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SPECTRAL ANALYSIS AND STABILIZATION OF A COUPLED WAVE-ODE SYSTEM*

ZHAO Dongxia · WANG Junmin

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Abstract In this paper, an interconnected wave-ODE system with K-V damping in the wave equation and unknown parameters in the ODE is considered. It is found that the spectrum of the system operator is composed of two parts: Point spectrum and continuous spectrum. The continuous spectrum consists of an isolated point $-\frac{1}{d}$, and there are two branches of the asymptotic eigenvalues: The first branch is accumulating towards $-\frac{1}{d}$, and the other branch tends to $-\infty$. It is shown that there is a sequence of generalized eigenfunctions, which forms a Riesz basis for the Hilbert state space. As a consequence, the spectrum-determined growth condition and exponential stability of the system are concluded.

Keywords Exponential stability, Kelvin-Voigt damping, Riesz basis, spectrum, wave equation.

1 Introduction

The coupled system, which has rich physical backgrounds such as electromagnetic coupling, mechanical coupling, and coupled chemical reactions, appeared in many practical control problems. For examples, a wave-ODE coupled system can describe the vertical displacement of the string and the lower rigid body for the model of two rigid bodies connected by a spring and hanging from an elastic string[1], and a heat-ODE system can model the solid-gas interaction of heat diffusion and chemical reaction[2]. There are many tools and methods for designing the feedback controller and proving the exponential stability of the PDE-ODE or PDE-PDE coupled system, such as backstepping method, Lyapunov function method, theory of semigroup, spectral analysis method, and Riesz basis method[3−6]. Many excellent results on controllability, observability, and stability of a coupled system have been achieved[7−10]. An

ZHAO Dongxia
Department of Mathematics, North University of China, Taiyuan 030051*, China; School of Mathematics, Beijing Institute of Technology, Beijing* 100081*, China.* Email: zhaodongxia6@sina.com.
WANG Junmin
School of Mathematics, Beijing Institute of Technology, Beijing 100081*, China.*
Email: jmwang@bit.edu.cn.
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interconnected system of Schrödinger and heat equations with boundary coupling is considered in [11], the asymptotic expressions of both eigenvalues and eigenfunctions are presented, and the exponential stability of the system are proved by spectral analysis method and Riesz basis property. In [12], a wave equation is applied to control an unstable linear ODE system, and the exponential stability is established by backstepping transformation and Lyapunov function method.

In this paper, we consider a coupled system of an ODE with unknown coefficients (although constants) and a wave equation with Kelvin-Voigt damping (see Figure 1), where the velocity of the wave equation enters the ODE through the variable $v_t(1,t)$, while the output $\dot{y}(t)$ is fed into the boundary of the wave equation.

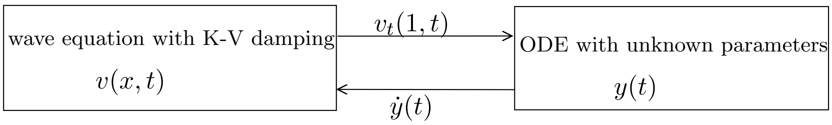

Figure 1 K-V damped wave PDE compensator for ODE with unknown parameters

To motivate our studies here, let us briefly review some ODE systems with a time delay. The pendulum system under position and delayed position feedbacks

$$\ddot{y}(t) + ky(t) = by(t-\tau) \tag{1}$$

is considered in [13, 14]. Since the time delay itself is a dynamical system, by a new variable $v(x,t) = y(t+x-\tau)$, the system (1) can be written as a coupled system of an ODE and a transport PDE:

$$\begin{cases} \ddot{y}(t) + ky(t) = bv(0,t), & t > 0,\ k, b \in \mathbb{R}, \\ v_t(x,t) = v_x(x,t), & x \in (0,\tau), t \geq 0, \\ v(\tau,t) = y(t). \end{cases} \tag{2}$$

The system (2) is asymptotically stable (see [14, 15]) if and only if k, b satisfy

$$\min\left\{k - \frac{j^2\pi^2}{\tau^2},\ \frac{(j+1)^2\pi^2}{\tau^2} - k\right\} > (-1)^j b > 0.$$

Furthermore, the exponential stability and spectral analysis of System (2) are obtained in [13]. On the other hand, the stabilization of PDE-ODE cascades has been extensively studied by Krstic and his co-authors[16], where the forms of the PDE subsystem include a transport PDE, a heat/wave equation, or even a diffusion-counter-convection PDE. So, the configuration in this paper arises as an extension of System (2), where we replace the transport PDE in (2) by a wave equation with Kelvin-Voigt damping:

$$\begin{cases} \ddot{y}(t) + ky(t) = bv_t(1,t), & t > 0, \\ v_{tt} = v_{xx} + dv_{xxt}, & x \in (0,1), t \geq 0, \\ v(0,t) = 0, \\ v_x(1,t) + dv_{tx}(1,t) = -b\dot{y}(t), \end{cases} \tag{3}$$

where $v(x,t)$ is the state of the PDE dynamics governed by a wave equation with Kelvin-Voigt damping, $t \geq 0$ is time, $x \in (0,1)$ is the spatial variable, $d > 0$ is a positive constant, and $k > 0, b \neq 0$ are two unknown constants. In System (3), wave PDE with K-V damping is considered as a dynamic controller to stabilize an ODE system with unknown constants. It will be shown (in Section 4) that no matter how much $k > 0$ and $b \neq 0$, System (3) will be exponentially stable under a K-V damped wave PDE compensator.

The energy function of (3) is given by

$$E(t) = \frac{1}{2}\left(ky^2(t) + \dot{y}^2(t) + \int_0^1 (v_t^2 + v_x^2)dx\right), \tag{4}$$

then we have

$$\begin{aligned} \dot{E}(t) &= ky\dot{y} + \dot{y}\ddot{y} + \int_0^1 (v_t v_{tt} + v_x v_{xt})dx \\ &= ky\dot{y} + \dot{y}\big(-ky + bv_t(1,t)\big) + \int_0^1 \big(v_t(v_{xx} + dv_{xxt}) + v_x v_{xt}\big)dx \\ &= b\dot{y}v_t(1,t) + v_t v_x\big|_0^1 - \int_0^1 v_x v_{tx}dx + dv_t v_{tx}\big|_0^1 - d\int_0^1 v_{tx}^2 dx + \int_0^1 v_x v_{xt} dx \\ &= b\dot{y}v_t(1,t) + v_t(1,t)v_x(1,t) + dv_t(1,t)v_{tx}(1,t) - d\int_0^1 v_{tx}^2 dx \\ &= v_t(1,t)\big(v_x(1,t) + dv_{tx}(1,t) + b\dot{y}\big) - d\int_0^1 v_{tx}^2 dx \\ &= -d\int_0^1 v_{tx}^2 dx < 0, \end{aligned}$$

and $E(t)$ is non-increasing, where $\dot{y}$ and $\ddot{y}$ denote the first and second derivative, respectively.

The paper is organized as follows. In Section 2, we formulate the system (3) into an abstract evolution equation and prove the well-posedness of the system. Section 3 is devoted to the spectrum analysis. In Section 4, the asymptotic expressions of eigenvalues and eigenfunctions are presented, and the exponential stability of (3) is established.

2 Well-Posedness of System (3)

We consider System (3) in the energy space

$$\mathcal{H} = \mathbb{C} \times \mathbb{C} \times H_L^1(0,1) \times L^2(0,1) \tag{5}$$

equipped with the usual inner product:

$$\langle Z_1, Z_2 \rangle = k\langle f_1, f_2\rangle_{\mathbb{C}} + \langle g_1, g_2\rangle_{\mathbb{C}} + \int_0^1 h_1'(x)\overline{h_2'(x)}dx + \int_0^1 q_1(x)\overline{q_2(x)}dx, \tag{6}$$

where $H_L^1(0,1)=\left\{h\in H^1(0,1)|h(0)=0\right\}, Z_1=(f_1,g_1,h_1,q_1), Z_2=(f_2,g_2,h_2,q_2)\in\mathcal{H}$, and $\langle f_1,f_2\rangle_{\mathbb{C}}=f_1\overline{f_2}$.

Define a linear operator $\mathcal{A}:D(\mathcal{A})(\subseteq\mathcal{H})\to\mathcal{H}$ by:

$$\mathcal{A}\left(f,g,h,q\right)=\left(g,-kf+bq(1),q,(h+dq)''\right), \tag{7}$$

with

$$D(\mathcal{A})=\left\{(f,g,h,q)\in\mathcal{H}|\mathcal{A}(f,g,h,q)\in\mathcal{H},h'(1)+dq'(1)+bg=0\right\}. \tag{8}$$

Then (3) can be written as an evolution equation in $\mathcal{H}$:

$$\begin{cases}\dot{Z}(t)=\mathcal{A}Z(t), \quad t>0,\\ Z(0)=Z_0,\end{cases} \tag{9}$$

where $Z(t)=(y(t),\dot{y}(t),v(\cdot,t),v_t(\cdot,t))$.

Now we have the following results on the properties of $\mathcal{A}$.

Lemma 2.1 *Let $\mathcal{A}$ be given by* (7) *and* (8). *Then its adjoint operator $\mathcal{A}^*$ has the following form:*

$$\mathcal{A}^*(\widetilde{f},\widetilde{g},\widetilde{h},\widetilde{q})=(-\widetilde{g},k\widetilde{f}-b\widetilde{q}(1),-\widetilde{q},-(\widetilde{h}-d\widetilde{q})''), \tag{10}$$

where

$$D(\mathcal{A}^*)=\left\{(\widetilde{f},\widetilde{g},\widetilde{h},\widetilde{q})\in\mathcal{H}|\mathcal{A}^*(\widetilde{f},\widetilde{g},\widetilde{h},\widetilde{q})\in\mathcal{H},\widetilde{h}'(1)-d\widetilde{q}'(1)+b\widetilde{g}=0\right\}. \tag{11}$$

Proof For each $F=(f,g,h,q)\in D(\mathcal{A}), G=(\widetilde{f},\widetilde{g},\widetilde{h},\widetilde{q})\in D(\mathcal{A}^*)$, a direct computation yields

$$\begin{aligned}\langle\mathcal{A}F,G\rangle&=\langle\big(g,-kf+bq(1),q,(h+dq)''\big),(\widetilde{f},\widetilde{g},\widetilde{h},\widetilde{q})\rangle\\&=kg\overline{\widetilde{f}}+(-kf+bq(1))\overline{\widetilde{g}}+\int_0^1 q'(x)\overline{\widetilde{h}'(x)}dx+\int_0^1(h+dq)''(x)\overline{\widetilde{q}(x)}dx\\&=kg\overline{\widetilde{f}}-kf\overline{\widetilde{g}}+bq(1)\overline{\widetilde{g}}+q(1)\overline{\widetilde{h}'(1)}-q(0)\overline{\widetilde{h}'(0)}-\int_0^1 q(x)\widetilde{h}''(x)dx+h'(1)\overline{\widetilde{q}(1)}\\&\quad-h'(0)\overline{\widetilde{q}(0)}-\int_0^1 h'(x)\overline{\widetilde{q}'(x)}dx+d(q'(1)\overline{\widetilde{q}(1)}-q'(0)\overline{\widetilde{q}(0)})\\&\quad-d(q(1)\overline{\widetilde{q}'(1)}-q(0)\overline{\widetilde{q}'(0)})+d\int_0^1 q(x)\overline{\widetilde{q}''(x)}dx\\&=kf\cdot(-\overline{\widetilde{g}})+g\cdot k\overline{\widetilde{f}}-\int_0^1 h'(x)\overline{\widetilde{q}'(x)}dx-\int_0^1 q(x)\overline{(\widetilde{h}-d\widetilde{q})''(x)}dx\\&\quad+q(1)(\overline{\widetilde{h}'(1)}-d\overline{\widetilde{q}'(1)}+b\overline{\widetilde{g}})-bg\overline{\widetilde{q}(1)}\\&=\langle F,\mathcal{A}^*G\rangle,\end{aligned} \tag{12}$$

and hence we obtain $\mathcal{A}^*$ that is given by (10) and (11). ∎

Lemma 2.2 *Let $\mathcal{A}$ be given by* (7) *and* (8), *$\mathcal{A}^*$ be given by* (10) *and* (11). *Then $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative in $\mathcal{H}$ and $\mathcal{A}$ generates a C_0-semigroup $\mathrm{e}^{\mathcal{A}t}$ of contractions on $\mathcal{H}$.*

Proof For each $Z = (f, g, h, q) \in D(\mathcal{A})$, we have

$$\begin{aligned}
\langle \mathcal{A}Z, Z\rangle &= \langle (g, -kf + bq(1), q, (h + dq)''), (f, g, h, q)\rangle \\
&= k\langle g, f\rangle_{\mathbb{C}} + \langle -kf + bq(1), g\rangle_{\mathbb{C}} + \int_0^1 q'(x)\overline{h'(x)}dx + \int_0^1 (h''(x) + dq''(x))\overline{q(x)}dx \\
&= k\langle g, f\rangle_{\mathbb{C}} - k\langle f, g\rangle_{\mathbb{C}} + b\langle q(1), g\rangle_{\mathbb{C}} + \int_0^1 q'(x)\overline{h'(x)}dx + \overline{q(x)}h'(x)\Big|_0^1 \\
&\quad - \int_0^1 h'(x)\overline{q'(x)}dx + d\overline{q(x)}q'(x)|_0^1 - d\int_0^1 |q'(x)|^2dx \\
&= kg\overline{f} - kf\overline{g} + bq(1)\overline{g} + \int_0^1 q'(x)\overline{h'(x)}dx - \int_0^1 h'(x)\overline{q'(x)}dx + \overline{q(1)}h'(1) \\
&\quad + d\overline{q(1)}q'(1) - d\int_0^1 |q'(x)|^2dx \\
&= kg\overline{f} - kf\overline{g} + bq(1)\overline{g} - b\overline{q(1)}g + \int_0^1 q'(x)\overline{h'(x)}dx - \int_0^1 h'(x)\overline{q'(x)}dx \\
&\quad - d\int_0^1 |q'(x)|^2dx,
\end{aligned}$$

so

$$\mathrm{Re}\langle \mathcal{A}Z, Z\rangle = -d\int_0^1 |q'(x)|^2dx \le 0. \tag{13}$$

Similarly, for any $\widetilde{Z} = (\widetilde{f}, \widetilde{g}, \widetilde{h}, \widetilde{q}) \in D(\mathcal{A}^*)$,

$$\mathrm{Re}\langle \mathcal{A}^*\widetilde{Z}, \widetilde{Z}\rangle = -d\int_0^1 |\widetilde{q}'(x)|^2dx \le 0. \tag{14}$$

Therefore, both $\mathcal{A}$ and $\mathcal{A}^*$ are strongly dissipative in $\mathcal{H}$. By the Lumer-Philips theorem[17], $\mathcal{A}$ generates a C_0-semigroup $\mathrm{e}^{\mathcal{A}t}$ of contractions on $\mathcal{H}$. ∎

3 Spectral Analysis for System (9)

In this section, we investigate the distribution of spectrum of $\mathcal{A}$ in the complex plane. Some analytic methods in [18, 19] will be adopted here. Firstly, we consider the eigenvalue problem of (9). By $\mathcal{A}Z = \lambda Z$, where $Z = (f, g, h, q) \in D(\mathcal{A})$, we have

$$\begin{cases} g = \lambda f, \quad -kf + bq(1) = \lambda g, \quad q = \lambda h, \\ (h + dq)'' = \lambda q, \quad h(0) = 0, \quad h'(1) + dq'(1) = -bg. \end{cases} \tag{15}$$

So,

$$\begin{cases} g = \lambda f, \quad -kf + bq(1) = \lambda^2 f, \quad q = \lambda h, \\ (1 + d\lambda)h'' = \lambda q, \quad h(0) = 0, \quad h'(1) + dq'(1) = -bg. \end{cases} \tag{16}$$

Theorem 3.1 following shows that $\sigma_r(\mathcal{A})$, the set of residual spectrum of $\mathcal{A}$, is empty. As usual, $\sigma_p(\mathcal{A})$ and $\sigma_c(\mathcal{A})$ denote the set of point spectrum and continuous spectrum, respectively.

Theorem 3.1 *Let $\mathcal{A}$ be given by* (7) *and* (8)*, then* $\sigma_r(\mathcal{A}) = \emptyset$.

Proof Since $\lambda \in \sigma_r(\mathcal{A})$ if and only if $\overline{\lambda} \in \sigma_p(\mathcal{A}^*)$, the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. From (10) and (11), the eigenvalue problem $\mathcal{A}^*F = \lambda F$, where $F = (f, g, h, q) \in D(\mathcal{A}^*)$ reads:

$$\begin{cases} -g = \lambda f, \ \ kf - bq(1) = \lambda g, \ \ -q(x) = \lambda h(x), \\ -(h - dq)'' = \lambda q, \ \ h(0) = 0, \ \ h'(1) - dq'(1) + bg = 0. \end{cases} \tag{17}$$

Obviously, (17) is the same as (15) by setting $\widetilde{g} = -g, \widetilde{h} = -h$. Hence, $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$. Since the eigenvalues of $\mathcal{A}$ are symmetric with the real axis, we have $\sigma_r(\mathcal{A}) = \emptyset$. ∎

Theorem 3.2 *Let* $\mathcal{A}$ *be given by* (7) *and* (8). *Then* $\mathcal{A}^{-1}$ *exists, and hence* $0 \in \rho(\mathcal{A})$, *the resolvent set of* $\mathcal{A}$.

Proof For any given $Z_1 = (f_1, g_1, h_1, q_1) \in \mathcal{H}$, solve

$$\mathcal{A}(f, g, h, q) = (g, -kf + bq(1), q, (h + dq)'') = (f_1, g_1, h_1, q_1), \ (f, g, h, q) \in D(\mathcal{A})$$

to get

$$\begin{cases} g = f_1, \ \ -kf + bq(1) = g_1, \ \ q = h_1, \\ (h + dq)'' = q_1, \ \ h(0) = 0, \ \ h'(1) + dq'(1) + bg = 0. \end{cases} \tag{18}$$

Then we have

$$\begin{cases} f = \dfrac{-1}{k}\big(g_1 - bh_1(1)\big), \quad g = f_1, \quad q(x) = h_1(x), \\ h(x) = (-dh_1'(1) - bf_1)x - \displaystyle\int_0^x s(q_1(s) - dh_1''(s))ds - x\int_x^1 (q_1(s) - dh_1''(s))ds. \end{cases} \tag{19}$$

Hence, $\mathcal{A}^{-1}$ exists, and $0 \in \rho(\mathcal{A})$. ∎

Lemma 3.3 *Let* $\mathcal{A}$ *be given by* (7) *and* (8). *Then* $-\frac{1}{d} \in \sigma_c(\mathcal{A})$.

Proof By Theorem 3.1, it suffices to show $-\frac{1}{d} \notin \sigma_p(\mathcal{A}) \cup \rho(\mathcal{A})$.

(i) Obviously, $0 \notin \sigma_p(\mathcal{A})$. Suppose $\lambda = -\frac{1}{d}$ is an eigenvalue of $\mathcal{A}$, then by the fourth equation of (16), $q(x) \equiv 0$, and hence $h(x) \equiv 0, f = g = 0$. This contradiction shows $-\frac{1}{d} \notin \sigma_p(\mathcal{A})$.

(ii) For any $G = (\widetilde{f}, \widetilde{g}, \widetilde{h}, \widetilde{q}) \in \mathcal{H}$, solve $(\lambda I - \mathcal{A})F = G, F = (f, g, h, q) \in D(\mathcal{A})$, that is,

$$\begin{cases} \lambda f - g = \widetilde{f}, \ \ \lambda g - (-kf + bq(1)) = \widetilde{g}, \ \ \lambda h - q = \widetilde{h}, \\ \lambda q - (h + dq)'' = \widetilde{q}, \ \ h(0) = 0, \ \ h'(1) + dq'(1) = -bg. \end{cases} \tag{20}$$

Substitute the third equation into the fourth equation, we have $\lambda^2 q - ((1 + d\lambda)q + \widetilde{h})'' = \lambda\widetilde{q}$. If $\lambda = -\frac{1}{d}$, it has $\lambda^2 q - \widetilde{h}'' = \lambda\widetilde{q}$, i.e.,

$$\lambda^2 q = \widetilde{h}'' + \lambda\widetilde{q}.$$

Since $q \in H_L^1(0, 1)$, the above equality holds true unless $\widetilde{h}'' + \lambda\widetilde{q} \in H_L^1(0, 1)$. This shows that $-\frac{1}{d} \notin \rho(\mathcal{A})$.

Combining the above two cases completes the proof. ∎

Theorem 3.4 *Let $\mathcal{A}$ be given by* (7) *and* (8), *and let*

$$\Delta(\lambda) = (\lambda^2 + k)\sqrt{1 + d\lambda}\cosh\sqrt{\frac{\lambda^2}{1 + d\lambda}} + b^2\lambda\sinh\sqrt{\frac{\lambda^2}{1 + d\lambda}}. \tag{21}$$

Then

$$\sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} | \Delta(\lambda) = 0\}, \tag{22}$$

and each $\lambda \in \sigma_p(\mathcal{A})$ is geometrically simple.

Proof The eigenvalue problem (15) is equivalent to

$$f = -\frac{1 + d\lambda}{b\lambda}h'(1), \quad g = -\frac{1 + d\lambda}{b}h'(1), \quad q = \lambda h \tag{23}$$

and h satisfies

$$h'' = \frac{\lambda^2}{1 + d\lambda}h, \quad h(0) = 0, \quad (1 + d\lambda)(\lambda^2 + k)h'(1) + b^2\lambda^2 h(1) = 0. \tag{24}$$

So $(f, g, h, q) \neq 0$ if and only if (24) has nonzero solution. Since the general solutions of the first two equations of (24) are of the form

$$h(x) = c\sinh\sqrt{\frac{\lambda^2}{1 + d\lambda}}x, \tag{25}$$

where c is a constant. Substitute (25) into the boundary condition of (24) to get

$$\left((\lambda^2 + k)\sqrt{1 + d\lambda}\cosh\sqrt{\frac{\lambda^2}{1 + d\lambda}} + b^2\lambda\sinh\sqrt{\frac{\lambda^2}{1 + d\lambda}}\right)c = 0. \tag{26}$$

Then $(f, g, h, q) \neq 0$ if and only if

$$(\lambda^2 + k)\sqrt{1 + d\lambda}\cosh\sqrt{\frac{\lambda^2}{1 + d\lambda}} + b^2\lambda\sinh\sqrt{\frac{\lambda^2}{1 + d\lambda}} = 0$$

has solutions. ∎

Theorem 3.5 *Let $\mathcal{A}$ be given by* (7) *and* (8). *Then $\sigma_c(\mathcal{A}) = \{-\frac{1}{d}\}$.*

Proof $\forall \lambda \notin \sigma_p(\mathcal{A})$, and $\lambda \neq 0, \lambda \neq -\frac{1}{d}$, it suffices to show that $\lambda \in \rho(\mathcal{A})$. In fact, for any $G = (\widetilde{f}, \widetilde{g}, \widetilde{h}, \widetilde{q}) \in \mathcal{H}$, solve $(\lambda I - \mathcal{A})F = G, F = (f, g, h, q) \in D(\mathcal{A})$, that is, (20), we rewrite it here for convenience:

$$\begin{cases} \lambda f - g = \widetilde{f}, \quad \lambda g - (-kf + bq(1)) = \widetilde{g}, \quad \lambda h - q = \widetilde{h}, \\ \lambda q - (h + dq)'' = \widetilde{q}, \quad h(0) = 0, \quad h'(1) + dq'(1) = -bg. \end{cases}$$

By the third equation of (20),

$$q = \lambda h - \widetilde{h}. \tag{27}$$

Substitute this into the fourth equation of (20), we can obtain the following problem

$$h'' - \frac{\lambda^2}{1 + d\lambda}h = \frac{d\widetilde{h}'' - \lambda\widetilde{h} - \widetilde{q}}{1 + d\lambda}, \quad h(0) = 0. \tag{28}$$

The solution of (28) is

$$h(x) = c\sinh\sqrt{\frac{\lambda^2}{1+d\lambda}}x + \sqrt{\frac{1+d\lambda}{\lambda^2}}\int_0^x \frac{d\widetilde{h}''(s) - \lambda\widetilde{h}(s) - \widetilde{q}(s)}{1+d\lambda}\sinh\sqrt{\frac{\lambda^2}{1+d\lambda}}(x-s)ds, \quad (29)$$

where c is a constant which is to be uniquely determined by the boundary condition of (20).

By the first equality of (20),

$$g = \lambda f - \widetilde{f}. \quad (30)$$

Substitute this into the second equation of (20), we have

$$(\lambda^2 + k)f - \lambda\widetilde{f} - bq(1) = \widetilde{g}. \quad (31)$$

(i) If $\lambda^2 + k = 0$, then $q(1) = -\frac{\lambda\widetilde{f}+\widetilde{g}}{b}$. Combining with (27), $h(1) = \frac{q(1)+\widetilde{h}(1)}{\lambda} = \frac{-\lambda\widetilde{f}-\widetilde{g}+b\widetilde{h}(1)}{b\lambda}$. Substitute this into (29), we can get the constant c:

$$c = \frac{\frac{-\lambda\widetilde{f}-\widetilde{g}+b\widetilde{h}(1)}{b\lambda} - \sqrt{\frac{1+d\lambda}{\lambda^2}}\int_0^1 \frac{d\widetilde{h}''(s)-\lambda\widetilde{h}(s)-\widetilde{q}(s)}{1+d\lambda}\sinh\sqrt{\frac{\lambda^2}{1+d\lambda}}(1-s)ds}{\sinh\sqrt{\frac{\lambda^2}{1+d\lambda}}}. \quad (32)$$

Hence, by (27), the expression of $q(x)$ is obtained. By the sixth equation of (20), we can get the expression of g : $g = -\frac{h'(1)+dq'(1)}{b}$, and then $f = \frac{g+\widetilde{f}}{\lambda}$.

(ii) If $\lambda^2 + k \neq 0$, then $f = \frac{\lambda\widetilde{f}+\widetilde{g}+bq(1)}{\lambda^2+k}$. Substitute this into (30),

$$g = \frac{-k\widetilde{f} + \lambda\widetilde{g} + b\lambda^2 h(1) - b\lambda\widetilde{h}(1)}{\lambda^2 + k}. \quad (33)$$

Then the sixth equation of (20) becomes

$$(1+d\lambda)h'(1) + \frac{b^2\lambda^2}{\lambda^2+k}h(1) = d\widetilde{h}'(1) + \frac{b^2\lambda}{\lambda^2+k}\widetilde{h}(1) - \frac{b(-k\widetilde{f}+\lambda\widetilde{g})}{\lambda^2+k}. \quad (34)$$

Similarly, substitute this into (29), we can get the expression of the constant c, and then the expression of (f, g, h, q) is followed.

Therefore, in all cases, we can get the unique solution (f, g, h, q) of (20) by a series of calculation. It shows that $(\lambda I - \mathcal{A})^{-1}$ exists and is bounded, or $\lambda \in \rho(\mathcal{A})$. ∎

Theorem 3.6 *Let* $\mathcal{A}$ *be given by* (7) *and* (8). *Then for each* $\lambda \in \sigma_p(\mathcal{A})$, *we have* $\mathrm{Re}\,\lambda < 0$.

Proof By Lemma 2.2, we have $\mathrm{Re}\,\lambda \leq 0$, $\forall\lambda \in \sigma_p(\mathcal{A})$. So we only need to show there is no eigenvalues on the imaginary axis. Let $\lambda = \mathrm{i}a \in \sigma_p(\mathcal{A})$, $a \in \mathbb{R}$ and let $Z = (f, g, h) \in D(\mathcal{A})$ be its associated eigenfunction of $\mathcal{A}$. Then by (13), we have

$$\mathrm{Re}\langle \mathcal{A}Z, Z\rangle = -d\int_0^1 |q'(x)|^2 dx = 0, \quad (35)$$

and hence $q'(x) = 0$. By $q(0) = 0$, we have $q \equiv 0$. Moreover, it follows from (15) that $h = f = g = 0$. Therefore, there is no eigenvalues on the imaginary axis. ∎

4 Exponential Stability for System (9)

In this section, we will give the asymptotic expressions of eigenvalues and eigenfunctions, and establish the exponential stability for System (9).

Proposition 4.1 *Let $\mathcal{A}$ be given by* (7) *and* (8)*, and let* $\Delta(\lambda)$ *be given by* (21)*. There are two branches of eigenvalues, the first branch is*

$$\lambda_{n1} = \left(-\frac{1}{d}\right)^{-} + \varepsilon_n, \quad \textit{where} \quad \varepsilon_n = \frac{1}{2d - n^2\pi^2 d^3} = \mathcal{O}(n^{-2}),\ n > N, \tag{36}$$

which means λ_{n1} *is accumulating towards* $-\frac{1}{d}$ *from the left of* $-\frac{1}{d}$ *as* $n \to \infty$*; The other branch of eigenvalues have the following asymptotic expressions:*

$$\lambda_{n2} = -\left(n - \frac{1}{2}\right)^2 \pi^2 d + \frac{1}{d} + \mathcal{O}(n^{-2}),\ n > N, \tag{37}$$

where N *is a positive number. Therefore,* $\mathrm{Re}\,\lambda_{n2} \to -\infty$*, as* $n \to \infty$.

Proof By $\Delta(\lambda) = 0$, we get

$$(\lambda^2 + k)\sqrt{1 + d\lambda}\left(\mathrm{e}^{\sqrt{\frac{\lambda^2}{1+d\lambda}}} + \mathrm{e}^{-\sqrt{\frac{\lambda^2}{1+d\lambda}}}\right) + b^2\lambda\left(\mathrm{e}^{\sqrt{\frac{\lambda^2}{1+d\lambda}}} - \mathrm{e}^{-\sqrt{\frac{\lambda^2}{1+d\lambda}}}\right) = 0,$$

or

$$\left[(\lambda^2 + k)\sqrt{1 + d\lambda} + b^2\lambda\right]\mathrm{e}^{2\sqrt{\frac{\lambda^2}{1+d\lambda}}} + \left[(\lambda^2 + k)\sqrt{1 + d\lambda} - b^2\lambda\right] = 0,$$

which yields

$$\mathrm{e}^{2\sqrt{\frac{\lambda^2}{1+d\lambda}}} = -\frac{(\lambda^2 + k)\sqrt{1 + d\lambda} - b^2\lambda}{(\lambda^2 + k)\sqrt{1 + d\lambda} + b^2\lambda} = -1 + \frac{2b^2}{\lambda\sqrt{1 + d\lambda} + k\frac{\sqrt{1+d\lambda}}{\lambda} + b^2}. \tag{38}$$

It is obvious that $\lambda_{n1} = \left(-\frac{1}{d}\right)^{-} + \frac{1}{2d - n^2\pi^2 d^3}$, $n > N$, are asymptotic expressions of solutions of (38).

On the other hand, (38) can be written as

$$\mathrm{e}^{2\sqrt{\frac{\lambda^2}{1+d\lambda}}} = -1 + \mathcal{O}(|\lambda|^{-\frac{3}{2}}), \tag{39}$$

if $|\lambda| \to \infty$. A direct computation gives

$$\sqrt{\frac{\lambda_{n2}^2}{1 + d\lambda_{n2}}} = \mathrm{i}\left(n - \frac{1}{2}\right)\pi + \mathcal{O}(n^{-3}), \quad n > N, \tag{40}$$

where N is a positive number. Therefore,

$$\frac{\lambda_{n2}^2}{1 + d\lambda_{n2}} = -\left(n - \frac{1}{2}\right)^2 \pi^2 + \mathcal{O}(n^{-2}), \quad n > N. \tag{41}$$

Hence,

$$\frac{1}{d}\lambda_{n2} - \frac{1}{d^2} = -\left(n - \frac{1}{2}\right)^2 \pi^2 + \mathcal{O}(n^{-2}), \quad n > N, \tag{42}$$

and then

$$\lambda_{n2} = -\left(n - \frac{1}{2}\right)^2 \pi^2 d + \frac{1}{d} + \mathcal{O}(n^{-2}), \quad n > N. \tag{43}$$

The proof is then completed. ∎

Remark 4.2 From the asymptotic expression (37), it is found that the principal parts of the spectrum of operator $\mathcal{A}$ with large modulus have the same expressions with comparison to the principal parts of the K-V damped wave equation with no coupling ODE.

Proposition 4.3 *Let* $\{\lambda_{n1}, n \in \mathbb{N}\}$ *be the eigenvalues of* $\mathcal{A}$ *with* λ_{n1} *being given by* (36). *Then the corresponding eigenfunctions* $\{(f_{n1}, g_{n1}, h_{n1}, q_{n1}), n \in \mathbb{N}\}$ *have the following asymptotic expressions:*

$$\begin{cases} f_{n1} = \mathcal{O}(n^{-2}), \\ g_{n1} = \mathcal{O}(n^{-2}), \\ h_{n1} = \dfrac{1}{n\pi} \sin n\pi x + \mathcal{O}(n^{-2}), \\ q_{n1} = \mathcal{O}(n^{-1}), \end{cases} \qquad n > N, \tag{44}$$

where N *is a positive number.*

Proof It is found from (36) that

$$\frac{\lambda_{n1}^2}{1 + d\lambda_{n1}} = \frac{\frac{1}{d^2} + 2\varepsilon_n(-\frac{1}{d}) + \varepsilon_n^2}{d\varepsilon_n} = -n^2\pi^2 + \mathcal{O}(n^{-2}). \tag{45}$$

Therefore, combining with (25), we can choose

$$h_n(x) = -\mathrm{i} \sinh \mathrm{i} n\pi x + \mathcal{O}(n^{-2}) = \sin n\pi x + \mathcal{O}(n^{-2}), \quad n > N. \tag{46}$$

Moreover, by the third equality of (23), we get that

$$q_n(x) = \lambda_{n1} h_n(x) = -\frac{1}{d} \sin n\pi x + \mathcal{O}(n^{-2}). \tag{47}$$

Since

$$h_n'(1) = n\pi \cos n\pi + \mathcal{O}(n^{-1}) = (-1)^n n\pi + \mathcal{O}(n^{-1}), \tag{48}$$

thus, by the second equality of (23), we can get g_n as following:

$$g_n = -\frac{d\varepsilon_n}{b} h_n'(1) = \mathcal{O}(n^{-1}), \tag{49}$$

and hence

$$f_n = \frac{g_n}{\lambda_{n1}} = \mathcal{O}(n^{-1}). \tag{50}$$

Moreover, normalize the above eigenfunctions in $\mathcal{H}$, we can get the approximate unit eigenfunctions (44). ∎

Proposition 4.4 *Let* $\{\lambda_{n2}, n \in \mathbb{N}\}$ *be the eigenvalues of* $\mathcal{A}$ *with* λ_{n2} *being given by* (37). *Then the corresponding eigenfunctions* $\{(f_{n2}, g_{n2}, h_{n2}, q_{n2}), n \in \mathbb{N}\}$ *have the following asymptotic expressions:*

$$\begin{cases} f_{n2} = \mathcal{O}(n^{-4}), \\ g_{n2} = \mathcal{O}(n^{-2}), \\ h_{n2} = \mathcal{O}(n^{-1}), \\ q_{n2} = \sin\left(n - \dfrac{1}{2}\right)\pi x + \mathcal{O}(n^{-3}), \end{cases} \qquad n > N, \tag{51}$$

where N *is a positive number.*

Proof It is found from (25) and (40) that for λ_{n2}, its corresponding eigenfunction has the asymptotic form:

$$h_n(x) = -\mathrm{i}\sinh\sqrt{\frac{\lambda_{n2}^2}{1+d\lambda_{n2}}}x = \sin\left(n - \frac{1}{2}\right)\pi x + \mathcal{O}(n^{-3}), \quad n > N. \tag{52}$$

Moreover, by the third equality of (23), we get that

$$q_n(x) = \lambda_{n2}h_n(x) = \left[-\left(n - \frac{1}{2}\right)^2 \pi^2 d + \frac{1}{d}\right] \sin\left(n - \frac{1}{2}\right)\pi x + \mathcal{O}(n^{-1}). \tag{53}$$

Since

$$h_n'(1) = \left(n - \frac{1}{2}\right)\pi \cos\left(n - \frac{1}{2}\right)\pi + \mathcal{O}(n^{-2}) = \mathcal{O}(n^{-2}), \tag{54}$$

thus, by the second equality of (23),

$$g_n = -\frac{1+d\lambda_{n2}}{b}h_n'(1) = \mathcal{O}(1), \tag{55}$$

and hence

$$f_n = \frac{g_n}{\lambda_{n2}} = \mathcal{O}(n^{-2}). \tag{56}$$

Moreover, normalize the above eigenfunctions in $\mathcal{H}$, we can get the approximate unit eigenfunctions (51). ∎

Now we show the Riesz basis generation and exponential stability.

Theorem 4.5 *Let* $\mathcal{A}$ *be given by* (7) *and* (8). *Then there is a set of generalized eigenfunctions of* $\mathcal{A}$ *which forms a Riesz basis for* $\mathcal{H}$ *and each eigenvalue of* $\mathcal{A}$ *with large modulus is algebraically simple. Moreover,* $\mathrm{e}^{\mathcal{A}t}$, *generated by* $\mathcal{A}$, *is an analytic semigroup for* $\mathcal{H}$.

Proof Let $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, and

$$F_{n1} = \left(0, 0, \frac{1}{n\pi}\sin n\pi x, 0\right), \quad F_{n2} = (0,0,0,\sin(n - 1/2)\pi x), \quad n \in \mathbb{N}.$$

Then $\{e_1, e_2, F_{n1}, F_{n2}, n \in \mathbb{N}\}$ form an orthogonal basis for $\mathcal{H}$. Let $G_{ni} = (f_{ni}, g_{ni}, h_{ni}, q_{ni})$, $n \in \mathbb{N}$, $i = 1,2$, where $(f_{ni}, g_{ni}, h_{ni}, q_{ni})$ are given by (44) and (51), then

$$\sum_{n=N}^{\infty}\sum_{i=1}^{2}||F_{ni} - G_{ni}||_{\mathcal{H}}^2 = \sum_{n=N}^{\infty}\left(\sum_{i=1}^{2}(|f_{ni}|^2 + |g_{ni}|^2) + |q_{n1}|^2 + |h_{n2}|^2\right) = \sum_{n=N}^{\infty}\mathcal{O}(n^{-2}) < \infty. \tag{57}$$

By Theorem 6.3 of [5], a modified classical Bari's theorem, there is a set of generalized eigenfunctions of $\mathcal{A}$ which forms a Riesz basis for $\mathcal{H}$ and each eigenvalue of the system operator $\mathcal{A}$ with large modulus is algebraically simple. Finally, by Theorem 13 of [20], $\mathcal{A}$ generates an analytic semigroup $e^{\mathcal{A}t}$. ∎

Theorem 4.6 *Let* $\mathcal{A}$ *be given by* (7) *and* (8). *Then*

1) *the spectrum-determined growth condition holds true for* $e^{\mathcal{A}t}$, *that is,* $s(\mathcal{A}) = \omega(\mathcal{A})$, *where* $s(\mathcal{A}) := \sup\{\text{Re}\lambda | \lambda \in \sigma(\mathcal{A})\}$ *is the spectral bound of* $\mathcal{A}$, *and* $\omega(\mathcal{A}) := \inf\{\omega | \exists M > 0$ *such that* $||e^{\mathcal{A}t}|| \leq Me^{\omega t}\}$ *denotes the growth bound of* $e^{\mathcal{A}t}$.

2) *the system* (9) *is exponentially stable, that is, there exist two positive constants* M *and* σ *such that the* C_0*-semigroup* $e^{\mathcal{A}t}$ *generated by* $\mathcal{A}$ *satisfies* $||e^{\mathcal{A}t}|| \leq Me^{-\sigma t}$.

Proof By Theorem 4.5, we have that $\mathcal{A}$ generates an analytical semigroup. Hence, as a conclusion, the spectrum-determined growth condition holds. By Theorems 3.1, 3.5, and 3.6, we have

$$\text{Re}\lambda < 0, \quad \forall \lambda \in \sigma(\mathcal{A}).$$

Hence, $e^{\mathcal{A}t}$ is exponentially stable. ∎

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