)}^2, \quad \forall\, Z = (w, y, u) \in \mathcal{H}_1.$$

Remark 5 It is well-known that $a(w) + \|w\|_{L^2(\Gamma_1)}^2 + \|\partial_{\nu_1} w\|_{L^2(\Gamma_1)}^2$ is an equivalent norm of $H^2(\Omega_1)$ provided that meas$(\Gamma_1) \neq \emptyset$. We refer this point to [1].

The system operator $\mathcal{A}_1 : D(\mathcal{A}_1)(\subset \mathcal{H}_1) \to \mathcal{H}_1$ associated with (5.1) is given by

$$\begin{cases} \mathcal{A}_1 Z & = (y, -\Delta^2 w, \Delta u),\ Z = (w, y, u) \in D(\mathcal{A}_1) \\ D(\mathcal{A}_1) & = \{Z = (w, y, u) \in \mathcal{H}_1 |\ \Delta^2 w \in L^2(\Omega_1), y \in H^2(\Omega_1), u \in H^2(\Omega_2) \cap H^1_{\Gamma_2}(\Omega_2), \\ \mathcal{B}_1 w & = -\partial_{\nu_1} w, \mathcal{B}_2 w = w + ky \text{ on } \Gamma_1, u = 0 \text{ on } \Gamma_2, \\ \mathcal{B}_1 w & = 0, \mathcal{B}_2 w = \partial_{\nu_2} u, y = u \text{ on } \gamma\ \}, \end{cases} \tag{5.6}$$

where $H^1_{\Gamma_2}(\Omega_2) = \{u \in H^1(\Omega_2)|\ u = 0 \text{ on } \Gamma_2\}$. With the operator $\mathcal{A}_1$ at hand, system (5.1) is written as an abstract evolution equation in $\mathcal{H}_1$:

$$\dot{Z}(t) = \mathcal{A}_1 Z(t),\ Z(0) = Z_0 = (w_0, w_1, u_0) \in \mathcal{H}_1, \tag{5.7}$$

where $Z(t) = (w(\cdot, t), w_t(\cdot, t), u(\cdot, t))$.

Remark 6 It is known from [7] that $\|\nabla u\|_{L^2(\Omega_2)}$ is equivalent to the usual H^1 norm of $H^1_{\Gamma_2}(\Omega_2)$ if meas$(\Gamma_2) \neq \emptyset$.

Since for each $i\,(i = 1, 2)$, the boundary of Ω_i is a curvilinear polygon, to prove the well-posedness of system (5.1), we need some regularity results about the second-order elliptic and fourth-order elliptic problems on the non-smooth domain. It is well-known that the solutions of the elliptic boundary problems on the non-smooth domain can be divided into a regular part and a singular part (see, e.g., [6,9,13,34]). First, we present the following Lemma 8 to describe the regularity of the solution of the Laplacian equation with certain boundary conditions, which comes from [1] or [9, Theorem 23.3].

Lemma 8 *Suppose that the domain Ω_2 is a bounded open-connected convex domain in $\mathbb{R}^2$, whose boundary $\partial\Omega_2 = \gamma \cup \Gamma_2$ is a curvilinear polygon of class $C^{1,1}$. If $u \in H^1(\Omega_2)$ satisfies the following elliptic boundary problem:*

$$\begin{cases} \Delta u = f \in L^2(\Omega_2) & \text{in } \Omega_2, \\ u = 0 & \text{on } \Gamma_2, \\ u = g \in H^{3/2}(\gamma) & \text{on } \gamma, \end{cases} \tag{5.8}$$

that is,

$$\int_{\Omega_2} u \Delta\varphi \mathrm{d}x - \int_{\gamma} g \frac{\partial \varphi}{\partial \nu_2} \mathrm{d}x = \int_{\Omega_2} f \varphi dx, \quad \forall\, \varphi \in H^2(\Omega_2) \cap H^1_0(\Omega_2).$$

Then $u \in H^2(\Omega_2)$*, and*

$$\|u\|_{H^2(\Omega_2)} \leq C\left[\|f\|_{L^2(\Omega_2)} + \|g\|_{H^{3/2}(\gamma)}\right] \tag{5.9}$$

for some constant $C > 0$ *independent of* f, g.

Next, to analyze the regularity of the fourth-order elliptic boundary problem, we need additional assumption on Ω_1.

Assumption (G) Denote by ω_j $(j = 1, ..., N)$ the interior angles at the corners of Ω_1 between two consecutive curve. There exists an ω_0 such that $\omega_j < \omega_0$ for all $j = 1, .., N$, where the minimal angle ω_0 depends on Poisson ratio μ in the sense of [6, Theorem 2], for example, $\omega_0 \simeq 77.753311...^\circ$ when $\mu = 0.3$.

The Assumption (G) is also used in [1] but our connections in (5.1), different to those in [1], are proposed in the point of view for the 1-d problem discussed in previous sections.

Lemma 9 [1,6] *Suppose that the domain* Ω_1 *satisfies Assumption (G). If* $w \in H^2(\Omega_1)$ *satisfies*

$$\begin{cases} \Delta^2 w \in L^2(\Omega_1) \\ \mathcal{B}_1 w, \mathcal{B}_2 w \in L^2(\partial\Omega_1), \end{cases} \tag{5.10}$$

then the following Green's formula holds true:

$$\int_{\Omega_1} \Delta^2 w\, \tilde{w} \mathrm{d}x = a(w,\ \tilde{w}) + \int_{\partial\Omega_1} B_2 w\, \tilde{w} \mathrm{d}\Gamma - \int_{\partial\Omega_1} B_1 w\, \partial_{\nu_1} \tilde{w} \mathrm{d}\Gamma,\ \forall\, \tilde{w} \in H^2(\Omega_1). \tag{5.11}$$

Theorem 11 *Suppose that the domain* Ω_1 *satisfies Assumption (G) and* Γ_i *has positive measure,* $i = 1, 2$*. Then* $\mathcal{A}_1$ *generates a* C_0*-semigroup of contractions on* $\mathcal{H}_1$*, and* $0 \in \rho(\mathcal{A}_1)$*, the resolvent of* $\mathcal{A}_1$.

Proof For any $Z = (w, y, u) \in D(\mathcal{A}_1)$,

$$\begin{aligned} \mathcal{R}e\langle \mathcal{A}_1 Z, Z\rangle_{\mathcal{H}_1} &= \mathcal{R}e\left[\langle y, w\rangle_{H^2(\Omega_1)} - \langle \Delta^2 w, y\rangle_{L^2(\Omega_1)} + \langle \Delta u, u\rangle_{L^2(\Omega_2)}\right] \\ &= \mathcal{R}e\left[a(y, w) + \int_{\Gamma_1} \left(\partial_{\nu_1} y \partial_{\nu_1} w + yw\right)\mathrm{d}\Gamma - a(w, y)\right. \\ &\quad \left. - \int_{\partial\Omega_1} (\mathcal{B}_2 w y - \mathcal{B}_1 w \partial_{\nu_1} y)\mathrm{d}\Gamma - \int_{\Omega} |\nabla u|^2 \mathrm{d}x + \int_{\partial\Omega_2} \partial_{\nu_2} u u \mathrm{d}\Gamma\right] \end{aligned} \tag{5.12}$$

$$= \mathcal{R}e\left[-k\int_{\Gamma_1}|y|^2\mathrm{d}\Gamma - \int_{\gamma}\mathcal{B}_2 w y\mathrm{d}\Gamma - \int_{\Omega_2}|\nabla u|^2\mathrm{d}x + \int_{\gamma}\partial_{\nu_2}u u\mathrm{d}\Gamma\right]$$

$$= -k\int_{\Gamma_1}|y|^2\mathrm{d}\Gamma - \int_{\Omega_2}|\nabla u|^2\mathrm{d}x \le 0,$$

which shows that $\mathcal{A}_1$ is dissipative. Now solve $\mathcal{A}_1 Z = (f_1, f_2, f_3) \in \mathcal{H}_1$ to obtain $y = f_1 \in H^2(\Omega_1)$, and

$$\begin{cases} -\Delta^2 w = f_2, \\ \mathcal{B}_1 w|_{\Gamma_1} = -\partial_{\nu_1} w|_{\Gamma_1}, \mathcal{B}_2 w|_{\Gamma_1} = (w + kf_1)|_{\Gamma_1}, \mathcal{B}_1 w|_{\gamma} = 0, \mathcal{B}_2 w|_{\gamma} = \partial_{\nu_2} u|_{\gamma}, \\ \Delta u = f_3, \\ u|_{\Gamma_2} = 0,\ u|_{\gamma} = f_1|_{\gamma} \in H^{3/2}(\gamma). \end{cases} \tag{5.13}$$

Since $f_3 \in L^2(\Omega_2)$, it follows from Lemma 8 that there exists a unique solution $u \in H^2(\Omega_2)$ to the Laplacian equation of u in (5.13). For any $\varphi \in H^2(\Omega_1)$, multiply the first equation in (5.13) by φ and apply Green's formula (5.11) to get

$$a(w,\varphi) + \int_{\Gamma_1}\left(\partial_{\nu_1} w \partial_{\nu_1}\varphi + w\varphi\right)\mathrm{d}\Gamma = -\int_{\Omega_1} f_2\varphi dx - k\int_{\Gamma_1} f_1\varphi\ \mathrm{d}\Gamma - \int_{\gamma}\partial_{\nu_2} u\varphi\ \mathrm{d}\Gamma. \tag{5.14}$$

By the Lax–Millgram Theorem, there exists a unique solution $w \in H^2(\Omega_1)$ to (5.14), and w satisfies (5.13) in terms of [35, Theorem 5.2]. So for any $(f_1, f_2, f_3) \in \mathcal{H}_1$, there exist a unique $(w, y, u) \in D(\mathcal{A}_1)$ such that $\mathcal{A}_1(w, y, u) = (f_1, f_2, f_3)$. By virtue of the closed graph theorem, the inverse $\mathcal{A}_1^{-1}$ is bounded, and hence $0 \in \rho(\mathcal{A}_1)$. So $\mathcal{A}$ is dissipative and generates a C_0-semigroup of contractions on $\mathcal{H}_1$ by the Lumer–Philips theorem ([39, p. 14]). The proof is complete. □

6 Exponential stability of the 2-d problem

As we indicate that the exponential stability for multi-dimensional system requires the measure of control boundary and connection boundary. The following assumption is about the boundary condition where γ is assumed to be a segment which means that Ω_i has a curvilinearly polygonal boundary, $i = 1, 2$ (see [3,19] or recently [1, (1.19)–(1.20)]).

Assumption (H) $m(x) \cdot \nu_1(x) \ge \delta > 0$ for all $x \in \Gamma_1$ and $m(x) \cdot \nu_1(x) = 0$ for all $x \in \gamma$, where $m = x - x_0$, x_0 is a fixed point of $\mathbb{R}^2$. In addition, we suppose δ satisfies $\delta < \frac{4C_{\Omega_1}}{1-\mu}$ where $C_\Omega = \sup\{|m(x)||\ x \in \Gamma_1\}$.

Remark 7 In this section, we use the frequency method and multipliers technique to get the exponential stability based on the geometric conditions (G) and (H). Figure 2

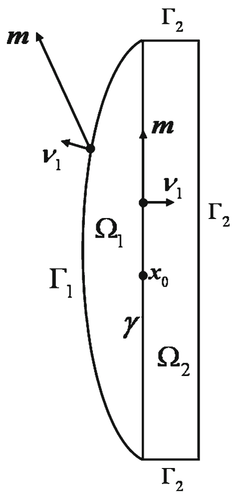

Fig. 2 Relations of Ω_1, Ω_2 and the connected interface γ

gives an example of the domain satisfying (G) and (H). The question whether the geometric conditions for the control or stabilization of wave/plate systems is sharp or not remains open (see [3] or [26]).

Lemma 10 [19,40] *Suppose that the domain Ω_1 satisfies the Assumptions (G) and (H). If $w \in H^2(\Omega_1)$ satisfies the condition for Green's formula (5.11) that $\Delta^2 w \in L^2(\Omega_1)$, $\mathcal{B}_1 w, \mathcal{B}_2 w \in L^2(\partial\Omega_1)$, then the following equality holds*

$$\begin{aligned} \int\limits_{\Omega_1} \Delta^2 w (m \cdot \nabla w) \mathrm{d}x &= a(w) + \int\limits_{\partial\Omega_1} \left[\mathcal{B}_2 w (m \cdot \nabla w) - \mathcal{B}_1 w \partial_{\nu_1} (m \cdot \nabla w)\right] \mathrm{d}\Gamma \\ &\quad + \frac{1}{2} \int\limits_{\partial\Omega_1} (m \cdot \nu_1) b(w) \mathrm{d}\Gamma, \end{aligned} \tag{6.1}$$

where

$$b(w) = \left|\frac{\partial^2 w}{\partial x_1^2}\right|^2 + \left|\frac{\partial^2 w}{\partial x_2^2}\right|^2 + 2\mu \frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} + 2(1-\mu)\left|\frac{\partial^2 w}{\partial x_1 \partial x_2}\right|^2.$$

Now, we are in a position to state the main result of this section.

Theorem 12 *Suppose that Ω_1 satisfies Assumptions (G) and (H), Γ_i has positive measure, $i = 1, 2$. Then the energy of system (5.1) decays exponentially, i.e., there exist positive constants M and ω such that*

$$E_1(t) \leq M \mathrm{e}^{-\omega t} E_1(0), \quad \forall\, t \geq 0,$$

where E_1 is defined by (5.3).

Proof Since $\mathcal{A}_1$ is dissipative by Huang–Prüss' theorem ([17,38] or [32, p.139]), it suffices to show that there exists a $r > 0$ such that

$$\inf_{\lambda\in\mathbb{R}} \left\{\|\mathrm{i}\lambda Z - \mathcal{A}_1 Z\|_{\mathcal{H}_1} \mid Z \in D(\mathcal{A}_1),\ \|Z\|_{\mathcal{H}_1} = 1\right\} \geq r. \tag{6.2}$$

We assume (6.2) is not true and obtain a contradiction. Actually, if (6.2) is false, then there exist a sequence of real numbers λ_n satisfying $|\lambda_n| > [2\|\mathcal{A}_1^{-1}\|]^{-1}$ and a sequence of functions $Z_n = (w_n, y_n, u_n) \in D(\mathcal{A}_1)$ with $\|Z_n\|_{\mathcal{H}_1} = 1$ such that

$$\|(\mathrm{i}\lambda_n I - \mathcal{A}_1)Z_n\|_{\mathcal{H}_1} \to 0; \tag{6.3}$$

here we may suppose without loss of generality that $\lambda_n > 0$. It then follows from (6.3), as $n \to \infty$, that

$$\begin{aligned} &\mathrm{i}\lambda_n w_n - y_n = f_{1,n} \to 0 && \text{in } H^2(\Omega_1),\\ &\mathrm{i}\lambda_n y_n + \Delta^2 w_n = f_{2,n} \to 0 && \text{in } L^2(\Omega_1),\\ &\mathrm{i}\lambda_n u_n - \Delta u_n = f_{3,n} \to 0 && \text{in } L^2(\Omega_2). \end{aligned} \tag{6.4}$$

Furthermore, by (6.3) and the fact that $\mathcal{R}e\langle\mathcal{A}_1 Z, Z\rangle_{\mathcal{H}_1} = -k\|y\|^2_{L^2(\Gamma_1)} - \|\nabla u\|^2_{L^2(\Omega_2)}$, we have

$$\|\nabla u_n\|_{L^2(\Omega_2)} \to 0, \tag{6.5}$$

and

$$\|y_n\|_{L^2(\Gamma_1)} \to 0. \tag{6.6}$$

Since $u_n|_{\Gamma_2} = 0$, it follows from (6.5) and Remark 6 that

$$\|u_n\|_{L^2(\Omega_2)} \to 0. \tag{6.7}$$

The proof will be split into two steps.

Step 1. From the second equality of (6.4) and the assumption $\|Z_n\| = 1$, we have

$$\langle \mathrm{i}\lambda_n y_n, m \cdot \nabla w_n\rangle_{L^2(\Omega_1)} + \left\langle \Delta^2 w_n, m \cdot \nabla w_n\right\rangle_{L^2(\Omega_1)} \to 0. \tag{6.8}$$

By the first equality of (6.4),

$$\langle y_n, \mathrm{i}\lambda_n m \cdot \nabla w_n\rangle_{L^2(\Omega_1)} - \langle y_n, m \cdot \nabla y_n\rangle_{L^2(\Omega_1)} \to 0. \tag{6.9}$$

Combining (6.8) with (6.9) yields

$$-\langle y_n, m \cdot \nabla y_n\rangle_{L^2(\Omega_1)} + \left\langle \Delta^2 w_n, m \cdot \nabla w_n\right\rangle_{L^2(\Omega_1)} \to 0. \tag{6.10}$$

Owing to (6.1) and the fact $(m \cdot \nu_1)|_\gamma = 0$, we have

$$\mathcal{R}e \left\{ a(w_n) + \|y_n\|^2_{L^2(\Omega_1)} - \frac{1}{2} \int_{\Gamma_1} (m \cdot \nu_1)|y_n|^2 \mathrm{d}\Gamma + \int_{\Gamma_1} [(w_n + k y_n)(m \cdot \nabla w_n) \right.$$
$$\left. + \partial_{\nu_1} w_n \partial_{\nu_1}(m \cdot \nabla w_n)\big] \mathrm{d}\Gamma + \int_\gamma \mathcal{B}_2 w_n (m \cdot \nabla w_n) \mathrm{d}\Gamma + \frac{1}{2} \int_{\Gamma_1} (m \cdot \nu_1) b(w_n) \mathrm{d}\Gamma \right\} \to 0. \tag{6.11}$$

Since

$$\|w_n\|_{L^2(\Gamma_1)} \to 0 \tag{6.12}$$

that comes from the first equality of (6.4) and (6.6), and

$$b(w_n) \geq (1-\mu) \left(\left| \frac{\partial^2 w_n}{\partial x_1^2} \right|^2 + \left| \frac{\partial^2 w_n}{\partial x_2^2} \right|^2 + 2 \left| \frac{\partial^2 w_n}{\partial x_1 \partial x_2} \right|^2 \right),$$

it follows from (6.6), (6.11) and (6.12) that

$$a(w_n) + \|y_n\|^2_{L^2(\Omega_1)} + \mathcal{R}e \int_{\Gamma_1} \partial_{\nu_1} w_n \partial_{\nu_1}(m \cdot \nabla w_n) \mathrm{d}\Gamma + \mathcal{R}e \int_\gamma \mathcal{B}_2 w_n (m \cdot \nabla w_n) \mathrm{d}\Gamma$$
$$+ \frac{1-\mu}{2} \int_{\Gamma_1} (m \cdot \nu_1) \left(\left| \frac{\partial^2 w_n}{\partial x_1^2} \right|^2 + \left| \frac{\partial^2 w_n}{\partial x_2^2} \right|^2 + 2 \left| \frac{\partial^2 w_n}{\partial x_1 \partial x_2} \right|^2 \right) \mathrm{d}\Gamma \to 0. \tag{6.13}$$

By a direct computation, we have

$$\left|\partial_{\nu_1} w_n \partial_{\nu_1}(m \cdot \nabla w_n)\right| \geq \frac{1}{2} \left|\partial_{\nu_1} w_n\right|^2 - 2C_\Omega \left(\left| \frac{\partial^2 w_n}{\partial x_1^2} \right|^2 + \left| \frac{\partial^2 w_n}{\partial x_2^2} \right|^2 + 2 \left| \frac{\partial^2 w_n}{\partial x_1 \partial x_2} \right|^2 \right).$$

Therefore, it follows from (6.13) and Assumption (H) that

$$a(w_n) + \|y_n\|^2_{L^2(\Omega_1)} + \frac{1}{2} \|\partial_{\nu_1} w_n\|^2_{L^2(\Gamma_1)} + \mathcal{R}e \int_\gamma \mathcal{B}_2 w (m \cdot \nabla w) \mathrm{d}\Gamma \to 0. \tag{6.14}$$

Step 2. From the first identity of (6.4) and the boundary conditions on the interface, we have

$$\int_\gamma \mathcal{B}_2 w_n (m \cdot \nabla w_n) \mathrm{d}\Gamma = \mathrm{i}\lambda_n^{-1} \int_\gamma \partial_{\nu_2} u_n (m \cdot \nabla(u_n + f_{1,n})) \mathrm{d}\Gamma. \tag{6.15}$$

Apply Lemma 8 and the trace theorem to the variable $\lambda_n^{-1} u_n$ to get

$$\begin{aligned}\|\lambda_n^{-1} u_n\|_{H^2(\Omega_2)} \le C[\|\mathrm{i}u_n - \lambda_n^{-1} f_{3,n}\|_{L^2(\Omega_2)} \\ + \|\mathrm{i}w_n - \lambda_n^{-1} f_{1,n}\|_{H^2(\Omega_1)}] \quad \text{for some } C > 0. \end{aligned} \tag{6.16}$$

In what follows, we use C to denote the constant that is independent of n although they may vary in different context. Since $\lambda_n > [2\|\mathcal{A}^{-}\|]^{-1}$, by (6.7) and the first and the third identities of (6.4), we get from (6.16) that

$$\lim_{n\to\infty} \|\lambda_n^{-1} u_n\|_{H^2(\Omega_2)} \le C \lim_{n\to\infty} \|w_n\|_{H^2(\Omega_1)} \le C, \quad C > 0. \tag{6.17}$$

This together with (6.5) gives

$$\lambda_n^{-1} \|u_n\|^2_{H^1(\gamma)} \le \|\lambda_n^{-1} u_n\|_{H^2(\Omega_2)} \|u_n\|_{H^1(\Omega_2)} \to 0. \tag{6.18}$$

Combining (6.18) and (6.15) gives

$$\lim_{n\to\infty} \int_\gamma \mathcal{B}_2 w_n (m \cdot \nabla w_n) \mathrm{d}\Gamma = 0. \tag{6.19}$$

This together with (6.14) leads to

$$a(w_n) + \|y_n\|^2_{L^2(\Omega_1)} + \frac{1}{2} \|\partial_{\nu_1} w_n\|^2_{L^2(\Gamma_1)} \to 0. \tag{6.20}$$

(6.20), (6.7) and (6.12) show that $\|Z_n\|_{\mathcal{H}_1} \to 0$. This contradicts the assumption that $\|Z_n\|_{\mathcal{H}_1} = 1$. The proof is thus complete.

Remark 8 We should point out that even for the 1-d problem, the multiplier method also cannot achieve the exponential stability in the case of $k = 0$. This is because, as we say in the proof of Theorem 12, that in applying the multiplier method, we need the dissipative term after taking the derivative of the energy function. How to relax this requirement in the multiplier methods proposes a technical problem in mathematics as well.

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